On the diameter of proportional sections of a symmetric convex body

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Abstract

Let K be a symmetric convex body in $\mathbb{R}^n$. Given any $\lambda \in (\frac{1}{2}, 1)$, we give lower and upper bounds for the diameter of a random $[\lambda n]$-dimensional section of K. We are interested in a description of the bounds which might be useful from the computational geometry point of view. Our approach is based on the function $M_K^*(r) = \frac{1}{r} M^*(K \cap rD)$ which is easily computable, and makes use of the low M^*-estimate, a new conditional low M-estimate and Borsuk's antipodal theorem. In the case of an α-regular body in M-position, the ratio of our bounds is independent of K and n.

1 Introduction

Let K be a symmetric convex body in $\mathbb{R}^n$. In this paper we study the following question:

Given any $\lambda \in (\frac{1}{2}, 1)$, *find an interval* $I = I_K(\lambda) = [r_1, r_2]$ *where* $r_i = r_i(K, \lambda)$, $i = 1, 2$, *such that most of the* $[\lambda n]$*-dimensional sections of* K *have diameter in* I *and* r_2/r_1 *is as small as possible.*

One naturally has to make precise the meaning of "most": we are interested in an estimate of the form

$$\nu_{n,[\lambda n]} \left(E \in G_{n,[\lambda n]} : \mathrm{diam}(K \cap E) \in I \right) \geq 1 - h(\lambda, n),$$

for some function h tending as fast as possible to 0 when $n \to \infty$, where $G_{n,k}$ is the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,k}$.

We were led to the formulation of this question by discussions with L. Lovasz and M. Simonovits on the computational problems arising when one wants to determine the diameter of a symmetric convex body in $\mathbb{R}^n$: it is known that it is impossible to give a good estimate of the diameter in less than exponential (in the dimension) time. Therefore, dealing with our question, we are at the same time interested in a description of the bounds r_1 and r_2 which might be useful from the computational geometry point of view.

Our method is to push to its limit a well-known and crucial inequality of the asymptotic theory of finite dimensional normed spaces, the Low M^*-estimate [M2], [PT], [Go]. In order to describe our approach we need to introduce a few related notions: If W is a symmetric convex body in $\mathbb{R}^n$ we write $\|.\|_W$ for the norm induced to $\mathbb{R}^n$ by W and define $M(W) = \int_{S^{n-1}} \|x\|_W \ \sigma(dx)$, where σ is the rotationally invariant probability measure on the Euclidean unit sphere S^{n-1}. Also, if W^o is the polar body of W, let $M^*(W) = M(W^o)$ (this quantity has a natural geometric meaning, being half of the mean width of W). The Low M^*-estimate states that there is a function $f_2 : (0,1) \to (0,1)$ – one can actually choose $f_2(\lambda) = c_2\sqrt{1-\lambda}$ for some absolute constant $c_2 > 0$ – such that for every W and every $\lambda \in (0,1)$,

$$\text{diam}(W \cap E) \leq \frac{2M^*(W)}{f_2(\lambda)} \tag{1.1}$$

for most $E \in G_{n,[\lambda n]}$.

This essentially gives an upper bound for the diameter of the proportional sections of an arbitrary body K: From (1.1) we easily deduce (Theorem 2.1) that for every K and every $\lambda \in (0,1)$, if $r > 0$ satisfies the inequality $\frac{1}{r}M^*(K \cap rD) \leq f_2(\lambda)$ where D is the Euclidean unit ball in $\mathbb{R}^n$, then

$$\text{diam}(K \cap E) \leq 2r$$

for most $E \in G_{n,[\lambda n]}$.

It turns out that this application of the Low M^*-estimate leads to bounds which are "already exact": there exists a second function $f_1 : (0,1) \to (0,1)$ such that for every K and every $\lambda \in (\frac{1}{2}, 1)$, if $r > 0$ satisfies the inequality $\frac{1}{r}M^*(K \cap rD) \geq f_1(\lambda)$, then

$$\text{diam}(K \cap E) \geq 2g(\lambda)r$$

for most $E \in G_{n,[\lambda n]}$ (Theorem 2.7). One can actually see that $f_1(\lambda)$ and $g(\lambda)$ may be chosen to be two absolute constants c_1 and c_1' in $(0,1)$ (which can be written down explicitely and work for all $\lambda \in (\frac{1}{2}, 1)$). What is of importance is of course that both functions f_1 and f_2 are independent of the body K and the dimension n.

In view of the above, let us associate to each symmetric convex body K the function $M_K^* : (0, \infty) \to (0, 1]$ defined by

$$M_K^*(r) = \frac{M^*(K \cap rD)}{r}.$$

The function M_K^* is onto $(0,1]$ and decreasing, and if ρ_1, ρ_2 are the radii of the inscribed and circumscribed balls of K, then $M_K^*(r) = 1$ on $(0, \rho_1]$ and $M_K^*(r) =$

$M^*(K)/r$ on $[\rho_2, \infty)$. Now, we can qualitatively describe our main result in terms of M_K^* as follows:

General Statement: *There exist three functions f_1, f_2 and $g : (0,1) \to (0,1)$ such that the following holds: Given a symmetric convex body K in $\mathbb{R}^n$ and any $\lambda \in (\frac{1}{2}, 1)$, let $r_i = r_i(K, \lambda)$, $i = 1, 2$, be the solutions of the equations*

$$M_K^*(r) = f_i(\lambda), \quad i = 1, 2, \tag{1.2}$$

in r. Then, we have

$$\mathrm{diam}(K \cap E) \in [2g(\lambda) r_1(K, \lambda), 2r_2(K, \lambda)],$$

for all $E \in \mathcal{L}_{n,k}$, where $\mathcal{L}_{n,k}$ is a subset of $G_{n,k}$ of measure $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - h(\lambda, n)$, $k = [\lambda n]$, and $h(\lambda, n) \to 0$ exponentially fast as $n \to \infty$.

Note that the simplest example of the Euclidean unit ball D in $\mathbb{R}^n$ shows that the function g is really needed in the statement above: we have $M_D^*(r) = \frac{1}{r}$ on $[1, \infty)$, hence for any function $f_1 : (0,1) \to (0,1)$ and for any λ, the solution of $M_D^*(r) = f_1(\lambda)$ in r will be greater than 1 while obviously $\mathrm{diam}(D \cap E) = 2$ for every $E \in G_{n,[\lambda n]}$.

The use of the function M_K^* meets the requirement of an effective determination of the bounds r_1 and r_2 in our original question. The reason is that, for any symmetric convex body K, one can "compute" with high probability $M_K^*(r)$ effectively to any given degree of accuracy: The empirical distribution method (described in a similar setting e.g in [BLM]) shows that given any δ and ζ in $(0,1)$, a random choice of $N = [c\frac{\log(\frac{2}{\delta})}{\zeta^2}] + 1$ points $x_1, \ldots, x_N$ in S^{n-1} satisfies

$$\left|M^*(K \cap rD) - \frac{1}{N}\sum_{i=1}^{N} \|x_i\|_{(K\cap rD)^\circ}\right| < \zeta M^*(K \cap rD) \tag{1.3}$$

with probability exceeding $1 - \delta$, where $c > 0$ is an absolute constant. One can therefore assume that $M_K^*(r)$ can be easily determined for every r. Since M_K^* is decreasing, one can then solve the equation $M_K^*(r) = \alpha$ for any given $\alpha < 1$. The number of steps needed depends, for example, on a rough estimate of the ratio ρ_2/ρ_1 of the radii of the circumscribed and the inscribed ball of K.

A second point which is of interest is that our general statement

$$2g(\lambda)(M_K^*)^{-1}(f_1(\lambda)) \leq \mathrm{diam}(K \cap E) \leq 2(M_K^*)^{-1}(f_2(\lambda)) \tag{1.4}$$

may be viewed as an *asymptotic formula* connecting the diameter of a random $[\lambda n]$-dimensional section of K with a quite simple average parameter of K. Compare with the following result obtained recently in [MS2]: Let $k = k(K)$ be the largest integer for which

$$\nu_{n,k}\left(\{E \in G_{n,k} : \frac{M(K)}{2}|x| \leq \|x\|_K \leq 2M(K)|x| \text{ for all } x \in E\}\right) > 1 - \frac{1}{n}.$$

It is a well-known fact [M1] that $k \geq cn(\frac{M(K)}{\mathrm{diam}(K^\circ)})^2$ for some absolute constant $c > 0$. Rather surprisingly, it is observed in [MS2] that the reverse inequality is also true: $k(K) \simeq n(\frac{M(K)}{\mathrm{diam}(K^\circ)})^2$. Again, proving a basic inequality to be exact gives rise to an asymptotic formula connecting the local structure of an arbitrary body K with some of its global parameters. It is an important direction to enrich this list of high dimensional formulas.

The main part of the paper is organized as follows: In Section 2.1 we give the proof of the general statement with an exact description of the functions f_1, f_2 and g, corresponding to one among many interpretations of the requirement that $h(\lambda, n) \to 0$ fast as $n \to \infty$. Our argument for the lower bound makes use of a new "conditional low M-estimate". We also make use of Borsuk's antipodal theorem in an essential way, and this is what forces us to restrict ourselves to the case $\lambda \in (\frac{1}{2}, 1)$.

What is interesting is of course the ratio r_2/r_1 and this makes it clear that the dependence of $(M_K^*)^{-1}$ on $f_1(\lambda)$ and $f_2(\lambda)$ for a given λ is quite important. In Section 2.2 we give an example of an ellipsoid with highly incomparable semiaxes which shows that the behavior of $(M_K^*)^{-1}$ can be very irregular: the interval I may be huge even if the ratio $f_2(\lambda)/f_1(\lambda)$ is very close to 1. This indicates that one cannot expect a completely satisfactory answer with this "one step" determination of I.

On the other hand, what seems to cause problems in our approach is not the geometry of the body K but the fact that K may be in a very bad "position" (like the ellipsoid in our example), in which case even the question doesn't make much sense. In fact, our original goal can be achieved if we allow a linear transformation in order to bring the body K in some kind of a more "regular" position. In Section 2.3 we assume that K is in M-position with parameter α (in the terminology of [Pi]), and we show that for every $\lambda \in (\frac{1}{2}, 1)$ and for most $E \in G_{n,[\lambda n]}$

$$\mathrm{diam}(K \cap E) \simeq (M_K^*)^{-1}(\frac{1}{2}\sqrt{1-\lambda}) \tag{1.5}$$

up to $\psi_\alpha(\lambda)$, where $\psi_\alpha : (0,1) \to \mathbb{R}^+$ is a fixed function depending only on α. Since every body K has an affine image which is in M-position, in this regular but general enough case (1.4) becomes a real asymptotic formula with $f_1(\lambda) = f_2(\lambda) = \frac{1}{2}\sqrt{1-\lambda}$.

We use the standard notation from [MS1]: In paricular, $|.|$ is a fixed Euclidean norm, the Lévy median of $\|.\|_W$ on S^{n-1} is denoted by m or $m(W)$, the boundary of W is denoted by $\mathrm{bd}(W)$, $|N|$ denotes the cardinality of a finite set N, and the letter c is reserved for absolute positive constants.

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2 Upper and lower bounds for the diameter of a random proportional section

2.1. Let us agree that a property of a random k-dimensional section of the body K in $\mathbb{R}^n$ is one that holds for all sections $K \cap E$ with E in a subset $\mathcal{L}_{n,k}$ of $G_{n,k}$ of measure $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - \zeta^n$, where $\zeta = \zeta(\frac{k}{n}) \in (0,1)$. There is nothing specific about this choice of the function h in our general statement: we want to examine more closely the dependence on the other parameters involved in the problem, in particular the ratio $f_2(\frac{k}{n})/f_1(\frac{k}{n})$. Obvious modifications of the arguments given below lead to various other possible estimates depending on what is of interest in each case.

The upper bound for $\text{diam}(K \cap E)$ is a well-known consequence of the low M^*-estimate [M2], [PT], [Go]. We give the statement in the spirit of our present discussion with a brief sketch of the estimates involved in the proof:

Theorem 2.1 *Let $\lambda \in (0,1)$ and $\varepsilon \in (0,1)$. There exist $n_0 = n_0(\varepsilon, \lambda)$ and $\zeta = \zeta(\varepsilon, \lambda) \in (0,1)$ with the following property: for every symmetric convex body K in $\mathbb{R}^n$, $n \geq n_0$, we can find a set $\mathcal{L}_{n,k} \subseteq G_{n,k}$, where $k = [\lambda n]$, of measure $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - \zeta^n$, such that* $\text{diam}(K \cap E) \leq 2r$ *for every $E \in \mathcal{L}_{n,k}$, where r is the solution of the equation*

$$M_K^*(r) = (1-\varepsilon)\sqrt{1-\lambda}.$$

Sketch of the proof: Define $a_s = \sqrt{2}\Gamma(\frac{s+1}{2})/\Gamma(\frac{s}{2})$. One can check that $\frac{a_{n-k}}{a_n} \geq (\frac{n-k-1}{n})^{1/2}$, and this implies that for $n \geq n_0(\varepsilon, \lambda)$ we have

$$\frac{(1-\frac{\varepsilon}{2})a_{n-k}}{a_n(1-\varepsilon)\sqrt{1-\lambda}} \geq 1 + \frac{\varepsilon}{2}. \tag{2.1}$$

Suppose that r satisfies the equation $\frac{M^*(K \cap rD)}{r} = (1-\varepsilon)\sqrt{1-\lambda}$. Then, Gordon's proof of the low M^*-estimate [Go, Corollary 3.4] shows that

$$\|x\|_{K \cap rD} \geq \frac{(1-\frac{\varepsilon}{2})a_{n-k}}{a_n M^*(K \cap rD)}|x| \geq \frac{1+\frac{\varepsilon}{2}}{r}|x|, \ \ x \in E \tag{2.2}$$

for all E in a subset $\mathcal{L}_{n,k}$ of $G_{n,k}$ of measure $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - \frac{7}{2}\exp(-\frac{1}{72}a_{n-k}^2\varepsilon^2)$. Since $\|x\|_{K \cap rD} = \max\{\|x\|_K, \frac{1}{r}|x|\}$, this shows that actually

$$\|x\|_K \geq \frac{1}{r}|x|, \ \ x \in E, \tag{2.3}$$

for every $E \in \mathcal{L}_{n,k}$, and this completes the proof since $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - \zeta^n(\varepsilon, \lambda)$ (observe that $a_{n-k}^2 \simeq n-k$). □

Our lower bound is based on a conditional low M-estimate. We start with the following geometric lemma:

Lemma 2.2 *Let W be a symmetric convex body in $\mathbb{R}^n$ such that $W \supseteq D$. Consider the function $\|.\|_W$ on S^{n-1} and denote its median by m. Then, for every $R > \frac{1}{m}$ we have*

$$\sigma_R(W \cap RS^{n-1}) \leq 1 - \sigma_R(B(\frac{\pi}{2} + \theta_0)),$$

where $\theta_0 \in (0, \frac{\pi}{2})$ is defined by

$$\sin\theta_0 = \frac{m}{R}\left([R^2 - 1]^{\frac{1}{2}} - [(1/m)^2 - 1]^{\frac{1}{2}}\right).$$

Here, σ_R denotes the rotationally invariant probability measure on RS^{n-1}, while $B(\frac{\pi}{2} + \theta_0)$ is a cap of angular radius $\frac{\pi}{2} + \theta_0$ in RS^{n-1}.

Proof: Let $\mathcal{A} = W^c \cap \frac{1}{m} S^{n-1}$ and consider an arbitrary point α on the boundary of $\mathcal{A}$. We then clearly have that $\alpha \in bd(W)$. If $H(\alpha)$ is any hyperplane that supports W at α, let $\delta = P_{H(\alpha)}(o)$ be the orthogonal projection of the origin o onto $H(\alpha)$.

Assume first that the points o, α and δ determine a two-dimensional plane $\Pi(\alpha)$. Write β, γ for the points in $\Pi(\alpha)$ where the lines $\overline{\delta\alpha}$ and $\overline{o\alpha}$ meet RS^{n-1}. Let also $y \geq 1$ be the distance from o to δ. If $\theta = \widehat{\beta o \gamma}$, $\varphi = \widehat{o\alpha\delta}$, and $\eta = \widehat{o\beta\delta}$, we have $\theta = \varphi - \eta$, therefore $\sin\theta = \sin\varphi\cos\eta - \cos\varphi\sin\eta$, and simple trigonometry shows that

$$\sin\theta = \frac{y}{1/m}\frac{[R^2 - y^2]^{\frac{1}{2}}}{R} - \frac{[(1/m)^2 - y^2]^{\frac{1}{2}}}{1/m}\frac{y}{R} \tag{2.4}$$
$$= \frac{my}{R}\left([R^2 - y^2]^{\frac{1}{2}} - [(1/m)^2 - y^2]^{\frac{1}{2}}\right).$$

We easily check that this is an increasing function of y on $[1, \frac{1}{m}]$, and this shows that

$$\sin\theta \geq \frac{m}{R}\left([R^2 - 1]^{\frac{1}{2}} - [(1/m)^2 - 1]^{\frac{1}{2}}\right) = \sin\theta_0. \tag{2.5}$$

If this is not the case, then we actually have that $H(\alpha)$ is uniquely determined and $\alpha = \delta$. Let γ be the point where $\overline{o\alpha}$ meets RS^{n-1}, and for any two-dimensional plane $\Pi(\alpha)$ containing $o\alpha$ write β for the point where the line in $\Pi(\alpha)$ perpendicular to $o\alpha$ at α meets RS^{n-1}. If $\theta = \widehat{\beta o \gamma}$, we readily see that

$$\sin\theta = \frac{[R^2 - (1/m)^2]^{\frac{1}{2}}}{R} \geq \sin\theta_0. \tag{2.6}$$

Observe that, in both cases, $H(\alpha)$ separates the cap $B(\gamma, \theta_0)$ in RS^{n-1} from W. Since the points $\gamma = \gamma(\alpha)$, $\alpha \in bd(\mathcal{A})$, form the boundary of $(Rm)\mathcal{A}$, we conclude that $W \cap ((Rm)\mathcal{A})_{\theta_0} = \emptyset$, where

$$((Rm)\mathcal{A})_{\theta_0} = \{z \in RS^{n-1} : z \in B(x, \theta_0) \text{ for some } x \in (Rm)\mathcal{A}\}.$$

On the other hand, by the definition of the median m we have $\sigma_{1/m}(\mathcal{A}) \geq \frac{1}{2}$ and hence $\sigma_R((Rm)\mathcal{A}) \geq \frac{1}{2}$. From the isoperimetric inequality on the sphere RS^{n-1}

(see [FLM] or [MS1]) it follows that $\sigma_R(((Rm)\mathcal{A})_{\theta_0}) \geq \sigma_R(B(\frac{\pi}{2}+\theta_0))$, and this means that

$$\sigma_R(W \cap RS^{n-1}) \leq 1 - \sigma_R(B(\frac{\pi}{2}+\theta_0)). \quad \Box \tag{2.7}$$

This Lemma shows that if m is close to 1, and if R is chosen suitably large, then a big part of W stays inside RD. In the next Lemma we make the dependence on the various parameters more precise in order to extract sections of W of (any) proportional dimension inside RD:

Lemma 2.3 *Let $\lambda \in (0,1)$ and $k = [\lambda n]$. There exists $n_0 = n_0(\lambda)$ for which the following holds: If $\zeta < 1$ and $\varepsilon \leq \varepsilon_0(\zeta, \lambda) = \frac{2}{5}[\frac{1}{2}(\frac{\zeta}{3})^{\lambda}]^{\frac{2}{1-\lambda}}$, then for every symmetric convex body W in $\mathbb{R}^n$, $n \geq n_0$, with $W \supseteq D$ and $m(W) \geq 1-\varepsilon$, we can find $\mathcal{L}_{n,k} \subseteq G_{n,k}$ of measure $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - \zeta^k$, such that*

$$W \cap E \subseteq 5(\frac{3}{\zeta})^{\frac{\lambda}{1-\lambda}} 2^{\frac{1}{1-\lambda}} D \cap E$$

for every $E \in \mathcal{L}_{n,k}$.

Proof: We assume from the beginning that $\varepsilon < \frac{1}{2}$. Let $R = R(\zeta, \lambda)$ be a function of ζ and λ to be determined, and define θ_0 by the equation $\sin\theta_0 = \frac{m}{R}([R^2-1]^{\frac{1}{2}} - [(1/m)^2 - 1]^{\frac{1}{2}})$. This is an increasing function of m, therefore

(1) $\sin\theta_0 \geq (1-\varepsilon)[1 - \frac{1}{R^2}]^{\frac{1}{2}} - \frac{1}{R}[\varepsilon(2-\varepsilon)]^{\frac{1}{2}} \geq c_1$, provided that, say, $R \geq 2$.

A computation analogous to the one in (2.4) shows that

(2) $\cos\theta_0 \leq [1 - \frac{1}{R^2}]^{\frac{1}{2}}[1-m^2]^{\frac{1}{2}} + \frac{m}{R} \leq \sqrt{2\varepsilon} + \frac{1}{R}$.

Let $J_n = \int_0^{\pi/2} \cos^{n-2} t\, dt$. By Lemma 2.2 we know that

$$\sigma_R(W \cap RS^{n-1}) \leq \frac{1}{2J_n} \int_{\theta_0}^{\frac{\pi}{2}} \cos^{n-2} t\, dt \leq \frac{1}{2J_n(n-1)} \frac{\cos^{n-1}\theta_0}{\sin\theta_0}, \tag{2.8}$$

and since $J_n \geq c_2/\sqrt{n}$, we arrive at

$$\sigma_R(W \cap RS^{n-1}) \leq \frac{c_3}{\sqrt{n}}[\sqrt{2\varepsilon} + \frac{1}{R}]^{n-1}. \tag{2.9}$$

Consider now a $\frac{4}{5R}$-net N on S^{k-1}. This can be done with $|N| \leq (1+\frac{5}{2}R)^k \leq [3R]^k$. A standard argument shows that if

$$|N|\sigma_R(W \cap RS^{n-1}) \leq \zeta^k, \tag{2.10}$$

then there exists $\mathcal{L}_{n,k} \subseteq G_{n,k}$ with $\nu_{n,k}(\mathcal{L}_{n,k}) \geq 1 - \zeta^k$ such that for every $E \in \mathcal{L}_{n,k}$ there exists a $\frac{4}{5}$-net of $E \cap RS^{n-1}$ disjoint from W. This means that if

$x \in E \cap RS^{n-1}$, we can find $y \in RS^{n-1}$ for which $\|y\|_W \geq 1$ and $|x-y| \leq \frac{4}{5}$, therefore

$$\|x\|_W \geq \|y\|_W - \|x-y\|_W \geq 1 - \frac{4}{5} = \frac{1}{5R}|x|$$

or, equivalently,

$$W \cap E \subseteq 5RD \cap E. \tag{2.11}$$

For n large enough, our condition on ε, λ, and R thus becomes:

$$\left(\sqrt{2\varepsilon} + \frac{1}{R}\right)^{n-1} [3R]^k \leq \zeta^k. \tag{2.12}$$

Let $\rho = \frac{k}{n-1}$. Then, (2.10) will be true if

$$\sqrt{2\varepsilon} R^{\rho} + \frac{1}{R^{1-\rho}} \leq (\frac{\zeta}{3})^{\rho}. \tag{2.13}$$

Choose $R = 2^{\frac{1}{1-\lambda}} (\frac{3}{\zeta})^{\frac{\lambda}{1-\lambda}}$. If $\varepsilon \leq \varepsilon_0(\zeta, \lambda)$ and if n is large enough (in which case we may practically assume that $\rho = \lambda$), then one can easily check that (2.13) is satisfied. □

Remark 2.4 Observe that our method cannot produce R smaller than $(\frac{3}{\zeta})^{\frac{\lambda}{1-\lambda}}$ even if we are allowed to choose ε arbitrarily close to 0 (this follows immediately from (2.13)). It is not clear if a better argument might give that $R(\varepsilon, \zeta, \lambda) \to 1$ as $\varepsilon \to 0$ for every fixed $\lambda \in (0,1)$.

Using Lemma 2.3 we can easily prove the following conditional low M-estimate:

Theorem 2.5 *Let $\zeta < 1$, $\lambda \in (0,1)$ and K be a symmetric convex body in $\mathbb{R}^n$, $n \geq n_0(\lambda, \zeta)$. Find $r > 0$ for which*

$$M(\text{co}(rK \cup D)) = 1 - \frac{1}{3}[\frac{1}{2}(\frac{\zeta}{3})^{\lambda}]^{\frac{2}{1-\lambda}}.$$

Then, we can find $\mathcal{L}_{n,[\lambda n]} \subseteq G_{n,[\lambda n]}$ with $\nu_{n,[\lambda n]}(\mathcal{L}_{n,[\lambda n]}) \geq 1 - \zeta^{[\lambda n]}$ such that

$$K \cap E \subseteq \left(5(\frac{3}{\zeta})^{\frac{\lambda}{1-\lambda}} 2^{\frac{1}{1-\lambda}}\right) \frac{1}{r}(D \cap E) = \frac{R(\lambda, \zeta)}{r}(D \cap E),$$

for every $E \in \mathcal{L}_{n,[\lambda n]}$.

Proof: Let $W = \text{co}(rK \cup D)$. From our choice of r we have $M(W) = 1 - \frac{1}{3}[\frac{1}{2}(\frac{\zeta}{3})^{\lambda}]^{\frac{2}{1-\lambda}}$, and since $\|.\|_W$ is 1-Lipschitz on S^{n-1} a standard argument from [M1] (see also [FLM] or [MS1]) shows that for every $\delta \in (0,1)$

$$\sigma\left(\{y \in S^{n-1} : |\, \|y\|_W - m(W)\, | > \delta\}\right) < 4e^{-n\delta^2/2}, \tag{2.14}$$

which means that

$$(2.15)\qquad 1-\frac{1}{3}[\frac{1}{2}(\frac{\zeta}{3})^{\lambda}]^{\frac{2}{1-\lambda}}=\int_{S^{n-1}}\|y\|_W\ \sigma(dy)\leq m(W)+\delta+4e^{-n\delta^2/2},$$

therefore, for $n\geq n_0(\lambda,\zeta)$ the right choice of δ gives

$$m(W)\geq 1-\varepsilon_0(\zeta,\lambda).$$

We now apply Lemma 2.3 for $\varepsilon=\varepsilon_0$ to find $\mathcal{L}_{n,[\lambda n]}\subseteq G_{n,[\lambda n]}$ of measure $\nu_{n,[\lambda n]}(\mathcal{L}_{n,[\lambda n]})\geq 1-\zeta^{[\lambda n]}$, such that

$$(2.16)\qquad W\cap E\subseteq 5(\frac{3}{\zeta})^{\frac{\lambda}{1-\lambda}}2^{\frac{1}{1-\lambda}}D\cap E=R(\lambda,\zeta)D\cap E,$$

for every $E\in\mathcal{L}_{n,[\lambda n]}$. Since $rK\subseteq W$, the proof is complete. □

If n is large enough, one can choose ζ almost equal to 1 and still achieve "almost full measure" for $\mathcal{L}_{n,[\lambda n]}$. In order to give the flavor of the statement, we rewrite the low M-estimate given by Theorem 2.5 in a less precise form:

Conditional Low M-estimate: *There exist two absolute positive constants $c<1, C>1$ such that if K is a symmetric convex body in $\mathbb{R}^n$, n large enough, and if $r>0$ satisfies*

$$M_K^*(r)\geq 1-c^{\frac{1}{1-\lambda}},$$

then

$$\mathrm{diam}(K^o\cap E)\leq\frac{20}{r}C^{\frac{\lambda}{1-\lambda}}$$

for all $\lambda\in(0,1)$ and all E in a subset $\mathcal{L}_{n,[\lambda n]}$ of $G_{n,[\lambda n]}$ with $\nu_{n,[\lambda n]}(\mathcal{L}_{n,[\lambda n]})\geq 1-c^{[\lambda n]}$. □

Compare with the version of the Low M^*-estimate which was used in the proof of Theorem 2.1:

Low M^*-estimate: *If K is a symmetric convex body in $\mathbb{R}^n$ and if $r>0$ satisfies*

$$M_K^*(r)\leq\frac{1}{2}\sqrt{1-\lambda},$$

then

$$\mathrm{diam}(K\cap E)\leq 2r$$

for all $\lambda\in(0,1)$ and all E in a subset $\mathcal{L}_{n,[\lambda n]}$ of $G_{n,[\lambda n]}$ of almost full measure.

We proceed to the lower bound for the diameter of $[\lambda n]$-dimensional sections of K. Besides Theorem 2.5, our proof is also based on the following application of Borsuk's antipodal theorem:

Lemma 2.6 *Let K be a symmetric convex body in $\mathbb{R}^n$. For every subspace E with $\dim E>\dim E^{\perp}$ we can find $y\in\mathrm{bd}(\mathcal{P}_E(K))\cap K$, where $\mathcal{P}_E$ denotes the orthogonal projection onto E and $\mathrm{bd}(\mathcal{P}_E(K))$ is the boundary of $\mathcal{P}_E(K)$.*

Proof: Without loss of generality we may assume that K is strictly convex. For every $y \in \mathrm{bd}(\mathcal{P}_E(K))$ there exists unique $t(y) \in \mathrm{bd}(K)$ such that $\mathcal{P}_E(t(y)) = y$. Define the map $T : \mathrm{bd}(\mathcal{P}_E(K)) \to E^\perp$ with $T(y) = t(y) - y$. Then, T is continuous and antisymmetric, and since $\dim E > \dim E^\perp$ we can apply Borsuk's theorem to find $y \in \mathrm{bd}(\mathcal{P}_E(K))$ with $t(y) = y$. □

Theorem 2.7 below gives a lower bound for the diameter of $[\lambda n]$-dimensional sections of K, $\lambda \in (\frac{1}{2}, 1)$. Adding this information to Theorem 2.1 which gave upper bounds in exactly the same spirit, we complete the proof of our General Statement:

Theorem 2.7 *Let $\zeta < 1$, $\lambda \in (\frac{1}{2}, 1)$, and K be a symmetric convex body in $\mathbb{R}^n$, $n \geq n_0(\lambda, \zeta)$. Find $r > 0$ for which*

$$M_K^*(r) = \frac{M^*(K \cap rD)}{r} = 1 - \frac{1}{48}(\frac{\zeta}{3})^2.$$

Then, we can find $\mathcal{L}_{n,[\lambda n]} \subseteq G_{n,[\lambda n]}$ with $\nu_{n,[\lambda n]}(\mathcal{L}_{n,[\lambda n]}) \geq 1 - \zeta^{\frac{n}{2}}$, such that

$$\mathrm{diam}(K \cap E) \geq \frac{1}{10}(\frac{\zeta}{3})r$$

for every $E \in \mathcal{L}_{n,[\lambda n]}$.

Proof: Apply Theorem 2.5 to K^o with any $\lambda_0 > \frac{1}{2}$. We can find $\mathcal{L}_{n,[\lambda_0 n]} \subseteq G_{n,[\lambda_0 n]}$ with $\nu_{n,[\lambda_0 n]}(\mathcal{L}_{n,[\lambda_0 n]}) \geq 1 - \zeta^{[\lambda_0 n]}$, for which

$$K^o \cap E \subseteq \frac{R(\lambda_0, \zeta)}{r} D \cap E \tag{2.17}$$

for every $E \in \mathcal{L}_{n,[\lambda_0 n]}$. Let $E \in \mathcal{L}_{n,[\lambda_0 n]}$. Passing to polars in E we get

$$\mathcal{P}_E(K) \supseteq \frac{r}{R(\lambda_0, \zeta)} D \cap E. \tag{2.18}$$

Since $\lambda_0 > \frac{1}{2}$, assuming that $n \geq n_0(\lambda_0)$ we have $\dim E > \dim E^\perp$. Therefore, we can apply Lemma 2.6 to find $y \in \mathrm{bd}(\mathcal{P}_E(K)) \cap K$. In particular, $y \in K \cap E$ and $|y| \geq \frac{r}{R(\lambda_0,\zeta)}$ which means that

$$\mathrm{diam}(K \cap E) \geq \frac{2r}{R(\lambda_0, \zeta)}. \tag{2.19}$$

For n large enough, we can assume that (2.19) is true with $\lambda_0 = \frac{1}{2}$, which gives the theorem in the special case of $\lambda = \frac{1}{2}$. Now, let $\lambda > \frac{1}{2}$, and define

$$\mathcal{L}_{n,[\lambda n]} = \{F \in G_{n,[\lambda n]} : \text{ there is } E \in \mathcal{L}_{n,[\frac{n}{2}]+1} \text{ with } E \leq F\}.$$

Claim: $\nu_{n,[\lambda n]}(\mathcal{L}_{n,[\lambda n]}) \geq \nu_{n,[\frac{n}{2}]+1}(\mathcal{L}_{n,[\frac{n}{2}]+1})$.

[This is a general fact: Fix $E_0 \subseteq F_0$, with $\dim E_0 = [\frac{n}{2}] + 1$ and $\dim F_0 = [\lambda n]$. By the definition of $\mathcal{L}_{n,[\lambda n]}$, if for some $T \in O_n$ we have $TE_0 \in \mathcal{L}_{n,[\frac{n}{2}]+1}$, then $TF_0 \in \mathcal{L}_{n,[\lambda n]}$. It follows that

$$\nu_{n,[\lambda n]}(\mathcal{L}_{n,[\lambda n]}) = \mu\left(T \in O_n : TF_0 \in \mathcal{L}_{n,[\lambda n]}\right) \tag{2.20}$$

$$\geq \mu\left(T \in O_n : TE_0 \in \mathcal{L}_{n,[\frac{n}{2}]+1}\right) = \nu_{n,[\frac{n}{2}]+1}(\mathcal{L}_{n,[\frac{n}{2}]+1}) \geq 1 - \zeta^{\frac{n}{2}}. \ ]$$

On the other hand, it is clear that if $F \in \mathcal{L}_{n,[\lambda n]}$, then for some $E \subseteq F$ in $\mathcal{L}_{n,[\frac{n}{2}]+1}$ we have

$$\operatorname{diam}(K \cap F) \geq \operatorname{diam}(K \cap E) \geq \frac{1}{10}(\frac{\zeta}{3})r, \tag{2.21}$$

which completes the proof. □

2.2. An example on the behavior of M_K^*. To show that M_K^* may behave in a quite irregular way, we study the behavior of the function $M_E^*(r) = \frac{1}{r}M^*(E \cap rD)$ for an ellipsoid with highly incomparable semiaxes. Let $\varepsilon \in (0,1)$ be a very small positive number, and define

$$E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \varepsilon^{2i} x_i^2 \leq 1\}.$$

Given any $r > 0$, one easily checks that $E \cap rD$ is $\sqrt{2}$–isomorphic to the ellipsoid

$$E'(r) = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} (\varepsilon^{2i} + \frac{1}{r^2}) x_i^2 \leq 1\}.$$

In particular, if $M_2^*(W) = \left(\int_{S^{n-1}} \|x\|_{W^\circ}^2 \sigma(dx)\right)^{1/2}$, we have $M_2^*(E'(r)) \leq M_2^*(E \cap rD) \leq \sqrt{2} M_2^*(E'(r))$ for every $r > 0$. Consider the function $F(r) = \frac{1}{r} M_2^*(E'(r))$. It is easy to see that

$$F(r) = \frac{1}{r}\left[\frac{1}{n}\sum_{i=1}^{n} \frac{r^2}{r^2\varepsilon^{2i} + 1}\right]^{1/2} = \left[\frac{1}{n}\sum_{i=1}^{n} \beta_i(r)\right]^{1/2}, \tag{2.22}$$

where $\beta_i(r) = 1/(r^2\varepsilon^{2i} + 1)$. We shall estimate $F(\varepsilon^{-k}),\ k = 1, 2, \ldots, n$:

(1) If $i < k$, then $0 \leq \beta_i(\varepsilon^{-k}) \leq \varepsilon^2$.
(2) If $i = k$, then $\beta_i(\varepsilon^{-k}) = \frac{1}{2}$.
(3) If $i > k$, then $(1 + \varepsilon^2)^{-1} \leq \beta_i(\varepsilon^{-k}) \leq 1$.

It follows that e.g for all $k \in [\frac{n}{3}, \frac{2n}{3}]$,

$$\frac{1}{3} + \frac{1}{2n} - \varepsilon^2 \leq F^2(\varepsilon^{-k}) \leq \frac{2}{3} + \frac{1}{2n} + \varepsilon^2. \tag{2.23}$$

Since $M^*(E \cap rD) \leq M_2^*(E \cap rD) \leq \sqrt{2} M^*(E \cap rD)$, M_E^* satisfies the inequality $\frac{1}{\sqrt{2}} F(r) \leq M_E^*(r) \leq \sqrt{2} F(r)$, and this shows that if ε is small enough then for every pair of $k, l \in [\frac{n}{3}, \frac{2n}{3}]$ we have $M_E^*(\varepsilon^{-k})/M_E^*(\varepsilon^{-l}) \leq c$ for some absolute constant $c > 0$. It follows that for some $k \in [\frac{n}{3}, \frac{2n}{3}]$ we must have $M_E^*(\varepsilon^{-k})/M_E^*(\varepsilon^{-k-1}) \leq c_1^{1/n}$, where c_1 is some other absolute constant. Hence, if n is large and if ε is too small, we can have r_1, r_2 with r_1/r_2 arbitrarily large and $M_E^*(r_1)/M_E^*(r_2)$ arbitrarily close to 1. Note that this happens in the "interesting" interval of the range of M_E^*.

2.3. Diameter of the sections of a body in M-position. It is well-known that every symmetric convex body can be put in a "regular" position by means of a linear transformation [M3]. We use this result in the formulation of Pisier [Pi]: For every $\alpha > \frac{1}{2}$ any body K has a linear image $\overline{K}$ which is α-regular: If $\rho_{\overline{K}} = (|\overline{K}|/|D|)^{\frac{1}{n}}$ is the volume radius of $\overline{K}$, and if $N(U,V)$ denotes the covering number of U by V i.e the minimal cardinality of a set $\{x_1, \dots, x_N\} \subseteq U$ for which $U \subseteq \bigcup_{i \leq N}(x_i + V)$, then

$$\max\{[N(\overline{K}, t\rho_{\overline{K}}D)]^{\frac{1}{n}}, [N(\rho_{\overline{K}}D, t\overline{K})]^{\frac{1}{n}}\} \leq c\exp(c_1(\alpha)t^{-\frac{1}{\alpha}}) \tag{2.24}$$

for every $t \geq 1$, where $c > 0$ is an absolute constant and $c_1(\alpha)$ is a positive constant depending only on α.

Moreover, it can be proved that for every K there exists a linear image $\overline{K}$ such that both $\overline{K}$ and $\overline{K}^0$ (as well as any orthogonal images of them) are α-regular. Also, if $r_1, r_2 > 0$ and $W = \mathrm{co}((\overline{K} \cap r_1 D) \cup r_2 D)$, then both W and W^o are α-regular with some possibly different (but independent from r_1 and r_2) constants $c', c'(\alpha)$.

Assume that K is α-regular in the strong sense defined above and consider any $\lambda \in (\frac{1}{2}, 1)$. Apply Theorem 2.1 with $\varepsilon = \frac{1}{2}$ to find $r > 0$ for which $M_K^*(r) = \frac{1}{2}\sqrt{1-\lambda}$. Then, for most $[\lambda n]$-dimensional subspaces E of $\mathbb{R}^n$ (most in the sense of §2.1) we have

$$\mathrm{diam}(K \cap E) \leq 2r. \tag{2.25}$$

Set $K_1 = (K \cap rD)^o$. Then, $M(K_1) = \frac{1}{2}\sqrt{1-\lambda}r$ and $\|x\|_{K_1} \leq r|x|$ for every $x \in \mathbb{R}^n$. By [BLM] we can find orthogonal transformations $u_1, \dots, u_s$ with $s \leq \frac{c_1}{1-\lambda}$ such that

$$\frac{1}{4}(1-\lambda)^{\frac{1}{2}} rD \subseteq \frac{1}{s}\sum_{i=1}^{s} u_i(K_1^o) \subseteq (1-\lambda)^{\frac{1}{2}} rD. \tag{2.26}$$

Since K_1^o is also α-regular, the inverse Brunn-Minkowski inequality [M3], [Pi] shows that

$$\frac{1}{4}(1-\lambda)^{\frac{1}{2}} r \leq c_2(\alpha) s^{\alpha} \left(\frac{|K_1^o|}{|D|}\right)^{\frac{1}{n}}. \tag{2.27}$$

Now choose $R > 0$ for which $M^*(K_1 \cap RD) = R/2\sqrt{2}$. Applying Theorem 2.1 once more (this time for $\lambda = \frac{1}{2}$), we see that for most $([\frac{n}{2}]+1)$-dimensional subspaces F of $\mathbb{R}^n$ we have

$$\mathrm{diam}(K_1 \cap F) \leq 2R, \tag{2.28}$$

and repeating the argument above we see that

$$\frac{1}{4\sqrt{2}}R \le c_3 \left(\frac{|K_1 \cap RD|}{|D|}\right)^{\frac{1}{n}} \le c_3 \left(\frac{|K_1|}{|D|}\right)^{\frac{1}{n}}. \tag{2.29}$$

Multiplying (2.27) and (2.29), and making use of the Blaschke-Santaló inequality and of the estimate on s, we obtain

$$rR \le \frac{c_4(\alpha)}{(1-\lambda)^{\alpha+\frac{1}{2}}}. \tag{2.30}$$

From (2.28), taking polars in F we have $\mathcal{P}_F(K) \supseteq \mathcal{P}_F(K \cap rD) \supseteq \frac{1}{R}D \cap F$, and applying Borsuk's theorem as in Theorem 2.7 we see that $\mathrm{diam}(K \cap F) \ge \frac{2}{R}$ (we assume that n is large enough). Exactly the same lower bound is true for most $[\lambda n]$-dimensional subspaces, $\lambda \in (\frac{1}{2}, 1)$. Thus, we have proved the following:

Theorem 2.8 *Let $\lambda \in (\frac{1}{2}, 1)$, $\alpha > \frac{1}{2}$, and K be an α-regular symmetric convex body in $\mathbb{R}^n$, $n \ge n_0(\lambda)$. Find $r > 0$ for which*

$$M_K^*(r) = \frac{M^*(K \cap rD)}{r} = \frac{1}{2}\sqrt{1-\lambda}.$$

Then, for most $[\lambda n]$-dimensional subspaces E of $\mathbb{R}^n$ we have

$$\mathrm{diam}(K \cap E) \in [2c(\alpha)(1-\lambda)^{\alpha+\frac{1}{2}}r, 2r],$$

where $c(\alpha) > 0$ is a constant depending only on α. □

References

[BLM] J. Bourgain, J. Lindenstrauss and V.D. Milman, *Minkowski sums and symmetrizations*, Lecture Notes in Mathematics **1317** (1988), 44-66.

[FLM] T. Figiel, J. Lindenstrauss and V.D. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. **139**, No. 1-2 (1977), 53-94.

[Go] Y. Gordon, *On Milman's inequality and random subspaces which escape through a mesh in $\mathbb{R}^n$*, Lecture Notes in Mathematics **1317** (1988), 84-106.

[M1] V.D. Milman, *A new proof of the theorem of A. Dvoretzky on sections of convex bodies*, Functional Analysis and its Applications **5**, No. 4 (1971), 28-37.

[M2] V.D. Milman, *Random subspaces of proportional dimension of finite dimensional normed spaces: approach through the isoperimetric inequality*, Lecture Notes in Mathematics **1166** (1985), 106-115.

[M3] V.D. Milman, *Inegalité de Brunn–Minkowski inverse et applications à la théorie locale des espaces normés*, C. R. Acad. Sci. Paris **302**, Sér 1 (1986), 25-28.

[MS1] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite–Dimensional Normed Spaces*, Lecture Notes in Mathematics **1200** (1986).

[MS2] V.D. Milman and G. Schechtman, *Global vs Local Asymptotic Theories of Finite Dimensional Normed Spaces*, Preprint.

[Pi] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Math. **94** (1989).

[PT] A. Pajor and N. Tomczak-Jaegermann, *Subspaces of small codimension of finite dimensional Banach spaces*, Proc. Amer. Math. Soc. **97** (1986), 637-642.

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