Geometry of random sections of isotropic convex bodies

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Abstract

Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. We show that a subspace $F \in G_{n,n-k}$ of codimension $k = \gamma n$, where $\gamma \in (1/\sqrt{n}, 1)$, satisfies

$$K \cap F \subseteq \frac{c}{\gamma}\sqrt{n}L_K(B_2^n \cap F)$$

with probability greater than $1-\exp(-\sqrt{n})$. Using a different method we study the same question for the L_q-centroid bodies $Z_q(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$. For every $1 \leqslant q \leqslant n$ and $\gamma \in (0,1)$ we show that a random subspace $F \in G_{n,(1-\gamma)n}$ satisfies $Z_q(\mu) \cap F \subseteq c_2(\gamma)\sqrt{q}\, B_2^n \cap F$. We also give bounds on the diameter of random projections of $Z_q(\mu)$ and using them we deduce that if K is an isotropic convex body in $\mathbb{R}^n$ then for a random subspace F of dimension $(\log n)^4$ one has that all directions in F are sub-Gaussian with constant $O(\log^2 n)$.

1 Introduction

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, its center of mass is at the origin (we call these convex bodies "centered"), and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.1}$$

for every θ in the Euclidean unit sphere S^{n-1}. For every centered convex body K in $\mathbb{R}^n$ there exists an invertible linear transformation $T \in GL(n)$ such that $T(K)$ is isotropic. This isotropic image of K is uniquely determined up to orthogonal transformations. A well-known problem in asymptotic convex geometry asks if there exists an absolute constant $C_1 > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C_1 \tag{1.2}$$

for all $n \geqslant 1$ (see Section 2 for background information on isotropic convex bodies and log-concave probability measures). Bourgain proved in [5] that $L_n \leqslant c\sqrt[4]{n}\log n$, and Klartag [19] improved this bound to $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [21].

Recall that the inradius $r(K)$ of a convex body K in $\mathbb{R}^n$ with $0 \in \text{int}(K)$ is the largest $r > 0$ for which $rB_2^n \subseteq K$, while the radius $R(K) := \max\{\|x\|_2 : x \in K\}$ of K is the smallest $R > 0$ for which $K \subseteq RB_2^n$. It is not hard to see that the inradius and the radius of an isotropic convex body K in $\mathbb{R}^n$ satisfy the bounds $c_1L_K \leqslant r(K) \leqslant R(K) \leqslant c_2nL_K$, where $c_1, c_2 > 0$ are absolute constants. In fact, Kannan, Lovász and Simonovits [17] have proved that

$$R(K) \leqslant (n+1)L_K. \tag{1.3}$$

Radius of random sections of isotropic convex bodies. The first question that we discuss in this article is to give sharp upper bounds for the radius of a random $(n-k)$-dimensional section of K. A natural "guess" is that the following question has an affirmative answer.

Question 1.1. *There exists an absolute constant* $\bar{c}_0 > 0$ *with the following property: for every isotropic convex body* K *in* $\mathbb{R}^n$ *and for every* $1 \leqslant k \leqslant n-1$, *a random subspace* $F \in G_{n,n-k}$ *satisfies*

$$R(K \cap F) \leqslant \bar{c}_0 \sqrt{n/k}\, \sqrt{n} L_K. \tag{1.4}$$

It was proved in [23] that if K is a symmetric convex body in $\mathbb{R}^n$ then a random $F \in G_{n,n-k}$ satisfies

$$R(K \cap F) \leqslant c(n/k)^{3/2} \tilde{M}(K), \tag{1.5}$$

where $c > 0$ is an absolute constant and

$$\tilde{M}(K) := \frac{1}{|K|} \int_K \|x\|_2 dx. \tag{1.6}$$

In the case of an isotropic convex body one has $|K| = 1$ and

$$\tilde{M}(K) \leqslant \left(\int_K \|x\|_2^2 dx \right)^{1/2} = \sqrt{n} L_K, \tag{1.7}$$

therefore (1.5) implies that a random $F \in G_{n,n-k}$ satisfies

$$R(K \cap F) \leqslant \bar{c}_1 (n/k)^{3/2} \sqrt{n} L_K, \tag{1.8}$$

where $\bar{c}_1 > 0$ is an absolute constant.

Our first main result shows that one can have a bound of the order of $\gamma^{-1}\sqrt{n} L_K$ when the codimension k is greater than γn.

Theorem 1.2. *Let* K *be an isotropic symmetric convex body in* $\mathbb{R}^n$ *and let* $1 \leqslant k \leqslant n-1$. *A random subspace* $F \in G_{n,n-k}$ *satisfies*

$$R(K \cap F) \leqslant \frac{\bar{c}_0 n}{\max\{k, \sqrt{n}\}} \sqrt{n} L_K \tag{1.9}$$

with probability greater than $1 - \exp(-\sqrt{n})$, *where* $\bar{c}_0 > 0$ *is an absolute constant.*

The proof is given in Section 3. Note that Theorem 1.2 gives non-trivial information when $k > \sqrt{n}$. In this case, writing $k = \gamma n$ for some $\gamma \in (1/\sqrt{n}, 1)$ we see that

$$R(K \cap F) \leqslant \frac{\bar{c}_0}{\gamma} \sqrt{n} L_K \tag{1.10}$$

with probability greater than $1 - \exp(-\sqrt{n})$ on $G_{n,(1-\gamma)n}$. The result of [23] establishes a $\gamma^{-3/2}$-dependence on $\gamma = k/n$.

A standard approach to Question 1.1 would have been to combine the low M^*-estimate with an upper bound for the mean width

$$w(K) := \int_{S^{n-1}} h_K(x)\, d\sigma(x), \tag{1.11}$$

of an isotropic convex body K in $\mathbb{R}^n$, that is, the L_1-norm of the support function of K with respect to the Haar measure on the sphere. This last problem was open for a number of years. The upper bound $w(K) \leqslant cn^{3/4} L_K$ appeared in the Ph.D. Thesis of Hartzoulaki [16]. Other approaches leading to the same bound can be found in Pivovarov [32] and in Giannopoulos, Paouris and Valettas [15]. Recently, E. Milman showed in [26] that if K is an isotropic symmetric convex body in $\mathbb{R}^n$ then

$$w(K) \leqslant c_3 \sqrt{n} (\log n)^2 L_K. \tag{1.12}$$

In fact, it is not hard to see that his argument can be generalized to give the same estimate in the not necessarily symmetric case. The dependence on n is optimal up to the logarithmic term. From the sharp version of V. Milman's low M^*-estimate (due to Pajor and Tomczak-Jaegermann [28]; see [2, Chapter 7] for complete references) one has that, for every $1 \leqslant k \leqslant n-1$, a subspace $F \in G_{n,n-k}$ satisfies

$$R(K \cap F) \leqslant c_4 \sqrt{n/k}\, w(K) \tag{1.13}$$

with probability greater than $1 - \exp(-c_5 k)$, where $c_4, c_5 > 0$ are absolute constants. Combining (1.13) with E. Milman's theorem we obtain the folowing estimate:

Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$, a subspace $F \in G_{n,n-k}$ satisfies

$$R(K \cap F) \leqslant \frac{\overline{c}_2 n (\log n)^2 L_K}{\sqrt{k}} \tag{1.14}$$

with probability greater than $1 - \exp(-\overline{c}_3 k)$, where $\overline{c}_2, \overline{c}_3 > 0$ are absolute constants.

Note that the upper bound of Theorem 1.2 has some advantages when compared to (1.14): If k is proportional to n (say $k \geqslant \gamma n$ for some $\gamma \in (1/\sqrt{n}, 1)$) then Theorem 1.2 guarantees that $R(K \cap F) \leqslant c(\gamma)\sqrt{n} L_K$ for a random $F \in G_{n,n-k}$. More generally, for all $k \geqslant \frac{c_6 n}{(\log n)^4}$ we have

$$\frac{\overline{c}_0 n \sqrt{n}}{\max\{k, \sqrt{n}\}} \leqslant \frac{\overline{c}_2 n (\log n)^2}{\sqrt{k}}, \tag{1.15}$$

and hence the estimate of Theorem 1.2 is stronger than (1.14). Nevertheless, we emphasize that our bound is not optimal and it would be very interesting to decide whether (1.4) holds true; this would be optimal for all $1 \leqslant k \leqslant n$.

Radius of random sections of L_q-centroid bodies and their polars. In Section 4 we study the diameter of random sections of the L_q-centroid bodies $Z_q(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$. Recall that a measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set and the restriction of $\log f$ on it is concave. It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then μ is absolutely continuous with respect to the Lebesgue measure and its density f_μ is log-concave; see [4]. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that the indicator function $\mathbf{1}_K$ of K is the density of a log-concave measure.

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if its barycenter $\mathrm{bar}(\mu)$ is at the origin and

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \, d\mu(x) = 1$$

for all $\theta \in S^{n-1}$. Note that the normalization is different from the one in (1.1); in particular, a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic if and only if the log-concave probability measure μ_K with density $x \mapsto L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic.

The L_q-centroid bodies $Z_q(\mu)$, $q \geqslant 1$, are defined through their support function

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y \rangle\|_{L_q(\mu)} = \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}, \tag{1.16}$$

and have played a key role in the study of the distribution of linear functionals with respect to the measure μ. For every $1 \leqslant q \leqslant n$ we obtain sharp upper bounds for the radius of random sections of $Z_q(\mu)$ of dimension proportional to n, thus extending a similar result of Brazitikos and Stavrakakis which was established only for $q \in [1, \sqrt{n}]$.

Theorem 1.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $1 \leqslant q \leqslant n$. Then:*

(i) *If $k = \gamma n$ for some $\gamma \in (0,1)$, then, with probability greater than $1 - e^{-\bar{c}_4 k}$, a random $F \in G_{n,n-k}$ satisfies*

$$R(Z_q(\mu) \cap F) \leqslant \bar{c}_5(\gamma)\sqrt{q}, \tag{1.17}$$

where $\bar{c}_4$ is an absolute constant and $\bar{c}_5(\gamma) = O(\gamma^{-2}\log^{5/2}(c/\gamma))$ is a positive constant depending only on γ.

(ii) *With probability greater than $1 - e^{-n}$, a random $U \in O(n)$ satisfies*

$$Z_q(\mu) \cap U(Z_q(\mu)) \subseteq (\bar{c}_6\sqrt{q})\, B_2^n, \tag{1.18}$$

where $\bar{c}_6 > 0$ is an absolute constant.

The method of proof is based on estimates (from [26] and [11]) for the Gelfand numbers of symmetric convex bodies in terms of their volumetric parameters; combining these general estimates with fundamental (known) properties of the family of the centroid bodies $Z_q(\mu)$ of an isotropic log-concave probability measure μ we provide estimates for the *minimal* radius of a k-codimensional section of $Z_q(\mu)$. Then, we pass to bounds for the radius of random k-codimensional sections of $Z_q(\mu)$ using known results from [12], [34] and [24]. We conclude Section 4 with a discussion of the same questions for the polar bodies $Z_q^\circ(\mu)$ of the centroid bodies $Z_q(\mu)$.

Using the same approach we study the diameter of random sections of convex bodies which have *maximal isotropic constant*. Set

$$L_n' := \max\{L_K : K \text{ is an isotropic symmetric convex body in } \mathbb{R}^n\}. \tag{1.19}$$

It is known that $L_n \leqslant cL_n'$ for some absolute constant $c > 0$ (see [9, Chapter 3]). We prove the following:

Theorem 1.4. *Assume that K is an isotropic symmetric convex body in $\mathbb{R}^n$ with $L_K = L_n'$. Then:*

(i) *A random $F \in G_{n,n/2}$ satisfies*

$$R(K \cap F) \leqslant \bar{c}_7\sqrt{n} \tag{1.20}$$

and

$$L_{K\cap F} \leqslant \bar{c}_8 \tag{1.21}$$

with probability greater than $1 - e^{-\bar{c}_9 n}$, where $\bar{c}_i > 0$ are absolute constants.

(ii) *A random $U \in O(n)$ satisfies*

$$K \cap U(K) \subseteq (\bar{c}_{10}\sqrt{n})\, B_2^n, \tag{1.22}$$

with probability greater than $1 - e^{-n}$, where $\bar{c}_{10} > 0$ is an absolute constant.

The same arguments work if we assume that K has almost maximal isotropic constant, i.e. $L_K \geqslant \beta L_n'$ for some (absolute) constant $\beta \in (0,1)$. We can obtain similar results, with the constants $\bar{c}_i$ now depending only on β. It should be noted that Alonso-Gutiérrez, Bastero, Bernués and Paouris [1] have proved that every convex body K has a section $K \cap F$ of dimension $n-k$ with isotropic constant

$$L_{K\cap F} \leqslant c\sqrt{\frac{n}{k}}\log\Big(\frac{en}{k}\Big). \tag{1.23}$$

For the proof of this result they considered an α-regular M-position of K. In Theorem 1.4 we consider convex bodies in the isotropic position and the estimates (1.20) and (1.21) hold for a random subspace F.

Radius of random projections of L_q-centroid bodies and sub-Gaussian subspaces of isotropic convex bodies. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. We say that a direction $\theta \in S^{n-1}$ is a ψ_α-direction (where $1 \leqslant \alpha \leqslant 2$) for K with constant $b > 0$ if

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_\alpha}(K)} \leqslant b\|\langle \cdot, \theta\rangle\|_2, \tag{1.24}$$

where

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_\alpha}(K)} := \inf\left\{t > 0 : \int_K \exp\left((|\langle x, \theta\rangle|/t)^\alpha\right) dx \leqslant 2\right\}. \tag{1.25}$$

Markov's inequality implies that if K satisfies a ψ_α-estimate with constant b in the direction of θ then for all $t \geqslant 1$ we have $|\{x \in K : |\langle x, \theta\rangle| \geqslant t\|\langle \cdot, \theta\rangle\|_2\}| \leqslant 2e^{-t^a/b^\alpha}$. Conversely, one can check that tail estimates of this form imply that θ is a ψ_α-direction for K.

It is well-known that every $\theta \in S^{n-1}$ is a ψ_1-direction for K with an absolute constant C. An open question is if there exists an absolute constant $C > 0$ such that every K has at least one sub-Gaussian direction (ψ_2-direction) with constant C. It was first proved by Klartag in [20] that for every centered convex body K of volume 1 in $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant ct\|\langle \cdot, \theta\rangle\|_2\}| \leqslant e^{-\frac{t^2}{[\log(t+1)]^{2a}}} \tag{1.26}$$

for all $t \geqslant 1$, where $a = 3$ (equivalently, $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^a\|\langle \cdot, \theta\rangle\|_2$). This estimate was later improved by Giannopoulos, Paouris and Valettas in [14] and [15] (see also [13]) who showed that the body $\Psi_2(K)$ with support function $y \mapsto \|\langle \cdot, y\rangle\|_{L_{\psi_2}(K)}$ has volume

$$c_1 \leqslant \left(\frac{|\Psi_2(K)|}{|Z_2(K)|}\right)^{1/n} \leqslant c_2\sqrt{\log n}. \tag{1.27}$$

From (1.27) it follows that there exists at least one sub-Gaussian direction for K with constant $b \leqslant C\sqrt{\log n}$.

Brazitikos and Hioni in [7] proved that if K is isotropic then logarithmic bounds for $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)}$ hold true with probability polynomially close to 1: For any $a > 1$ one has

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^{3/2} \max\left\{\sqrt{\log n}, \sqrt{a}\right\} L_K$$

for all θ in a subset Θ_a of S^{n-1} with $\sigma(\Theta_a) \geqslant 1 - n^{-a}$, where $C > 0$ is an absolute constant.

Here, we consider the question if one can have an estimate of this type *for all* directions θ of a subspace $F \in G_{n,k}$ of dimension k increasing to infinity with n. We say that $F \in G_{n,k}$ is a *sub-Gaussian subspace* for K with constant $b > 0$ if

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_\alpha}(K)} \leqslant b\|\langle \cdot, \theta\rangle\|_2 \tag{1.28}$$

for all $\theta \in S_F := S^{n-1} \cap F$. In Section 5 we show that if K is isotropic then a random subspace of dimension $(\log n)^4$ is sub-Gaussian with constant $b \simeq (\log n)^2$. More precisely, we prove the following.

Theorem 1.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $k \simeq (\log n)^4$ then there exists a subset Γ of $G_{n,k}$ with $\nu_{n,k}(\Gamma) \geqslant 1 - n^{-(\log n)^3}$ such that*

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K \tag{1.29}$$

for all $F \in \Gamma$ and all $\theta \in S_F$, where $C > 0$ is an absolute constant.

An essential ingredient of the proof is the good estimates on the radius of random projections of the L_q-centroid bodies $Z_q(K)$ of K, which follow from E. Milman's sharp bounds on their mean width $w(Z_q(K))$ (see Theorem 5.1).

2 Notation and preliminaries

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also if $A, D \subseteq \mathbb{R}^n$ we will write $A \simeq D$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 A \subseteq D \subseteq c_2 A$.

Convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex subset A of $\mathbb{R}^n$ with nonempty interior. We say that A is symmetric if $A = -A$. We say that A is centered if the center of mass of A is at the origin, i.e. $\int_A \langle x, \theta\rangle\, dx = 0$ for every $\theta \in S^{n-1}$.

The volume radius of A is the quantity $\mathrm{vrad}(A) = (|A|/|B_2^n|)^{1/n}$. Integration in polar coordinates shows that if the origin is an interior point of A then the volume radius of A can be expressed as

$$\mathrm{vrad}(A) = \left(\int_{S^{n-1}} \|\theta\|_A^{-n}\, d\sigma(\theta)\right)^{1/n}, \tag{2.1}$$

where $\|\theta\|_A = \min\{t > 0 : \theta \in tA\}$. The radial function of A is defined by $\rho_A(\theta) = \max\{t > 0 : t\theta \in A\}$, $\theta \in S^{n-1}$. The support function of A is defined by $h_A(y) := \max\bigl\{\langle x, y\rangle : x \in A\bigr\}$, and the mean width of A is the average

$$w(A) := \int_{S^{n-1}} h_A(\theta)\, d\sigma(\theta) \tag{2.2}$$

of h_A on S^{n-1}. The radius $R(A)$ of A is the smallest $R > 0$ such that $A \subseteq RB_2^n$. For notational convenience we write $\overline{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\overline{A} := |A|^{-1/n}A$.

The polar body A° of a convex body A in $\mathbb{R}^n$ with $0 \in \mathrm{int}(A)$ is defined by

$$A^\circ := \bigl\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in A\bigr\}. \tag{2.3}$$

The Blaschke-Santaló inequality states that if A is centered then $|A||A^\circ| \leqslant |B_2^n|^2$, with equality if and only if A is an ellipsoid. The reverse Santaló inequality of J. Bourgain and V. Milman [6] states that there exists an absolute constant $c > 0$ such that

$$(|A||A^\circ|)^{1/n} \geqslant c/n \tag{2.4}$$

whenever $0 \in \mathrm{int}(A)$.

For every centered convex body A of volume 1 in $\mathbb{R}^n$ and for every $q \in (-n, \infty) \setminus \{0\}$ we define

$$I_q(A) = \left(\int_A \|x\|_2^q dx\right)^{1/q}. \tag{2.5}$$

As a consequence of Borell's lemma (see [9, Chapter 1]) one has

$$I_q(A) \leqslant c_1 q I_2(A) \tag{2.6}$$

for all $q \geqslant 2$.

For basic facts from the Brunn-Minkowski theory and the asymptotic theory of convex bodies we refer to the books [33] and [2] respectively.

Log-concave probability measures. Let μ be a log-concave probability measure on $\mathbb{R}^n$. The density of μ is denoted by f_μ. We say that μ is centered and we write $\mathrm{bar}(\mu)=0$ if, for all $\theta\in S^{n-1}$,

$$\int_{\mathbb{R}^n}\langle x,\theta\rangle d\mu(x)=\int_{\mathbb{R}^n}\langle x,\theta\rangle f_\mu(x)dx=0. \tag{2.7}$$

The isotropic constant of μ is defined by

$$L_\mu:=\left(\frac{\sup_{x\in\mathbb{R}^n}f_\mu(x)}{\int_{\mathbb{R}^n}f_\mu(x)dx}\right)^{\frac{1}{n}}[\det\mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{2.8}$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij}:=\frac{\int_{\mathbb{R}^n}x_ix_jf_\mu(x)\,dx}{\int_{\mathbb{R}^n}f_\mu(x)\,dx}-\frac{\int_{\mathbb{R}^n}x_if_\mu(x)\,dx}{\int_{\mathbb{R}^n}f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n}x_jf_\mu(x)\,dx}{\int_{\mathbb{R}^n}f_\mu(x)\,dx}. \tag{2.9}$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if $\mathrm{bar}(\mu)=0$ and $\mathrm{Cov}(\mu)$ is the identity matrix. Note that a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic, i.e. it satisfies (1.1), if and only if the log-concave probability measure μ_K with density $x\mapsto L_K^n\mathbf{1}_{K/L_K}(x)$ is isotropic. Note that for every log-concave measure μ on $\mathbb{R}^n$ one has

$$L_\mu\leqslant\kappa L_n, \tag{2.10}$$

where $\kappa>0$ is an absolute constant (a proof can be found in [9, Proposition 2.5.12]).

We will use the following sharp result on the growth of $I_q(K)$, where K is an isotropic convex body in $\mathbb{R}^n$, proved by Paouris in [29] and [30].

Theorem 2.1 (Paouris)**.** *There exists an absolute constant $\delta>0$ with the following property: if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\frac{1}{\delta}\sqrt{n}L_K=\frac{1}{\delta}I_2(K)\leqslant I_{-q}(K)\leqslant I_q(K)\leqslant\delta I_2(K)=\delta\sqrt{n}L_K \tag{2.11}$$

for every $1\leqslant q\leqslant\sqrt{n}$.

For every $q\geqslant 1$ and every $y\in\mathbb{R}^n$ we set

$$h_{Z_q(\mu)}(y)=\left(\int_{\mathbb{R}^n}|\langle x,y\rangle|^qd\mu(x)\right)^{1/q}. \tag{2.12}$$

The L_q-centroid body $Z_q(\mu)$ of μ is the symmetric convex body with support function $h_{Z_q(\mu)}$. Note that μ is isotropic if and only if it is centered and $Z_2(\mu)=B_2^n$. If K is an isotropic convex body in $\mathbb{R}^n$ we define $Z_q(K)=L_KZ_q(\mu_K)$. From Hölder's inequality it follows that $Z_1(K)\subseteq Z_p(K)\subseteq Z_q(K)\subseteq Z_\infty(K)$ for all $1\leqslant p\leqslant q\leqslant\infty$, where $Z_\infty(K)=\mathrm{conv}\{K,-K\}$. Using Borell's lemma, one can check that

$$Z_q(K)\subseteq c_1\frac{q}{p}Z_p(K) \tag{2.13}$$

for all $1\leqslant p<q$. In particular, if K is isotropic, then $R(Z_q(K))\leqslant c_1qL_K$. One can also check that if K is centered, then $Z_q(K)\supseteq c_2Z_\infty(K)$ for all $q\geqslant n$.

It was shown by Paouris [29] that if $1\leqslant q\leqslant\sqrt{n}$ then

$$w\big(Z_q(\mu)\big)\simeq\sqrt{q}, \tag{2.14}$$

and that for all $1\leqslant q\leqslant n$,

$$\mathrm{vrad}(Z_q(\mu))\leqslant c_1\sqrt{q}. \tag{2.15}$$

Conversely, it was shown by B. Klartag and E. Milman in [21] that if $1 \leqslant q \leqslant \sqrt{n}$ then

$$\mathrm{vrad}(Z_q(\mu)) \geqslant c_2\sqrt{q}. \tag{2.16}$$

This determines the volume radius of $Z_q(\mu)$ for all $1 \leqslant q \leqslant \sqrt{n}$. For larger values of q one can still use the lower bound:

$$\mathrm{vrad}(Z_q(\mu)) \geqslant c_2\sqrt{q}\, L_\mu^{-1}, \tag{2.17}$$

obtained by Lutwak, Yang and Zhang in [25] for convex bodies and extended by Paouris and Pivovarov in [31] to the class of log-concave probability measures.

Let μ be a probability measure on $\mathbb{R}^n$ with density f_μ with respect to the Lebesgue measure. For every $1 \leqslant k \leqslant n-1$ and every $E \in G_{n,k}$, the marginal of μ with respect to E is the probability measure $\pi_E(\mu)$ on E, with density

$$f_{\pi_E(\mu)}(x) = \int_{x+E^\perp} f_\mu(y)dy. \tag{2.18}$$

It is easily checked that if μ is centered, isotropic or log-concave, then $\pi_E(\mu)$ is also centered, isotropic or log-concave, respectively. A very useful observation is that:

$$P_F\big(Z_q(\mu)\big) = Z_q\big(\pi_F(\mu)\big) \tag{2.19}$$

for every $1 \leqslant k \leqslant n-1$ and every $F \in G_{n,n-k}$.

If μ is a centered log-concave probability measure on $\mathbb{R}^n$ then for every $p > 0$ we define

$$K_p(\mu) := K_p(f_\mu) = \left\{ x : \int_0^\infty r^{p-1} f_\mu(rx)\, dr \geqslant \frac{f_\mu(0)}{p} \right\}. \tag{2.20}$$

From the definition it follows that $K_p(\mu)$ is a star body with radial function

$$\rho_{K_p(\mu)}(x) = \left(\frac{1}{f_\mu(0)} \int_0^\infty p r^{p-1} f_\mu(rx)\, dr \right)^{1/p} \tag{2.21}$$

for $x \neq 0$. The bodies $K_p(\mu)$ were introduced in [3] by K. Ball who showed that if μ is log-concave then, for every $p > 0$, $K_p(\mu)$ is a convex body.

If K is isotropic then for every $1 \leqslant k \leqslant n-1$ and $F \in G_{n,n-k}$, the body $\overline{K_{k+1}}(\pi_{F^\perp}(\mu_K))$ satisfies

$$|K \cap F|^{1/k} \simeq \frac{L_{\overline{K_{k+1}(\pi_{F^\perp}(\mu_K))}}}{L_K}. \tag{2.22}$$

For more information on isotropic convex bodies and log-concave measures see [9].

3 Random sections of isotropic convex bodies

The proof of Theorem 1.2 is based on Lemma 3.1 and Lemma 3.2 below. They exploit some ideas of Klartag from [18].

Lemma 3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ there exists a subset $\mathcal{A} := \mathcal{A}(n,k)$ of $G_{n,n-k}$ with $\nu_{n,n-k}(\mathcal{A}) \geqslant 1 - e^{-\sqrt{n}}$ that has the following property: for every $F \in \mathcal{A}$,*

$$|\{x \in K \cap F : \|x\|_2 \geqslant c_1\sqrt{n}L_K\}| \leqslant e^{-(k+\sqrt{n})}|K \cap F|, \tag{3.1}$$

where $c_1 > 0$ is an absolute constant.

Proof. Integration in polar coordinates shows that for all $q>0$

$$\int_{G_{n,n-k}}\int_{K\cap F}\|x\|_2^{k+q}dx\,d\nu_{n,n-k}(F)=\frac{(n-k)\omega_{n-k}}{n\omega_n}\int_K\|x\|_2^q dx=\frac{(n-k)\omega_{n-k}}{n\omega_n}I_q^q(K), \tag{3.2}$$

and an application of Markov's inequality shows that a random $F\in G_{n,n-k}$ satisfies

$$\int_{K\cap F}\|x\|_2^{k+q}dx\leqslant\frac{(n-k)\omega_{n-k}}{n\omega_n}(eI_q(K))^q \tag{3.3}$$

with probability greater than $1-e^{-q}$.

Fix a subspace $F\in G_{n,n-k}$ which satisfies (3.3). From (2.22) we have

$$|K\cap F|^{1/k}\geqslant c_2\frac{L_{\overline{K_{k+1}}(\pi_{F^\perp}(\mu_K))}}{L_K}\geqslant\frac{c_3}{L_K} \tag{3.4}$$

where $c_2,c_3>0$ are absolute constants. A simple computation shows that

$$\frac{(n-k)\omega_{n-k}}{n\omega_n}\leqslant(c_4\sqrt{n})^k \tag{3.5}$$

for an absolute constant $c_4>0$. Using also (2.11) with $q=\sqrt{n}$ we get

$$\begin{aligned}\frac{1}{|K\cap F|}\int_{K\cap F}\|x\|_2^{k+\sqrt{n}}dx&\leqslant\frac{1}{|K\cap F|}\frac{(n-k)\omega_{n-k}}{n\omega_n}(eI_{\sqrt{n}}(K))^{\sqrt{n}}\\&\leqslant(c_5L_K)^k(c_4\sqrt{n})^k(e\delta\sqrt{n}L_K)^{\sqrt{n}}\leqslant(c_6\sqrt{n}L_K)^{k+\sqrt{n}},\end{aligned} \tag{3.6}$$

where $c_6>0$ is an absolute constant. It follows that

$$|\{x\in K\cap F:\|x\|_2\geqslant ec_6\sqrt{n}L_K\}|\leqslant e^{-(k+\sqrt{n})}|K\cap F|. \tag{3.7}$$

and the lemma is proved with $c_1=ec_6$. □

The next lemma comes from [18].

Lemma 3.2 (Klartag)**.** *Let A be a symmetric convex body in $\mathbb{R}^m$. Then, for any $0<\varepsilon<1$ we have*

$$|\{x\in A:\|x\|_2\geqslant\varepsilon R(A)\}|\geqslant\frac{1}{2}(1-\varepsilon)^m|A|. \tag{3.8}$$

Proof. Let $x_0\in A$ such that $\|x_0\|_2=R(A)$ and define $v=x_0/\|x_0\|_2$. We consider the set A^+ defined as

$$A^+:=\{x\in A:\langle x,v\rangle\geqslant 0\}. \tag{3.9}$$

Since A is symmetric, we have $|A^+|=|A|/2$. Note that

$$\{x\in A:\|x\|_2\geqslant\varepsilon R(A)\}\supseteq\varepsilon x_0+(1-\varepsilon)A^+. \tag{3.10}$$

Therefore,

$$|\{x\in A:\|x\|_2\geqslant\varepsilon R(A)\}|\geqslant|\varepsilon x_0+(1-\varepsilon)A^+|=(1-\varepsilon)^m|A^+|=\frac{1}{2}(1-\varepsilon)^m|A|, \tag{3.11}$$

as claimed. □

Proof of Theorem 1.2. Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Applying Lemma 3.1 we find a subset $\mathcal{A}$ of $G_{n,n-k}$ with $\nu_{n,n-k}(\mathcal{A})\geqslant 1-e^{-\sqrt{n}}$ such that, for every $F\in\mathcal{A}$,

$$|\{x\in K\cap F:\|x\|_2\geqslant c_1\sqrt{n}L_K\}|\leqslant e^{-(k+\sqrt{n})}|K\cap F|. \tag{3.12}$$

We distinguish two cases:
Case 1. If $k > n/3$ then choosing $\varepsilon_0 = 1 - e^{-\frac{1}{3}}$ we get

$$\frac{1}{2}(1-\varepsilon_0)^{n-k}|K\cap F| = \frac{1}{2}e^{-\frac{n-k}{3}}|K\cap F| > e^{-\frac{n-k}{3}-1}|K\cap F| > e^{-(k+\sqrt{n})}|K\cap F|, \tag{3.13}$$

because $k + \sqrt{n} > \frac{n-k}{3} + 1$. By Lemma 3.2 and (3.12) we get that

$$|\{x\in K\cap F : \|x\|_2 \geqslant \varepsilon_0 R(K\cap F)\}| > |\{x \in K\cap F : \|x\|_2 \geqslant c_1\sqrt{n}L_K\}|, \tag{3.14}$$

therefore

$$R(K\cap F) < c_2\sqrt{n}L_K, \tag{3.15}$$

where $c_2 = \varepsilon_0^{-1}c_1 > 0$ is an absolute constant.
Case 2. If $k \leqslant n/3$ then we choose $\varepsilon_1 = \frac{k+\sqrt{n}}{6(n-k)}$. Note that $\varepsilon_1 < 1/2$. Using the inequality $1 - t > e^{-2t}$ on $(0, 1/2)$ we get

$$\frac{1}{2}(1-\varepsilon_1)^{n-k}|K\cap F| = \frac{1}{2}\left(1 - \frac{k+\sqrt{n}}{6(n-k)}\right)^{n-k}|K\cap F| > e^{-\frac{k+\sqrt{n}}{3}-1}|K\cap F| > e^{-(k+\sqrt{n})}|K\cap F|, \tag{3.16}$$

because $\frac{2(k+\sqrt{n})}{3} > 1$. By Lemma 3.2 this implies that

$$|\{x\in K\cap F : \|x\|_2 \geqslant \varepsilon_1 R(K\cap F)\}| > |\{x \in K\cap F : \|x\|_2 \geqslant c_1\sqrt{n}L_K\}|, \tag{3.17}$$

therefore

$$\varepsilon_1 R(K\cap F) < c_1\sqrt{n}L_K, \tag{3.18}$$

which, by the choice of ε_1 becomes

$$R(K\cap F) < \frac{c_3 n}{\max\{k, \sqrt{n}\}}\sqrt{n}L_K \tag{3.19}$$

for some absolute constant $c_3 > 0$. This completes the proof of the theorem (with a probability estimate $1 - e^{-\sqrt{n}}$ for all $1 \leqslant k \leqslant n-1$). □

Remark 3.3. It is possible to improve the probability estimate $1 - e^{-\sqrt{n}}$ in the range $k \geqslant \gamma n$, for any $\gamma \in (1/\sqrt{n}, 1)$. This can be done with the help of known results that demonstrate the fact that the existence of one s-dimensional section with radius r implies that random m-dimensional sections, where $m < s$, have radius of "the same order". This was first observed in [12], [34] and, soon after, in [24]. Let us recall this last statement.

> *Let A be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leqslant s < m \leqslant n-1$. If $R(A\cap F) \leqslant r$ for some $F \in G_{n,m}$ then a random subspace $E \in G_{n,s}$ satisfies*
>
> $$R(A\cap E) \leqslant r\left(\frac{c_2 n}{n-m}\right)^{\frac{n-s}{2(m-s)}} \tag{3.20}$$
>
> *with probability greater than $1 - 2e^{-(n-s)/2}$, where $c_2 > 0$ is an absolute constant.*

We apply this result as follows. Let $k = \gamma n \geqslant \sqrt{n}$ and set $t = \delta n$, where $\delta \simeq \gamma/\log(1 + 1/\gamma)$. From the proof of Theorem 1.2 we know that there exists $E \in G_{n,n-t}$ such that

$$R(K\cap E) \leqslant \frac{c_1 n}{t}\sqrt{n}L_K, \tag{3.21}$$

where $c_1 > 0$ is an absolute constant. Applying (3.20) with $s = n - k$ and $m = n - t$ we see that a random subspace $F \in G_{n,n-k}$ satisfies

$$R(K \cap F) \leqslant \left(\frac{c_2}{\delta}\right)^{\frac{3}{2}} R(K \cap E) = c_3(\gamma)\sqrt{n}L_K \tag{3.22}$$

with probability greater than $1 - 2e^{-k/2}$, where $c_3(\gamma) = O((\gamma^{-1}\log(1+1/\gamma))^{\frac{3}{2}})$.

Remark 3.4. It is also possible to give lower bounds of the order of $\sqrt{n}L_K$ for the diameter of $(n-k)$-dimensional sections, provided that the codimension k is small. Integration in polar coordinates shows that

$$\int_K \|x\|_2^{-q} dx = \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \int_{K\cap F} \|x\|_2^{k-q} dx\, d\nu_{n,n-k}(F) \tag{3.23}$$

for every $1 \leqslant k \leqslant n-1$ and every $0 < q < n$. It follows that

$$\int_{G_{n,n-k}} \int_{K\cap F} \|x\|_2^{k-q} dx\, d\nu_{n,n-k}(F) = \frac{(n-k)\omega_{n-k}}{n\omega_n} I_{-q}^{-q}(K), \tag{3.24}$$

and an application of Markov's inequality shows that a random $F \in G_{n,n-k}$ satisfies

$$\int_{K\cap F} \|x\|_2^{k-q} dx \leqslant \frac{(n-k)\omega_{n-k}}{n\omega_n} (e/I_{-q}(K))^q \tag{3.25}$$

with probability greater than $1 - e^{-q}$. Assuming that $q > k$, for any $F \in G_{n,n-k}$ satisfying (3.25) we have

$$|K\cap F|\, R(K\cap F)^{k-q} \leqslant \int_{K\cap F} \|x\|_2^{k-q} dx \leqslant \frac{(n-k)\omega_{n-k}}{n\omega_n} (e/I_{-q}(K))^q, \tag{3.26}$$

which implies

$$R(K\cap F) \geqslant \left(\frac{n\omega_n}{(n-k)\omega_{n-k}}\right)^{\frac{1}{q-k}} |K\cap F|^{\frac{1}{q-k}} \left(\frac{I_{-q}(K)}{e}\right)^{\frac{q}{q-k}} \geqslant \left(\frac{c_1}{\sqrt{n}L_K}\right)^{\frac{k}{q-k}} (c_2 I_{-q}(K))^{\frac{q}{q-k}}. \tag{3.27}$$

If $k \leqslant \sqrt{n}$ then we may choose $q = 2\sqrt{n}$ and use the fact that $I_{-2\sqrt{n}}(K) \geqslant c_3\sqrt{n}L_K$ by Theorem 2.1, to get:

Proposition 3.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant \sqrt{n}$ there exists a subset $\mathcal{A}$ of $G_{n,n-k}$ with $\nu_{n,n-k}(\mathcal{A}) \geqslant 1 - e^{-\sqrt{n}}$ such that, for every $F \in \mathcal{A}$,*

$$R(K\cap F) \geqslant c\sqrt{n}L_K, \tag{3.28}$$

where $c > 0$ is an absolute constant.

Remark 3.6. Choosing $k = \lfloor n/2 \rfloor$ in Theorem 1.2 we see that if K is an isotropic symmetric convex body in $\mathbb{R}^n$ then a subspace $F \in G_{n,\lceil n/2\rceil}$ satisfies

$$R(K\cap F) \leqslant c_1\sqrt{n}\, L_K \tag{3.29}$$

with probability greater than $1 - 2\exp(-c_2 n)$, where $c_1, c_2 > 0$ are absolute constants. A standard argument that goes back to Krivine (see [2, Proposition 8.6.2]) shows that there exists $U \in O(n)$ such that

$$K \cap U(K) \subseteq (c_3\sqrt{n}L_K)\, B_2^n, \tag{3.30}$$

where $c_3 > 0$ is an absolute constant. In fact, one can prove an analogue of (3.30) for a random $U \in O(n)$ using a result of Vershynin and Rudelson (see [34, Theorem 1.1]): There exist absolute constants $\gamma_0 \in (0, 1/2)$

and $c_1 > 0$ with the following property: if A and D are two symmetric convex bodies in $\mathbb{R}^n$ which have sections of dimensions at least k and $n - 2\gamma_0 k$ respectively whose radius is bounded by 1, then a random $U \in O(n)$ satisfies

$$R(A \cap U(D)) \leqslant c_1^{n/k} \tag{3.31}$$

with probability greater than $1 - e^{-n}$. As an application, setting $D = A$ and $k = n/2$ one has the following (see [8]). If

$$r_A := \min\{R(A \cap F) : \dim(F) = \lceil (1-\gamma_0)n \rceil\} \tag{3.32}$$

then $R(A \cap U(A)) \leqslant c_2 r_A$ with probability greater than $1 - e^{-n}$ with respect to $U \in O(n)$.

Choosing $k = \lfloor \gamma_0 n/2 \rfloor$ in Theorem 1.2 we see that if K is an isotropic symmetric convex body in $\mathbb{R}^n$ then

$$r_K \leqslant c_4 \sqrt{n} L_K \tag{3.33}$$

for some absolute constant $c_4 > 0$. This gives that a random $U \in O(n)$ satisfies

$$K \cap U(K) \subseteq (c_5 \sqrt{n} L_K)\, B_2^n, \tag{3.34}$$

with probability greater than $1 - e^{-n}$, where $c_5 > 0$ is an absolute constant.

4 Minimal and random sections of the centroid bodies of isotropic log-concave measures

In this section we discuss the case of the L_q-centroid bodies $Z_q(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$. Our method will be different from the one in the previous section.

In view of (3.20) we can give an upper bound for the radius of a random k-codimensional section of a symmetric convex body A in $\mathbb{R}^n$ if we are able to give an upper bound for the radius of *some* t-codimensional section of A, where $t \ll k$. This leads us to the study of the Gelfand numbers $c_t(A)$, which are defined by

$$c_t(A) = \min\{R(A \cap F) : F \in G_{n,n-t}\} \tag{4.1}$$

for every $t = 0, \ldots, n-1$. It was proved in [11] that if A is a symmetric convex body in $\mathbb{R}^n$ then, for any $t = 1, \ldots, \lfloor n/2 \rfloor$ there exists $F \in G_{n,n-2t}$ such that

$$A \cap F \subseteq c_1 \frac{n}{t} \log\left(e + \frac{n}{t}\right) w_t(A) B_2^n \cap F, \tag{4.2}$$

where

$$w_t(A) := \sup\{\mathrm{vrad}(A \cap E) : E \in G_{n,t}\}. \tag{4.3}$$

In other words,

$$c_{2t}(A) \leqslant c_1 \frac{n}{t} \log\left(e + \frac{n}{t}\right) w_t(A). \tag{4.4}$$

This is a refinement of a result of V. Milman and G. Pisier from [27], where a similar estimate was obtained, with the parameter $w_t(A)$ replaced by (the larger one)

$$v_t(A) := \sup\{\mathrm{vrad}(P_E(A)) : E \in G_{n,t}\}. \tag{4.5}$$

We shall apply this method to the bodies $Z_q(\mu)$. The main additional ingredient is the next fact, which combines results of Paouris and Klartag (see [26] or [9, Chapter 5] for precise references):

Theorem 4.1. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Then, for all $1 \leqslant t \leqslant n$ and $q \geqslant 1$ we have*

$$v_t(Z_q(\mu)) = \sup\{\text{vrad}(P_E(Z_q(\mu))) : E \in G_{n,t}\} \leqslant c_0\sqrt{\frac{q}{t}}\max\{\sqrt{q},\sqrt{t}\}\max_{E\in G_{n,t}}\det \text{Cov}(\pi_E(\mu))^{\frac{1}{2t}}, \tag{4.6}$$

where $c_0 > 0$ is an absolute constant.

We apply Theorem 4.1 as follows: for every $1 \leqslant t \leqslant n/2$ and every $E \in G_{n,t}$ we have that $\pi_E(\mu)$ is isotropic, and hence $\det \text{Cov}(\pi_E(\mu))^{\frac{1}{2t}} = 1$. Then,

$$w_t(Z_q(\mu)) \leqslant v_t(Z_q(\mu)) \leqslant c_0\sqrt{\frac{q}{t}}\max\{\sqrt{q},\sqrt{t}\}. \tag{4.7}$$

From (4.4) we get

Lemma 4.2. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $1 \leqslant t \leqslant \lfloor n/2\rfloor$ and $1 \leqslant q \leqslant n$. Then,*

$$c_{2t}(Z_q(\mu)) \leqslant c_2\frac{n}{t}\log\left(e+\frac{n}{t}\right)\sqrt{\frac{q}{t}}\max\{\sqrt{q},\sqrt{t}\}, \tag{4.8}$$

where $c_2 > 0$ is an absolute constant. □

Let $k \geqslant 4$ and let $t < k/2$. From Lemma 4.2 we know that there exists $E \in G_{n,n-2t}$ such that

$$R(Z_q(\mu)\cap E) \leqslant c_2\frac{n}{t}\log\left(e+\frac{n}{t}\right)\sqrt{\frac{q}{t}}\max\{\sqrt{q},\sqrt{t}\}, \tag{4.9}$$

where $c_2 > 0$ is an absolute constant. Applying (3.20) with $s = n-k$ and $m = n-2t$ we see that a random subspace $F \in G_{n,n-k}$ satisfies

$$R(Z_q(\mu)\cap F) \leqslant \left(\frac{c_2 n}{t}\right)^{\frac{k}{2(k-2t)}} R(Z_q(\mu)\cap E) \leqslant \left(\frac{c_3 n}{t}\right)^{\frac{3}{2}+\frac{t}{k-2t}}\log\left(e+\frac{n}{t}\right)\sqrt{\frac{q}{t}}\max\{\sqrt{q},\sqrt{k}\} \tag{4.10}$$

with probability greater than $1-2e^{-k/2}$, where $c_3 > 0$ is an absolute constant. In particular, if $k = \gamma n$ we can choose $t = \gamma n/\log(c/\gamma)$, for $c > e^2$, to get the following.

Theorem 4.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $\gamma \in (0,1)$ and $1 \leqslant q \leqslant n$. If $k \geqslant \gamma n$ then a random subspace $F \in G_{n,n-k}$ satisfies*

$$R(Z_q(\mu)\cap F) \leqslant c(\gamma)\sqrt{q} \tag{4.11}$$

with probability greater than $1-2e^{-\gamma n/2}$, where $c(\gamma) = O(\gamma^{-2}\log^{5/2}(c/\gamma))$ is a positive constant depending only on γ.

Next, we apply (3.31): choosing $t = \gamma_0 n/2$ in (4.8) we see that

$$r_{Z_q(\mu)} = c_{\gamma_0 n}(Z_q(\mu)) \leqslant c_4\sqrt{q} \tag{4.12}$$

for every $1 \leqslant q \leqslant n$, where $c_4 = c_4(\gamma_0) > 0$ is an absolute constant. Therefore, we have:

Theorem 4.4. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $1 \leqslant q \leqslant n$. Then, a random $U \in O(n)$ satisfies*

$$Z_q(\mu)\cap U(Z_q(\mu)) \subseteq (c\sqrt{q})\,B_2^n, \tag{4.13}$$

with probability greater than $1-e^{-n}$, where $c > 0$ is an absolute constant.

Note that Theorem 1.3 summarizes the contents of Theorem 4.3 and Theorem 4.4.

Remark 4.5. We can study the same question for the polar body $Z_q^\circ(\mu)$ of $Z_q(\mu)$. Note that

$$w_t(Z_q^\circ(\mu)) := \sup\{\mathrm{vrad}(Z_q^\circ(\mu)\cap E) : E\in G_{n,t}\} \simeq [\inf\{\mathrm{vrad}(P_E(Z_q(\mu))) : E\in G_{n,t}\}]^{-1} \tag{4.14}$$

by duality and by the Bourgain-Milman inequality. For any $1\leqslant t\leqslant n-1$ and any symmetric convex body A in $\mathbb{R}^n$ define

$$v_t^-(A) = \inf\{\mathrm{vrad}(P_E(A)) : E\in G_{n,t}\}. \tag{4.15}$$

In the case $A = Z_q(\mu)$ this parameter has been studied in [11]:

Lemma 4.6. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $q\geqslant 1$ and $1\leqslant k\leqslant n-1$ we have:*

$$v_k^-(Z_q(\mu)) \geqslant c_1\sqrt{\min(q,\sqrt{k})}. \tag{4.16}$$

If we assume that $\sup_n L_n\leqslant\alpha$ then we have

$$v_k^-(Z_q(\mu)) \geqslant \frac{c_2}{\alpha}\sqrt{\min(q,k)} \tag{4.17}$$

These estimates are leading to the next bounds on the minimal radius of a k-codimensional section of $Z_q^\circ(\mu)$. The following theorem is also from [11].

Theorem 4.7. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $q\geqslant 1$ and $1\leqslant k\leqslant n-1$ we have:*

(i) *There exists $F\in G_{n,n-k}$ such that:*

$$P_F(Z_q(\mu)) \supseteq \frac{1}{R_{k,q}}B_2^n\cap F \quad \textit{and hence} \quad R(Z_q^\circ(\mu)\cap F)\leqslant R_{k,q}, \tag{4.18}$$

where

$$R_{k,q} = \min\left\{1, c_3\frac{1}{\min(q^{1/2},k^{1/4})}\frac{n}{k}\log\left(e+\frac{n}{k}\right)\right\}. \tag{4.19}$$

(ii) *If we assume that $\sup_n L_n\leqslant\alpha$ then there exists $F\in G_{n,n-k}$ such that:*

$$P_F(Z_q(\mu)) \supseteq \frac{1}{R_{k,q,\alpha}}B_2^n\cap F \quad \textit{and hence} \quad R(Z_q^\circ(\mu)\cap F)\leqslant R_{k,q,\alpha}, \tag{4.20}$$

where

$$R_{k,q,\alpha} = \min\left\{1, c_4\alpha\frac{1}{\sqrt{\min(q,k)}}\frac{n}{k}\log\left(e+\frac{n}{k}\right)\right\}. \tag{4.21}$$

Assuming that $q\leqslant\sqrt{n}$ and choosing $k=\gamma_0 n$ we see from (4.18) and (4.19) that

$$c_{\gamma_0 n}(Z_q^\circ(\mu)) \leqslant c_1(\gamma_0)\frac{1}{\sqrt{q}} \tag{4.22}$$

where $c_1(\gamma_0)>0$ is an absolute constant. Then, we apply (3.20) with $s=n/2$ and $m=(1-\gamma_0)n$ to get that a random subspace $E\in G_{n,n/2}$ satisfies

$$R(Z_q^\circ(\mu)\cap E) \leqslant c_3\cdot c_{\gamma_0 n}(Z_q^\circ(\mu)) \leqslant c_2(\gamma_0)\frac{1}{\sqrt{q}} \tag{4.23}$$

with probability greater than $1-2e^{-n/4}$, where $c_2(\gamma_0)>0$ is an absolute constant. As usual, this implies that a random $U\in O(n)$ satisfies

$$Z_q^{\circ}(\mu)\cap U(Z_q^{\circ}(\mu))\subseteq \frac{c}{\sqrt{q}}\,B_2^n, \tag{4.24}$$

with probability greater than $1-e^{-n}$, where $c>0$ is an absolute constant. This estimate appears in [22] (and a second proof is given in [8]).

Assuming that $\sup_n L_n\leqslant \alpha$ we may apply the same reasoning for every $1\leqslant q\leqslant n$: choosing $k=\gamma_0 n$ we see from (4.20) and (4.21) that

$$c_{\gamma_0 n}(Z_q^{\circ}(\mu))\leqslant c_1(\gamma_0)\frac{\alpha}{\sqrt{q}}, \tag{4.25}$$

where $c_1(\gamma_0)>0$ is an absolute constant. Then, we apply (3.20) with $s=n/2$ and $m=(1-\gamma_0)n$ to get that a random subspace $E\in G_{n,n/2}$ satisfies

$$R(Z_q^{\circ}(\mu)\cap E)\leqslant c_3\cdot c_{\gamma_0 n}(Z_q^{\circ}(\mu))\leqslant c_2(\gamma_0)\frac{\alpha}{\sqrt{q}} \tag{4.26}$$

with probability greater than $1-2e^{-n/4}$, where $c_2(\gamma_0)>0$ is an absolute constant. Finally, this implies that a random $U\in O(n)$ satisfies

$$Z_q^{\circ}(\mu)\cap U(Z_q^{\circ}(\mu))\subseteq \frac{c\alpha}{\sqrt{q}}\,B_2^n, \tag{4.27}$$

with probability greater than $1-e^{-n}$, where $c>0$ is an absolute constant.

4.1 Random sections of bodies with maximal isotropic constant

Starting with an isotropic symmetric convex body K in $\mathbb{R}^n$ we can use the method of this section in order to estimate the quantities

$$c_t(K)=\min\{R(K\cap F):F\in G_{n,n-t}\} \tag{4.28}$$

for every $t=0,\dots,n-1$. From (2.22) we have

$$|K\cap E|^{\frac{1}{n-t}}\leqslant c_2\frac{L_{\overline{K_{k+1}}(\pi_{E^\perp}(\mu_K))}}{L_K}\leqslant \frac{c_3L_{n-t}}{L_K} \tag{4.29}$$

for every $E\in G_{n,t}$, therefore

$$w_t(K)\leqslant c_4\sqrt{t}\left(\frac{c_3L_{n-t}}{L_K}\right)^{\frac{n-t}{t}}. \tag{4.30}$$

Assume that K has maximal isotropic constant, i.e. $L_K=L_n'$ (the same argument works if we assume that L_K is almost maximal, i.e. $L_K\geqslant \beta L_n'$ for some absolute constant $\beta\in(0,1)$). It is known that $L_{n-t}\leqslant c_1L_n\leqslant c_2L_n'$ for all $1\leqslant t\leqslant n-1$, where $c_1,c_2>0$ are absolute constants. Therefore, we get:

Lemma 4.8. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$ such that $L_K=L_n'$, and let $1\leqslant t\leqslant \lfloor n/2\rfloor$. Then,*

$$c_{2t}(K)\leqslant c_1^{\frac{n-t}{t}}\frac{n}{\sqrt{t}}\log\left(e+\frac{n}{t}\right), \tag{4.31}$$

where $c>0$ is an absolute constant.

Then, we apply (3.20) with $s = n/2$ and $m = (1-\gamma_0)n$ to get that a random subspace $E \in G_{n,n/2}$ satisfies

$$R(K \cap E) \leqslant c_3 \cdot c_{\gamma_0 n}(K) \leqslant c_1(\gamma_0)\sqrt{n} \tag{4.32}$$

with probability greater than $1 - 2e^{-n/4}$, where $c_1(\gamma_0) > 0$ is an absolute constant.

Also, since $c_{\gamma_0 n}(K) \leqslant c(\gamma_0)\sqrt{n}$, we may apply (3.31) to get:

Theorem 4.9. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$ with $L_K = L_n'$. A random $U \in O(n)$ satisfies*

$$K \cap U(K) \subseteq (c_3\sqrt{n})\, B_2^n, \tag{4.33}$$

with probability greater than $1 - e^{-n}$, where $c_3 > 0$ is an absolute constant.

We can also prove the local analogue of this fact: random proportional sections of a body with maximal isotropic constant have bounded isotropic constant.

Theorem 4.10. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$ with $L_K = L_n'$. A random $F \in G_{n,n/2}$ satisfies*

$$L_{K\cap F} \leqslant c_4 \tag{4.34}$$

with probability greater than $1 - e^{-c_5 n}$, where $c_4, c_5 > 0$ are absolute constants.

Proof. It was proved in [10] (see also [9, Lemma 6.3.5]) that if $L_K = L_n'$ then

$$|K \cap F|^{\frac{1}{n}} \geqslant c_6 \tag{4.35}$$

for every $G_{n,n/2}$, where $c_6 > 0$ is an absolute constant. Since $R(K \cap F) \leqslant c_3\sqrt{n}$ for a random $F \in G_{n,n/2}$, for all these F we get

$$\frac{n}{2} L_{K\cap F}^2 \leqslant \frac{1}{|K \cap F|^{1+\frac{2}{n}}} \int_{K\cap F} \|x\|_2^2 dx \leqslant \frac{1}{|K\cap F|^{\frac{2}{n}}} R^2(K\cap F) \leqslant c_6^{-2} c_3^2 n, \tag{4.36}$$

which implies that

$$L_{K\cap F} \leqslant c_4, \tag{4.37}$$

where $c_4 = \sqrt{2} c_6^{-1} c_3$. □

5 Sub-Gaussian subspaces

In this section we prove Theorem 1.5. We will use E. Milman's estimates [26] on the mean width $w(Z_q(K))$ of the L_q-centroid bodies $Z_q(K)$ of an isotropic convex body K in $\mathbb{R}^n$.

Theorem 5.1 (E. Milman)**.** *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for all $q \geqslant 1$ one has*

$$w(Z_q(K)) \leqslant c_1 \log(1+q) \max\left\{ \frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q} \right\} L_K \tag{5.1}$$

where $c_1 > 0$ is an absolute constant.

We also use the next fact on the diameter of k-dimensional projections of symmetric convex bodies (see [2, Proposition 5.7.1]).

Proposition 5.2. *Let D be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leqslant k < n$ and $\alpha > 1$. Then there exists a subset $\Gamma_{n,k} \subset G_{n,k}$ with measure $\nu_{n,k}(\Gamma_{n,k}) \geqslant 1 - e^{-c_2\alpha^2 k}$ such that the orthogonal projection of D onto any subspace $F \in \Gamma_{n,k}$ satisfies*

$$R(P_F(D)) \leqslant c_3\alpha \max\{w(D), R(D)\sqrt{k/n}\}, \tag{5.2}$$

where $c_2 > 0, c_3 > 1$ are absolute constants.

Combining Proposition 5.2 with Theorem 5.1 and the fact that $R(Z_q(K)) \leqslant cqL_K$, we get:

Lemma 5.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Given $1 \leqslant q \leqslant n$ define $k_0(q)$ by the equation*

$$k_0(q) = \log^2(1+q)\max\{\log^2(1+q), n/q\}. \tag{5.3}$$

Then, for every $1 \leqslant k \leqslant k_0(q)$, a random $F \in G_{n,k}$ satisfies

$$R(P_F(Z_q(K))) \leqslant c_1\alpha \log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\} L_K \tag{5.4}$$

with probability greater than $1 - e^{-c_2\alpha^2 k_0(q)}$, where $c_1, c_2 > 0$ are absolute constants.

Proof. Since $R(Z_q(K)) \leqslant cqL_K$ we see that

$$\begin{aligned} \frac{R(Z_q(K))\sqrt{k_0(q)}}{\sqrt{n}} &\leqslant \frac{cq}{\sqrt{n}}\log(1+q)\max\left\{\log(1+q), \frac{\sqrt{n}}{\sqrt{q}}\right\} L_K \\ &= c\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\} L_K. \end{aligned} \tag{5.5}$$

From Theorem 5.1 we have an upper bound of the same order for $w(Z_q(K))$. Then, we apply Proposition 5.2 for $Z_q(K)$. □

Remark 5.4. Note that if $1 \leqslant s \leqslant k$ then the conclusion of Proposition 5.2 continues to hold for a random $F \in G_{n,s}$ with the same probability on $G_{n,s}$; this is an immediate consequence of Fubini's theorem and of the fact that $R(P_H(D)) \leqslant R(P_F(D))$ for every s-dimensional subspace H of a k-dimensional subspace F of $\mathbb{R}^n$.

Proof of Theorem 1.5. We define q_0 by the equation

$$q_0\log^2(1+q_0) = n. \tag{5.6}$$

Note that $q_0 \simeq n/(\log n)^2$ and $\log(1+q_0) \simeq \log n$. For every $2 \leqslant q \leqslant q_0$ we have $q\log^2(1+q) \leqslant n$, therefore

$$k_0(q) = \frac{n\log^2(1+q)}{q} \geqslant \frac{c_1 n\log^2(1+q_0)}{q_0} \tag{5.7}$$

for some absolute constant $c_1 > 0$, because $q \mapsto \log^2(1+q)/q$ is decreasing for $q \geqslant 4$. It follows that

$$k_0(q) \geqslant c_1\log^4(1+q_0) \geqslant c_2(\log n)^4 \tag{5.8}$$

for all $2 \leqslant q \leqslant q_0$.

Now, we fix $\alpha > 1$ and define

$$k_0 = c_1\log^4(1+q_0). \tag{5.9}$$

Using Lemma 5.3 and Remark 5.4, for every $q \leqslant q_0$ we can find a set $\Gamma_q \subseteq G_{n,k_0}$ with $\nu_{n,k_0}(\Gamma_q) \geqslant 1 - e^{-c\alpha^2 k_0}$ such that

$$R(P_F(Z_q(K))) \leqslant c_3\alpha \log(1+q) \max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\} L_K \leqslant c_3\alpha\sqrt{q}\log(1+q)L_K \tag{5.10}$$

for all $F \in G_{n,k_0}$. If $\Gamma := \bigcap_{s=1}^{\lfloor \log_2 q_0 \rfloor} \Gamma_{2^s}$, then

$$\nu_{n,k_0}\big(G_{n,k_0} \setminus \Gamma\big) \leqslant \nu_{n,k_0}\Big(G_{n,k_0} \setminus \bigcap_{s=1}^{\lfloor \log_2 n \rfloor} \Gamma_{2^s}\Big) \leqslant c(\log n)e^{-c\alpha^2 k_0} \leqslant \frac{1}{n^{\log^3 n}} \tag{5.11}$$

if $\alpha \simeq 1$ is chosen large enough. Then for every $F \in \Gamma$, for all $\theta \in S_F$ and for every $1 \leqslant s \leqslant \lfloor \log_2 q_0 \rfloor$ we have

$$\frac{h_{Z_{2^s}(K)}(\theta)}{\sqrt{2^s}} = \frac{h_{P_F(Z_{2^s}(K))}(\theta)}{\sqrt{2^s}} \leqslant c_3\alpha\log(1+2^s)L_K \leqslant c_4\alpha(\log n)L_K. \tag{5.12}$$

Taking into account the fact that if $2^s \leqslant q < 2^{s+1}$ then

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant \frac{h_{Z_{2^{s+1}}(K)}(y)}{2^{s/2}} = \sqrt{2}\frac{h_{Z_{2^{s+1}}(K)}(y)}{2^{(s+1)/2}}, \tag{5.13}$$

we see that

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant c_5\alpha(\log n)L_K \tag{5.14}$$

for every $F \in \Gamma$, for all $\theta \in S_F$ and for every $2 \leqslant q \leqslant q_0$.

Next, observe that if $q_0 \leqslant q \leqslant n$ then we may write

$$\begin{aligned}\frac{h_{Z_q(K)}(y)}{\sqrt{q}} &\leqslant \frac{c_6 q}{q_0}\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q}} = \frac{c_6\sqrt{q}}{\sqrt{q_0}}\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}} \leqslant \frac{c_6\sqrt{n}}{\sqrt{q_0}}\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}} \\ &= c_6\log(1+q_0)\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}} \leqslant c_7(\log n)\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}},\end{aligned} \tag{5.15}$$

and hence

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant c_7\alpha(\log n)^2 L_K \tag{5.16}$$

for every $F \in \Gamma$, for all $\theta \in S_F$ and for every $q_0 \leqslant q \leqslant n$.

Recall that $\Psi_2(K)$ is the convex body with support function $h_{\Psi_2(K)}(y) = \|\langle \cdot, y\rangle\|_{L_{\psi_2}(K)}$. One also has

$$h_{\Psi_2(K)}(y) \simeq \sup_{q \geqslant 2}\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \simeq \sup_{2 \leqslant q \leqslant n}\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \tag{5.17}$$

because $h_{Z_q(K)}(y) \simeq h_{Z_n(K)}(y)$ for all $q \geqslant n$. Then, (5.14) and (5.16) and the fact that $\alpha \simeq 1$ show that

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K \tag{5.18}$$

for every $F \in \Gamma$ and for all $\theta \in S_F$, where $C > 0$ is an absolute constant. □

References

[1] D. Alonso-Gutiérrez, J. Bastero, J. Bernués and G. Paouris, *High dimensional random sections of isotropic convex bodies*, J. Math. Appl. **361** (2010), 431-439.

[2] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Vol. I*, Mathematical Surveys and Monographs **202**, Amer. Math. Society (2015).

[3] K. M. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$, Studia Math. **88** (1988), 69-84.

[4] C. Borell, *Convex measures on locally convex spaces*, Ark. Mat. **12** (1974), 239-252.

[5] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, in Geom. Aspects of Funct. Analysis, Lecture Notes in Mathematics **1469**, Springer, Berlin (1991), 127-137.

[6] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$, Invent. Math. **88** (1987), 319-340.

[7] S. Brazitikos and L. Hioni, *Sub-Gaussian directions of isotropic convex bodies*, J. of Math. Anal. and Applications **425** (2015), 919-927.

[8] S. Brazitikos and P. Stavrakakis, *On the intersection of random rotations of a symmetric convex body*, Math. Proc. Cambridge Philos. Soc. **157** (2014), 13-30.

[9] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs **196**, Amer. Math. Society (2014).

[10] N. Dafnis and G. Paouris, *Small ball probability estimates,* ψ_2*-behavior and the hyperplane conjecture*, J. Funct. Anal. **258** (2010), 1933-1964.

[11] A. Giannopoulos and E. Milman, M*-estimates for isotropic convex bodies and their* L_q*-centroid bodies*, in Geom. Aspects of Funct. Analysis, Lecture Notes in Mathematics **2116** (2014), 159-182.

[12] A. Giannopoulos, V. D. Milman and A. Tsolomitis, *Asymptotic formulas for the diameter of sections of symmetric convex bodies*, Journal of Functional Analysis **223** (2005), 86-108.

[13] A. Giannopoulos, A. Pajor and G. Paouris, *A note on subgaussian estimates for linear functionals on convex bodies*, Proc. Amer. Math. Soc. **135** (2007), 2599-2606.

[14] A. Giannopoulos, G. Paouris and P. Valettas, *On the existence of subgaussian directions for log-concave measures*, Contemporary Mathematics **545** (2011), 103-122.

[15] A. Giannopoulos, G. Paouris and P. Valettas, *On the distribution of the* ψ_2*-norm of linear functionals on isotropic convex bodies*, in Geom. Aspects of Funct. Analysis, Lecture Notes in Mathematics **2050** (2012), 227-254.

[16] M. Hartzoulaki, *Probabilistic methods in the theory of convex bodies*, Ph.D. Thesis (March 2003), University of Crete.

[17] R. Kannan, L. Lovász and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), 541-559.

[18] B. Klartag, *A geometric inequality and a low* M*-estimate*, Proc. Amer. Math. Soc. **132** (2004), 2619-2628.

[19] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274-1290.

[20] B. Klartag, *Uniform almost sub-Gaussian estimates for linear functionals on convex sets*, Algebra i Analiz **19** (2007), 109-148.

[21] B. Klartag and E. Milman, *Centroid Bodies and the Logarithmic Laplace Transform – A Unified Approach*, J. Funct. Anal. **262** (2012), 10-34.

[22] B. Klartag and E. Milman, *Inner regularization of log-concave measures and small-ball estimates*, in Geom. Aspects of Funct. Analysis (Klartag-Mendelson-Milman eds.), Lecture Notes in Math. **2050** (2012), 267-278.

[23] A. E. Litvak, V. D. Milman and A. Pajor, *Covering numbers and "low* M^**-estimate" for quasi-convex bodies*, Proc. Amer. Math. Soc. **127** (1999), 1499-1507.

[24] A. E. Litvak, A. Pajor and N. Tomczak-Jaegermann, *Diameters of Sections and Coverings of Convex Bodies*, J. Funct. Anal. **231** (2006), 438-457.

[25] E. Lutwak, D. Yang and G. Zhang, *L^p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111-132.

[26] E. Milman, *On the mean width of isotropic convex bodies and their associated L_p-centroid bodies*, Int. Math. Res. Not. IMRN (2015), no. 11, 3408-3423.

[27] V. D. Milman and G. Pisier, *Gaussian processes and mixed volumes*, Annals of Probability **15** (1987), 292-304.

[28] A. Pajor and N. Tomczak-Jaegermann, *Subspaces of small codimension of finite dimensional Banach spaces*, Proc. Amer. Math. Soc. **97** (1986), 637-642.

[29] G. Paouris, *Concentration of mass in convex bodies*, Geometric and Functional Analysis **16** (2006), 1021-1049.

[30] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. **364** (2012), 287-308.

[31] G. Paouris and P. Pivovarov, *A probabilistic take on isoperimetric-type inequalities*, Adv. Math. **230** (2012), 1402-1422.

[32] P. Pivovarov, *On the volume of caps and bounding the mean-width of an isotropic convex body*, Math. Proc. Cambridge Philos. Soc. **149** (2010), 317-331.

[33] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Second expanded edition. Encyclopedia of Mathematics and Its Applications 151, Cambridge University Press, Cambridge, 2014.

[34] R. Vershynin, *Isoperimetry of waists and local versus global asymptotic convex geometries* (with an appendix by M. Rudelson and R. Vershynin), Duke Mathematical Journal **131** (2006), 1-16.

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