On some vector balancing problems

A. Giannopoulos

Abstract

Let V be an origin symmetric convex body in $\mathbb{R}^n$, $n \geq 2$, of Gaussian measure $\gamma_n(V) \geq \frac{1}{2}$. It is proved that for every choice $u_1, \ldots, u_n$ of vectors in the Euclidean unit ball B_n, there exist signs $\varepsilon_j \in \{-1, 1\}$ with $\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n \in c \log n\ V$. The method used can be modified to give simple proofs of several related results of J. Spencer and E.D. Gluskin.

1 Introduction

Let $\mathcal{C}_n$ denote the class of all origin symmetric convex bodies in $\mathbf{R}^n$, $n \geq 2$. Following W. Banaszczyk [2], for each pair $U, V \in \mathcal{C}_n$ we define $\beta(U, V)$ as the smallest $r > 0$ satisfying the following condition: given $u_1, \ldots, u_n \in U$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that $\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n \in rV$.

Several "vector balancing" results, proved by various authors for quite different purposes, can be described as estimates on $\beta(U, V)$ for specific choices of U, V, or both of them:

(a) W. Banaszczyk [2] established a general lower bound for $\beta(U, V)$ in terms of the volumes of U, V: for some absolute constant $c > 0$, and for any $U, V \in \mathcal{C}_n$, one has

$$\beta(U, V) \geq c\sqrt{n}(|U|/|V|)^{1/n}.$$

(b) I. Bárány and V.S. Grinberg [5] show that $\beta(U, U) \leq 2n$ for every $U \in \mathcal{C}_n$.

(c) The vector form of a well-known result of J. Beck and T. Fiala [6] states that $\beta(B_1^n, Q_n) \leq 2$, where B_1^n is the unit ball of ℓ_1^n and Q_n is the unit cube in $\mathbb{R}^n$.

(d) J. Spencer [11] and E.D. Gluskin [7] have proved independently that

$$\beta(Q_n, Q_n) \leq c\sqrt{n},$$

where $c > 0$ is an absolute constant.

(e) We write B_n for the Euclidean unit ball in $\mathbf{R}^n$. Suppose that $\mathcal{E} \in \mathcal{C}_n$ is an ellipsoid with principal semiaxes $a_1, \ldots, a_n$. W. Banaszczyk [3] proves that $\beta(B_n, \mathcal{E}) = (a_1^{-2} + \ldots + a_n^{-2})^{1/2}$.

A standard reference for many of these results, especially those of them motivated by combinatorial questions, is the book of J. Spencer [13].

J. Komlós conjectures that the sequence $\beta(B_n, Q_n)$ is bounded. The best known result on this problem (J. Spencer, [11]) states that $\beta(B_n, Q_n) = O(\log n)$ as $n \to \infty$.

Closely related to the Komlós conjecture is recent work of W. Banaszczyk and S.J. Szarek [4]. They define and study the quantity

$$\alpha(U,V) = \sup_L \frac{\mu(L,V)}{\lambda_n(L,U)} \tag{1.1}$$

where the supremum is taken over all lattices L in $\mathbb{R}^n$, $\mu(L,V)$ is the covering radius and $\lambda_n(L,U)$ is the n-th successive minimum of L with respect to V and U respectively. If γ_n denotes the standard Gaussian measure on $\mathbb{R}^n$ with density $(2\pi)^{-n/2}e^{-|x|^2/2}$, where $|x|$ is the Euclidean norm of x, the main result in [4] states that

$$\alpha(B_n, V) \leq \theta^{-1}, \tag{1.2}$$

for every closed convex set V in $\mathbb{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$, where θ is an absolute constant defined by the equation $\gamma_1([-\theta/2, \theta/2]) = \frac{1}{2}$. On the other hand, one can see that $\alpha(U,V) \leq \beta(U,V)$ for every $U, V \in \mathcal{C}_n$. This motivates the question if $\alpha(B_n, V)$ may be replaced by $\beta(B_n, V)$ in (1.2). More precisely, it is conjectured in [4] that for some function $f : (0,1) \to \mathbf{R}^+$ and for all $V \in \mathcal{C}_n$ one has $\beta(B_n, V) \leq f(\gamma_n(V))$. This would imply in particular that $\beta(B_n, Q_n) = O(\sqrt{\log n})$ as $n \to \infty$.

The purpose of this note is to discuss upper bounds for $\beta(B_n, V)$ when V is an arbitrary origin symmetric convex body in $\mathbb{R}^n$. In §2 we give a simple proof of the following fact:

Theorem 1. *Let $V \in \mathcal{C}_n$, with $\gamma_n(V) \geq \frac{1}{2}$. Then, $\beta(B_n, V) \leq 6 \log n$.*

Theorem 1 and standard estimates involving Sidák's lemma ([10], see also [7]) allow a reasonable upper bound for $\beta(B_n, V)$ when V is a body with few faces (in the terminology of [1]). As an example, consider the case of the intersection of N strips defined by unit vectors in $\mathbb{R}^n$:

Theorem 2. *Let $V = \{x \in \mathbf{R}^n : |\langle x, z_j\rangle| \leq 1,\ j = 1, \ldots, N\}$, where $z_j \in B_n, j \leq N$, are N vectors spanning $\mathbf{R}^n$. Then,*

$$\beta(B_n, V) \leq 9\ \log n\ \sqrt{\log(3N)}.$$

When $V = Q_n$, the estimate given by Theorem 2 is worse than Spencer's. In §3 we formulate a more precise version of Theorem 1, in which $\beta(B_n, V)$ is bounded in terms of the quantities $\inf_H \gamma_r(V \cap H)$, $r \leq n$, where the inf is taken over all r-dimensional subspaces of $\mathbf{R}^n$. Spencer's $\log n$-theorem is a consequence of this Theorem 3 and of the fact that $\inf_H \gamma_r(Q_n \cap H)$ is roughly "of the order" of 2^{-r}.

2 Proof of Theorems 1 and 2

For the proof of Theorem 1 we shall make use of two well known facts:

(I) If $V \in \mathcal{C}_n$ and $x \in \mathbb{R}^n$, then $\gamma_n(x+V) \geq e^{-|x|^2/2}\gamma_n(V)$. This is a simple consequence of the symmetry of V and the convexity of the exponential function.

(II) Consider the entropy function $F(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)$, on $(0, \frac{1}{2})$. If $\mathcal{A}$ is a subset of $\{-1,1\}^r$ with cardinality $|\mathcal{A}| \geq 2^{rF(\frac{1-\theta}{2})}$, $0 < \theta < 1$, then we can find $\varepsilon', \varepsilon'' \in \mathcal{A}$ with $|\{j \leq r : \varepsilon'_j = \varepsilon''_j\}| \leq \theta r$ (D. Kleitman, [9]).

Lemma 2.1. *Let $V \in \mathcal{C}_n$, $n \geq 7$, with $\gamma_n(V) \geq \frac{1}{2}$, and $u_1, \ldots, u_r \in B_n$, $7 \leq r \leq n$. There exists a subset σ of $\{1, \ldots, r\}$ with $|\sigma| \geq \frac{r}{2}$, and signs $\varepsilon_j \in \{-1,1\}$, $j \in \sigma$, such that*

$$\sum_{j\in\sigma} \varepsilon_j u_j \in 4V.$$

Proof: For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{-1,1\}^r$ we write $L(\varepsilon) = \varepsilon_1 u_1 + \cdots + \varepsilon_r u_r$. By the parallelogram law,

$$\text{Ave}_\varepsilon |L(\varepsilon)|^2 = \sum_{j=1}^r |u_j|^2 \leq r. \tag{2.1}$$

Consider the sets $\frac{L(\varepsilon)}{4} + V$, $\varepsilon \in \{-1,1\}^r$. Using fact (I) and the arithmetic–geometric means inequality, we obtain

$$\int_{\mathbb{R}^n} \left[\sum_\varepsilon \chi_{\frac{L(\varepsilon)}{4}+V}(y)\right] \gamma_n(dy) = \sum_\varepsilon \gamma_n\left(\frac{L(\varepsilon)}{4} + V\right) \tag{2.2}$$

$$\geq \left(\sum_\varepsilon e^{-\frac{|L(\varepsilon)|^2}{32}}\right)\gamma_n(V) \geq 2^r e^{-\frac{\text{Ave}_\varepsilon |L(\varepsilon)|^2}{32}}\gamma_n(V)$$

$$\geq 2^{(1-\frac{1}{32\log 2}-\frac{1}{7})r} \geq 2^{F(\frac{1}{4})r}.$$

It follows that for some subset $\mathcal{A}$ of $\{-1,1\}^r$ with cardinality $|\mathcal{A}| \geq 2^{F(\frac{1}{4})r}$ we must have

$$\bigcap_{\varepsilon\in\mathcal{A}} \left(\frac{L(\varepsilon)}{4} + V\right) \neq \emptyset. \tag{2.3}$$

Using Kleitman's result we find a pair ε', $\varepsilon'' \in \mathcal{A}$ for which:

$$|\{j \leq r : \varepsilon'_j = \varepsilon''_j\}| \leq \frac{r}{2} \quad \text{and} \quad \frac{L(\varepsilon') - L(\varepsilon'')}{4} \in 2V.$$

Setting $\sigma = \{j \leq r : \ \varepsilon'_j \neq \varepsilon''_j\}$ and $\varepsilon_j = \frac{\varepsilon'_j - \varepsilon''_j}{2}$, $j \in \sigma$, we conclude the proof. $\square$

Remark 2.2. Instead of Kleitman's result, in the final step of the proof of Lemma 2.1, one may use the Sauer–Shelah lemma or an even simpler argument based on the computation of the cardinality of a neighborhood of a point in $\{-1,1\}^r$. This would only affect the value of the constant in the statement.

Proof of Theorem 1: We first observe that if $\gamma_n(V) \geq \frac{1}{2}$, and if ρB_n is the largest origin symmetric ball inscribed in V, then V is contained in a symmetric strip of width 2ρ, hence $\gamma_1([-\rho,\rho]) \geq \gamma_n(V) \geq \frac{1}{2}$. It follows that $\rho > \frac{1}{2}$, that is, $\frac{1}{2}B_n \subseteq V$.

Case 1: If $2 \leq n \leq 6$, then $\beta(B_n, V) \leq \beta(B_n, \frac{1}{2}B_n) = 2\sqrt{n} < 6\log n$.

Case 2: If $n \geq 7$, we use an inductive argument. For the first step, set $r_0 = n$ and apply Lemma 2.1 to find $\sigma_1 \subseteq \{1,\ldots,n\}$ with $|\sigma_1| \geq \frac{n}{2}$, and signs $\varepsilon_j \in \{-1,1\}, j \in \sigma_1$, so that $\sum_{j\in\sigma_1} \varepsilon_j u_j \in 4V$. Then, define $\tau_1 = \sigma_1^c$ and $r_1 = |\tau_1| \leq \frac{r_0}{2} = \frac{n}{2}$.

If $\tau_{k-1}, r_{k-1} = |\tau_{k-1}|$ have been defined for some $k \geq 2$ and if $r_{k-1} \geq 7$, then, using Lemma 2.1 again, we find $\sigma_k \subseteq \tau_{k-1}$ with $|\sigma_k| \geq \frac{r_{k-1}}{2}$ and $\varepsilon_j \in \{-1,1\}, j \in \sigma_k$, with $\sum_{j\in\sigma_k} \varepsilon_j u_j \in 4V$. We define $\tau_k = \tau_{k-1}\backslash\sigma_k, r_k = |\tau_k| \leq \frac{r_{k-1}}{2} \leq \frac{n}{2^k}$, and continue in the same way until, for some k_0, $r_{k_0} < 7$. The number of steps needed does not exceed $\log_2(\frac{n}{6}) + 1$: if $r_k > 6$ then $2^k < \frac{n}{6}$. At this point we choose $\varepsilon_j, j \in \tau_{k_0}$, with $\sum_{j\in\tau_{k_0}} \varepsilon_j u_j \in \sqrt{r_{k_0}}\, B_n \subseteq 2\sqrt{6}\, V$.
The choice of signs $\bigcup_{k=1}^{k_0}\{\varepsilon_j : j \in \sigma_k\} \cup \{\varepsilon_j : j \in \tau_{k_0}\}$ satisfies

$$\sum_{j=1}^{n} \varepsilon_j u_j \in \left[4\left(\log_2(\frac{n}{6}) + 1\right) + 2\sqrt{6}\right] V \subseteq 6\ \log n\ V, \tag{2.4}$$

and this completes the proof. $\square$

Remark 2.3. Theorem 1 and a standard argument (see [11]) show that if $V \in \mathcal{C}_n$ with $\gamma_n(V) \geq \frac{1}{2}$, and if $u_1,\ldots,u_m \in B_n, m \geq n$, then there exist signs $\varepsilon_j \in \{-1,1\}$ such that $\varepsilon_1 u_1 + \ldots + \varepsilon_m u_m \in 12\log n\ V$.

There is nothing special about assuming that $\gamma_n(V) \geq \frac{1}{2}$. If $\gamma_n(V) = \alpha \in (0,1)$, one can find $c(\alpha)$ such that $\beta(B_n, V) \leq c(\alpha)\log n$. Moreover, the symmetry and the compactness of V are also not so important: if, say, V is any closed convex set in $\mathbb{R}^n$ with $\gamma_n(V) \geq \frac{3}{4}$, then

$$\beta(B_n, V) \leq \beta(B_n, V \cap (-V)) \leq c\log n. \tag{2.5}$$

Remark 2.4. It is easy to see that if the vectors $u_1,\ldots,u_n \in B_n$ are orthogonal and if V is a centered cube in $\mathbb{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$, then there exist signs ε_j for which $\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n \in cV$. A careful examination of the proof of (1.2) in [4] shows that the same is true for an arbitrary closed convex set V in $\mathbb{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$.

Remark 2.5. Let V be a symmetric convex set in $\mathbb{R}^n, n \geq 2$. If $x \in \mathbb{R}$, we write

$$V_x = \{(x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1} : (x_1,\ldots,x_{n-1},x) \in V\}.$$

From the log-concavity of γ_{n-1} and the symmetry of V, one easily deduces that $h(x) = \gamma_{n-1}(V_x)$ is an even log-concave function on $\{x : \gamma_{n-1}(V_x) > 0\}$, hence it attains its maximum value at 0. It follows that

$$\gamma_n(V) = \int_{\mathbb{R}} h(x)\gamma_1(dx) \leq \gamma_{n-1}(V \cap e_n^{\perp}), \tag{2.6}$$

where $e_n^{\perp} = \{x \in \mathbb{R}^n : x_n = 0\}$. Induction and the rotational invariance of the Gaussian measure show that whenever H, H_1 are subspaces of $\mathbb{R}^n$ with $H \subseteq H_1$, then $\gamma_{H_1}(V \cap H_1) \leq \gamma_H(V \cap H)$ (by γ_H we denote $\gamma_{\dim H}$ on H).

Another useful remark is that the set $I_s(h) = \{x \in \mathbb{R} : h(x) \geq s\}$ is a symmetric interval in $\mathbb{R}$ (possibly the empty set) for every $s > 0$.

Using these observations one can give a proof of Sidák's lemma starting from the following lemma:

Lemma 2.6. *Let K be a symmetric convex set in $\mathbb{R}^n$, $n \geq 1$, and $V_z = \{x \in \mathbb{R}^n : |\langle x, z\rangle| \leq 1\}, z \in \mathbb{R}^n$. Then*

$$\gamma_n(K \cap V_z) \geq \gamma_n(K)\ \gamma_n(V_z).$$

Proof: If $n = 1$ the inequality is trivially true since $K \cap V_z$ is either K or V_z, and γ_1 is a probability measure. Let $n \geq 2$. By the rotational invariance of γ_n we may assume that $z = \frac{1}{\lambda}e_n, \lambda > 0$. Then,

$$\begin{aligned} \gamma_n(K \cap V_z) &= \int_{-\lambda}^{\lambda} h(x)\gamma_1(dx) = \int_0^{\infty} \gamma_1(I_s(h) \cap [-\lambda, \lambda])ds \\ &\geq \left(\int_0^{\infty} \gamma_1(I_s(h))ds\right)\ \gamma_1([-\lambda, \lambda]) = \gamma_n(K)\ \gamma_n(V_z). \quad \Box \end{aligned} \tag{2.7}$$

Let $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2}dt,\ x \geq 0$. A simple inductive argument based on Lemma 2.6 provides a proof of

Sidák's Lemma: Let $z_j \in \mathbb{R}^n, j \leq N$, and $V_j = \{x : |\langle x, z_j\rangle| \leq 1\}$. Then,

$$\gamma_n(\bigcap_{j \leq N} V_j) \geq \prod_{j=1}^{N} \gamma_n(V_j) = \prod_{j=1}^{N} \Phi(1/|z_j|). \quad \Box \tag{2.8}$$

We include this proof of Lemma 2.6 because of its simplicity. The possibility of deducing Sidák's lemma in such an easy way became known to several people more or less at the same time (see e.g [14] where a much harder non-symmetric version of Lemma 2.6 is proved). The connection of Sidák's lemma with $\beta(B_n, V)$ is clear: given any $V \in \mathcal{C}_n$, we solve the equation $\gamma_n(\mu V) = \frac{1}{2}$ in μ, and then apply Theorem 1 to obtain $\beta(B_n, V) \leq c\mu \log n$. If V is an intersection of strips, we can easily find an upper bound for μ using (2.8). As an example, let us see what happens if all the strips have width 2:

Proof of Theorem 2: Let $\mu \geq 1$. Using the standard estimate

$$\Phi(x) \geq 1 - e^{-x^2/2} \geq \exp(-2e^{-x^2/2}) \qquad (x \geq 1),$$

we obtain

$$\gamma_n(\mu V) \geq [\Phi(\mu)]^N \geq \exp(-2Ne^{-\mu^2/2}).$$

It is clear that choosing $\mu = \sqrt{2\log\left(\frac{2N}{\log 2}\right)}$, we get $\gamma_n(\mu V) \geq \frac{1}{2}$. Therefore, Theorem 1 implies that

$$\beta(B_n, V) = \mu\beta(B_n, \mu V) \leq 6\sqrt{2}\log n\ \sqrt{\log(3N)}. \quad \square$$

Finally, let us mention one slightly more delicate application of the method:

Proposition 2.7. *Let $z_j \in \mathbb{R}^n$, $j \leq N$, with $|z_j| \leq 1/\sqrt{\log(j+1)}$. If $V = \{x : |\langle x, z_j\rangle| \leq 1,\ j \leq N\}$, then*

$$\beta(B_n, V) \leq c\log n,$$

where $c > 0$ is an absolute constant. $\square$

Note that the statement is independent of N.

3 A general upper bound for $\beta(B_n, V)$

When V is a parallelepiped in $\mathbb{R}^n$ which contains B_n, the estimate given by Theorem 2 is: $\beta(B_n, V) = O\left((\log n)^{3/2}\right)$ as $n \to \infty$. Spencer's result for the cube can be recovered by a more precise version of Theorem 1 which we now describe:

Definition: If $V \in \mathcal{C}_n, c \in (0,1)$, and $r \in \{1, \ldots, n\}$, we define

$$\varphi(V, c, r) = \min\{\rho > 0 : \inf_H \gamma_r(\rho V \cap H) \geq 2^{-cr}\},$$

where the inf is over all r-dimensional subspaces of $\mathbb{R}^n$. Note that $\varphi(V, c, r)$ is well defined since, for every H, $\gamma_r(\rho V \cap H) \geq \gamma_n(\rho V)$.

Our way to estimate $\beta(B_n, V)$ depends on an iteration scheme (similar to the one in the proof of Theorem 1), based on the following Lemma:

Lemma 3.1. *Let $V \in \mathcal{C}_n, r \leq n$, and $u_1, \ldots, u_r \in B_n$. We can find a subset σ of $\{1, \ldots, r\}$ with $|\sigma| \geq \frac{r}{2}$, and signs $\varepsilon_j \in \{-1, 1\}, j \in \sigma$, for which*

$$\sum_{j\in\sigma} \varepsilon_j u_j \in 4\,\varphi(V, \frac{1}{7}, r)V.$$

Proof: We may clearly assume that $u_1,\ldots,u_r$ are linearly independent. Consider the subspace $H=\mathrm{span}\{u_1,\ldots,u_r\}$ and set $\rho=\varphi(V,\frac{1}{7},r), L(\varepsilon)=\varepsilon_1u_1+\ldots+\varepsilon_ru_r,\ \varepsilon\in\{-1,1\}^r$. We estimate

$$\sum_{\varepsilon}\gamma_H\left(\frac{L(\varepsilon)}{4}+(\rho V\cap H)\right)\geq 2^{F(\frac{1}{4})r}, \tag{3.1}$$

and, exactly as in Lemma 2.1, we find $\sigma\subseteq\{1,\ldots,r\}, |\sigma|\geq\frac{r}{2}$, and a sequence of signs $\varepsilon_j\in\{-1,1\},\ j\in\sigma$, with

$$\sum_{j\in\sigma}\varepsilon_ju_j\in 4\rho V\cap H. \quad \square \tag{3.2}$$

Theorem 3. *Let $V\in\mathcal{C}_n$, and $\psi_l=\left[\frac{n}{2^l}\right],\ l=0,1,\ldots$ Then,*

$$\beta(B_n,V)\leq 8\sum_{l=0}^{[\log_2 n]}\varphi(V,\frac{1}{21},\psi_l).$$

Proof: Suppose that $u_1,\ldots,u_n\in B_n$. We set $\tau_0=\{1,\ldots,n\},r_0=n,\sigma_0=\emptyset$, and following the proof of Theorem 1 (with Lemma 3.1 playing now the role of Lemma 2.1), for $k\geq 1$ we choose σ_k,τ_k,r_k:

(i) $\sigma_k\subseteq\tau_{k-1},|\sigma_k|\geq\frac{|\tau_{k-1}|}{2}$, and there exist $\varepsilon_j\in\{-1,1\},j\in\sigma_k$, with

$$\sum_{j\in\sigma_k}\varepsilon_ju_j\in 4\ \varphi(V,\frac{1}{7},r_{k-1})V$$

(ii) $\tau_k=\tau_{k-1}\backslash\sigma_k, r_k=|\tau_k|$.

This procedure exhausts $\{1,\ldots,n\}$ in a finite number of steps: for some $m\leq[\log_2 n]+1$, we will have $r_m=0$.

Each $r_k,\ k=0,\ldots,m-1$, lies in an interval of the form $(\psi_{l+1},\psi_l]\,,\ l=0,\ldots,[\log_2 n]$, and at most two of them are in the same interval. If $\psi_{l+1}<r_k\leq\psi_l$, then it is easy to see that $\varphi(V,\frac{1}{7},r_k)\leq\varphi(V,\frac{1}{21},\psi_l)$: notice that if $\dim H=r_k$ and H_1 is any ψ_l-dimensional subspace of $\mathbb{R}^n$ with $H\subseteq H_1$, then $\gamma_{H_1}(\rho V\cap H_1)\geq 2^{-\frac{\psi_l}{21}}$ implies that $\gamma_H(\rho V\cap H)\geq\gamma_{H_1}(\rho V\cap H_1)\geq 2^{-\frac{\psi_l/3}{7}}\geq 2^{-\frac{r_k}{7}}$. It follows that

$$\sum_{k=0}^{m}\varphi(V,\frac{1}{7},r_k)\leq 2\sum_{l=0}^{[\log_2 n]}\varphi(V,\frac{1}{21},\psi_l). \tag{3.3}$$

Therefore, the sequence of signs ε_j chosen in our m steps satisfies

$$\sum_{j=1}^{n}\varepsilon_ju_j\in\left[8\sum_{l=0}^{[\log_2 n]}\varphi(V,\frac{1}{21},\psi_l)\right]V. \quad \square$$

We shall apply Theorem 3 in the case where $V = Q_n$:

Lemma 3.2. *For some absolute constant $c > 0$, and for every $r \leq n$, one has $\varphi(Q_n, \frac{1}{21}, r) \leq c$.*

Proof: Let H be an r-dimensional subspace of $\mathbb{R}^n$. Let also $\{w_1, \ldots, w_r\}$ be an orthonormal basis of H, and W be the $n \times r$ matrix with columns $w_j, j \leq r$. Then, for every $c > 0$,

$$\gamma_r(cQ_n \cap H) = \gamma_r\left(\{x \in \mathbb{R}^r : |\langle x, W^* e_i\rangle| \leq c, i = 1, \ldots, n\}\right),$$

where $\{e_i\}_{i \leq n}$ is the standard orthonormal basis of $\mathbb{R}^n$.

Claim. If $t_1, \ldots, t_n > 0$, then

$$\prod_{i=1}^{n} \Phi\left(\frac{1}{t_i}\right) \geq 2^{-\delta \sum_{i=1}^{n} t_i^2}, \tag{3.4}$$

where $\delta > 0$ is an absolute constant.

[*Proof of the claim:* We may assume that $t_1 \leq \ldots \leq t_s \leq 1 < t_{s+1} \leq \ldots \leq t_n$. We set $S = \sum_{i=1}^{n} t_i^2$, and $A_j = \{i \leq n : 2^{j-1} < t_i \leq 2^j\}, j = 1, 2, \ldots$ Note that $|A_j| \leq \frac{S}{2^{2j-2}}$.

We have the estimates:

$$\prod_{i=1}^{s} \Phi\left(\frac{1}{t_i}\right) \geq \exp\left(-2\sum_{i=1}^{s} e^{-\frac{1}{2t_i^2}}\right) \geq \exp\left(-4\sum_{i=1}^{s} t_i^2\right) \geq e^{-4S}. \tag{3.5}$$

$$\prod_{i \in A_j} \Phi\left(\frac{1}{t_i}\right) \geq \left(\sqrt{2/\pi e}\,\frac{1}{2^j}\right)^{|A_j|} \geq \left[\left(\sqrt{2/\pi e}\right)^{\frac{1}{2^{2j-2}}}\; 2^{-\frac{j}{2^{2j-2}}}\right]^S \tag{3.6}$$

and hence

$$\prod_{j} \prod_{i \in A_j} \Phi\left(\frac{1}{t_i}\right) \geq \left[\left(\sqrt{\pi e/2}\right)^{-\sum_{j=1}^{\infty} \frac{1}{2^{2j-2}}}\; 2^{-\sum_{j=1}^{\infty} \frac{j}{2^{2j-2}}}\right]^S. \tag{3.7}$$

From (3.5) and (3.7) it follows that

$$\prod_{i=1}^{n} \Phi\left(\frac{1}{t_i}\right) = \prod_{i=1}^{s} \Phi\left(\frac{1}{t_i}\right) \prod_{j} \prod_{i \in A_j} \Phi\left(\frac{1}{t_i}\right) \geq 2^{-\delta S},$$

for some absolute constant $\delta > 0$.]

By Sidák's lemma we have $\gamma_r(cQ_n \cap H) \geq \prod_{i=1}^{n} \Phi\left(\frac{c}{|W^* e_i|}\right)$, and since $\sum_{i=1}^{n} |W^* e_i|^2 = \sum_{i=1}^{r} |w_j|^2 = r$, our claim provides the inequality

$$\gamma_r(cQ_n \cap H) \geq 2^{-\delta \frac{r}{c^2}} \geq 2^{-\frac{r}{21}} \tag{3.8}$$

if $c = c(\delta) > 0$ has been chosen large enough (independent of n and r). $\square$

As a consequence of Theorem 3 and Lemma 3.2 one has Spencer's estimate on the Komlós conjecture:

Corollary 3.3. $\beta(B_n, Q_n) = O(\log n)$ as $n \to \infty$.

Remark 3.4. J. Spencer [11] and E.D. Gluskin [7] have proved that $\beta(Q_n, Q_n) = O(\sqrt{n})$ as $n \to \infty$, which is clearly optimal. The basic step towards this theorem is to prove the following:

Claim. If $u_1, \ldots, u_r \in Q_n, r \leq n$, then there exist a subset σ of $\{1, \ldots, r\}$ with cardinality $|\sigma| \geq \theta r$ and a choice of signs $\varepsilon_j, j \in \sigma$, such that

$$\sum_{j \in \sigma} \varepsilon_j u_j \in c\sqrt{r}\sqrt{\log(2n/r)}\, Q_n, \tag{3.9}$$

where $\theta \in (0,1)$ and $c > 0$ are absolute constants.

A modification of the proof of Lemma 2.1 gives a simple proof of this fact: define $K = \{x \in \mathbb{R}^r : |\langle x, W^* e_i\rangle| \leq 1,\ i \leq n\}$ where W is the $n \times r$ matrix with columns $u_j, j \leq r$. Note that $|W^* e_i| \leq \sqrt{r}, i = 1, \ldots, r$. Choosing an absolute constant $c > 0$ large enough and using Sidák's lemma one has the inequality

$$\sum_{\varepsilon} \gamma_r(\varepsilon + c\sqrt{r}\sqrt{\log(2n/r)}\, K) \geq 2^{F(\frac{\theta}{2})r},$$

where $\theta = \theta(c) \in (0,1)$ is some other absolute constant. The rest is as in Lemma 2.1: we find $\varepsilon \in \{-1, 0, 1\}^r$ with $|\{j : \varepsilon_j \neq 0\}| \geq \theta r$, and $\varepsilon \in c\sqrt{r}\sqrt{\log(2n/r)}\, K$.

This is equivalent to the claim, and an inductive argument analogous to the one in [7], [11] leads to the Spencer–Gluskin theorem. In this case, our method may be viewed as a (simplified) variation of Gluskin's method where Sidák's lemma was used for volume estimates and then combined with Minkowski's theorem from the geometry of numbers.

Another modification of Lemma 2.1, now combined with the binary blocks decomposition used by B.S. Kashin in [8], can give the following stronger result of J. Spencer [12]:

"If $u_1, \ldots, u_n \in Q_n$, then there exist signs $\varepsilon_j \in \{-1, 1\}$ for which

$$\max_{t \leq n} \|\sum_{j=1}^{t} \varepsilon_j u_j\|_\infty \leq c\sqrt{n}$$

where $c > 0$ is an absolute constant."

Acknowledgement: The author gratefully acknowledges the hospitality of the Institute for Advanced Study where this work was completed.

References

[1] K. Ball and A. Pajor, *Convex bodies with few faces*, Proc. Amer. Math. Soc. **110** (1990), 225-231.

[2] W. Banaszczyk, *Balancing vectors and convex bodies*, Studia Math. **106** (1993), 93-100.

[3] W. Banaszczyk, *A Beck–Fiala-type theorem for Euclidean norms*, Europ. J. Combin. **11** (1990), 497-500.

[4] W. Banaszczyk and S.J. Szarek, *Lattice coverings and Gaussian measures of n-dimensional convex bodies*, Discrete and Computational Geometry (to appear).

[5] I. Bárány and V.S. Grinberg, *On some combinatorial questions in finite dimensional spaces*, Linear Algebra Appl. **41** (1981), 1-9.

[6] J. Beck and T. Fiala, *Integer-making theorems*, Discrete Appl. Math. **3** (1981), 1-8.

[7] E.D. Gluskin, *Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces*, Math. USSR Sbornik **64** (1989), 85-96.

[8] B.S. Kashin, *Analogue of Menchoff's theorem "on correction" for discrete orthonormal systems*, Mat. Zametki **46**(6) (1989), 67-74.

[9] D. Kleitman, *On a combinatorial conjecture of Erdös*, J. Combin. Theory Ser.A **1** (1966), 209-214.

[10] Z. Sidák, *On multivariate normal probabilities of rectangles: their dependence on correlation*, Ann. Math. Statist. **39** (1968), 1425-1434.

[11] J. Spencer, *Six standard deviations suffice*, Trans. Amer. Math. Soc. **289** (1985), 679-706.

[12] J. Spencer, *Balancing vectors in the max norm*, Combinatorica **6**(1) (1986), 55-65.

[13] J. Spencer, *Ten Lectures on the Probabilistic Method*, Society for Industrial and Applied Mathematics, Philadelphia, Penn. 1994.

[14] S.J. Szarek and E. Werner, *Confidence regions for means of multivariate normal distributions and a non-symmetric correlation inequality for Gaussian measure*, Preprint.

Department of Mathematics, University of Crete, Iraklion, Crete, Greece
E-mail: deligia@talos.cc.uch.gr