Moments of the Cramér transform of log-concave probability measures

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Abstract

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ and let Λ_μ^* denote the Cramér transform of μ, i.e. $\Lambda_\mu^*(x) = \sup\{\langle x, \xi\rangle - \Lambda_\mu(\xi) : \xi \in \mathbb{R}^n\}$ where Λ_μ is the logarithmic Laplace transform of μ. We show that $\mathbb{E}_\mu \left[\exp\left(\frac{c_1}{n}\Lambda_\mu^*\right)\right] < \infty$ where $c_1 > 0$ is an absolute constant. In, particular, Λ_μ^* has finite moments of all orders. The proof, which is based on the comparison of certain families of convex bodies associated with μ, implies that $\|\Lambda_\mu^*\|_{L^2(\mu)} \leqslant c_2 n \ln n$. The example of the uniform measure on the Euclidean ball shows that this estimate is optimal with respect to n as the dimension n grows to infinity.

1 Introduction

In this work we study a question that was left open in [7] and concerns the existence of moments of the Cramér transform of log-concave probability measures. Let μ be a log-concave probability measure on $\mathbb{R}^n$. The log-Laplace transform of μ is defined by

$$\Lambda_\mu(\xi) = \ln\left(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} d\mu(z)\right).$$

Note that $\Lambda_\mu(0) = 0$ and that Λ_μ is convex by Hölder's inequality. If we also assume that the barycenter $\mathrm{bar}(\mu)$ of μ is at the origin then from Jensen's inequality we see that $\Lambda_\mu(\xi) \geqslant 0$ for all ξ. One can also check that the set $A(\mu) = \{\Lambda_\mu < \infty\}$ is open and Λ_μ is C^∞ and strictly convex on $A(\mu)$. The Cramér transform Λ_μ^* of μ is the Legendre transform of Λ_μ, defined by

$$\Lambda_\mu^*(x) = \sup_{\xi\in\mathbb{R}^n} \left\{\langle x, \xi\rangle - \Lambda_\mu(\xi)\right\},$$

and plays a key role in the theory of large deviations (see [11]).

Brazitikos, Pafis and the first named author develop in [7] an approach to the "threshold problem" for the expected measure of random polytopes whose vertices have an arbitrary log-concave distribution. To make this question precise, consider the random polytope $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$, where $N > n$ and $X_1, X_2, \ldots$ are independent random vectors distributed according to μ. Given $\delta \in \left(0, \frac{1}{2}\right)$, let

$$\varrho_1(\mu, \delta) = \sup\{r > 0 : \mathbb{E}_{\mu^N}[\mu(K_N)] \leqslant \delta \text{ for all } N \leqslant e^r\}$$

and

$$\varrho_2(\mu, \delta) = \inf\{r > 0 : \mathbb{E}_{\mu^N}[\mu(K_N)] \geqslant 1 - \delta \text{ for all } N \geqslant e^r\},$$

where μ^N is the product measure $\mu \times \cdots \times \mu$ (N times). Let also $\varrho(\mu, \delta) = \varrho_2(\mu, \delta) - \varrho_1(\mu, \delta)$. We say that μ exhibits a threshold around $\tau > 0$ for the expected measure $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of K_N if $\varrho_1(\mu, \delta) \leqslant \tau \leqslant \varrho_2(\mu, \delta)$ for all $\delta \in \left(0, \frac{1}{2}\right)$ and the "window" $\varrho(\mu, \delta)$ is "small" when compared with τ. The approach of [7] establishes, under some conditions, a sharp threshold with $\tau = \mathbb{E}_\mu(\Lambda_\mu^*)$. One should of course show that $\mathbb{E}_\mu(\Lambda_\mu^*)$ is finite, and in order to obtain a good bound for the window of the threshold it is necessary to prove that the second moment of Λ_μ^* is finite and in fact that the parameter $\beta(\mu) = \mathrm{Var}_\mu(\Lambda_\mu^*)/(\mathbb{E}_\mu(\Lambda_\mu^*))^2$ is small as the dimension n increases to infinity, ideally that $\beta(\mu) = o_n(1)$ independently from μ. More precisely, combining the results of [7] with subsequent estimates from [5] we know that if $\beta(\mu) \leqslant c_0$ then

$$\frac{\rho(\mu,\delta)}{\tau} \leqslant c_1 \sqrt{\beta(\mu)/\delta}$$

for every log-concave probability measure μ on $\mathbb{R}^n$ and any $0 < \delta < 1/2$, where $\tau = \mathbb{E}_\mu(\Lambda_\mu^*)$ and $c_0, c_1 > 0$ are absolute constants.

It was proved in [7] that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ and μ is a centered κ-concave probability measure, where $\kappa \in (0, 1/n]$, with $\mathrm{supp}(\mu) = K$ then $\mathbb{E}_\mu\big[\exp(\kappa\Lambda_\mu^*(x)/2)\big] < \infty$. In particular, this is true for the Lebesgue measure μ_K on K (with $\kappa = 1/n$) and implies that $\Lambda_{\mu_K}^*$ has finite moments of all orders. We obtain a similar integrability result in the broader setting of log-concave probability measures.

Theorem 1.1. *For every centered log-concave probability measure μ on $\mathbb{R}^n$ we have that*

$$\int_{\mathbb{R}^n} \exp\left(\frac{c}{n}\Lambda_\mu^*(x)\right) d\mu(x) < \infty$$

where $c > 0$ is an absolute constant. In particular, Λ_μ^* has finite moments of all orders.

Theorem 1.1 shows that $\|\Lambda_\mu^*\|_{L^2(\mu)} < +\infty$, and hence both $\mathbb{E}_\mu(\Lambda_\mu^*)$ and $\beta(\mu)$ are finite for any centered log-concave probability measure μ on $\mathbb{R}^n$. In fact, we can give the following upper bound for $\|\Lambda_\mu^*\|_{L^2(\mu)}$, which is optimal as one can check from the example of the uniform measure on the Euclidean ball.

Theorem 1.2. *For every centered log-concave probability measure μ on $\mathbb{R}^n$ we have that*

$$\|\Lambda_\mu^*\|_{L^2(\mu)} \leqslant cn \ln n$$

where $c > 0$ is an absolute constant.

The proof of Theorem 1.1 and Theorem 1.2 is presented in Section 3. It is based on a number of intermediate results regarding certain families of convex bodies that are naturally associated with a given centered log-concave probability measure μ on $\mathbb{R}^n$. First of all, it is natural to consider the family $\{B_t(\mu)\}_{t>0}$ of level sets of Λ_μ^*. For any $t > 0$ we define $B_t(\mu) = \{x \in \mathbb{R}^n : \Lambda_\mu^*(x) \leqslant t\}$. Then, the starting point for the proof of, say, Theorem 1.2 is the identity

$$\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^2 d\mu(x) = \int_0^\infty 2t\, \mu(\{x : \Lambda_\mu^*(x) > t\})\, dt = \int_0^\infty 2t\,(1 - \mu(B_t(\mu)))\, dt.$$

Therefore, we would like to know how fast $\mu(B_t)$ tends to 1 as $t \to \infty$. We consider a second family of convex bodies associated with μ. For any $t > 0$ we define $R_t(\mu) = \{x \in \mathbb{R}^n : f_\mu(x) \geqslant e^{-t} f_\mu(0)\}$, where f_μ is the density of μ. An estimate for the measure of $R_t(\mu)$ is essentially contained in Klartag's [19, Lemma 5.2]. His arguments lead to the estimate

$$\mu(R_t(\mu)) \geqslant 1 - e^{-t/4}$$

for t large enough with respect to n (see Proposition 2.1). We compare $R_t(\mu)$ and $B_t(\mu)$ in Lemma 3.5 where we show that for any $\delta \in (0, 1)$ and $t \geqslant t_n := n \ln n$ we have that

$$(1 - \delta) R_t(\mu) \subseteq B_{g(t,\delta)}(\mu) \tag{1.1}$$

where $g(t,\delta) \leqslant 2t + n\ln(1/\delta)$. We combine the above with the useful observation (see Lemma 3.1) that

$$\mu((1+\delta)A) \leqslant e^{2n\delta}\mu(A)$$

for any $\delta > 0$ and any Borel subset A of $\mathbb{R}^n$. Choosing $\delta = t^{-4}$ we get that

$$\mu(B_{10t}(\mu)) \geqslant 1 - 1/t^3$$

for all $t \geqslant t_n$, and Theorem 1.2 follows. For the proof of Theorem 1.1 we apply a similar reasoning, this time using (1.1) with $\delta = \exp(-ct)$.

Besides the families $\{B_t(\mu)\}_{t>0}$ and $\{R_t(\mu)\}_{t>0}$ we discuss and compare a number of other interesting families of convex bodies associated with μ: the family $\{K_t(\mu)\}_{t>0}$ of K. Ball's bodies, the family $\{Z_t^+(\mu)\}_{t>0}$ of one-sided L_t-centroid bodies of μ and the family $\{T_t(\mu)\}_{t>0}$ of "floating bodies" of μ, defined by $T_t(\mu) = \{x \in \mathbb{R}^n : \varphi_\mu(x) \geqslant e^{-t}\}$, where $\varphi_\mu(x) = \inf\{\mu(H) : H \text{ is a closed half-space containing } x\}$ is Tukey's half-space depth function of μ. This is done in Section 2, where we introduce all these bodies and obtain inclusion relations between them. As far as we know, sharp inclusions between various pairs of these families and sharp estimates on the rate at which the measure of these bodies tends to 1 when $t \to \infty$ do not appear in the literature, and we feel that a systematic and complete list of such results would be useful in other situations too.

We illustrate this point of view in Section 4 where we provide further applications of our approach. Our first result concerns the question to obtain uniform upper and lower thresholds for the expected measure of a random polytope defined as the convex hull of independent random points with a log-concave distribution. The general formulation of the problem is the following. Given a log-concave probability measure μ on $\mathbb{R}^n$ and the random polytope $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ as above, our aim is to find a constant $N_1(n)$, depending only on n and as large as possible, so that

$$\sup_{\mu} \Big(\sup \Big\{ \mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant N_1(n) \Big\} \Big) \longrightarrow 0$$

as $n \to \infty$ and a second constant $N_2(n)$, depending only on n and as small as possible, so that

$$\inf_{\mu} \Big(\inf \Big\{ \mathbb{E}_{\mu^N}[\mu(K_N)] : N \geqslant N_2(n) \Big\} \Big) \longrightarrow 1$$

as $n \to \infty$, where the supremum and the infimum are over all log-concave probability measures. We shall call the first type of result a "uniform upper threshold" and the second type a "uniform lower threshold". Results of this type were obtained by Chakraborti, Tkocz and Vritsiou in [9] for some general families of distributions and by Brazitikos, Pafis and the first named author in [6] for the class of all log-concave probability measures. An asymptotically optimal uniform upper threshold has been established with $N_1(n) = \exp(cn)$, but the available uniform lower threshold $N_2(n) = \exp(C(n\ln n)^2 u(n))$ where $u(n)$ is any function with $u(n) \to \infty$ as $n \to \infty$ is not optimal (see Section 4 for a more detailed description of earlier works on this topic). Here we prove the next theorem.

Theorem 1.3. *There exists an absolute constant $C > 0$ such that*

$$\inf_{\mu} \Big(\inf \Big\{ \mathbb{E}_{\mu^N}\big[\mu(K_N)\big] : N \geqslant \exp(Cn\ln n) \Big\} \Big) \longrightarrow 1$$

as $n \to \infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$.

The uniform lower threshold of Theorem 1.3 is of the right order. An exponential in the dimension lower threshold is not possible in full generality; actually, in the case where X_i are uniformly distributed in the

Euclidean ball one can check that $N \geqslant \exp(cn \ln n)$ points are needed so that the volume of a random K_N will be significantly large.

The second result of Section 4 concerns the distribution of the half-space depth function φ_μ. Our starting point is the estimate

$$\exp(-c_1 n) \leqslant \mathbb{E}_\mu(\varphi_\mu) \leqslant \exp(-c_2 n)$$

from [6] which holds true for every log-concave probability measure μ on $\mathbb{R}^n$. We consider negative moments of φ_μ and ask for the values of $p > 0$ for which $\mathbb{E}_\mu(\varphi_\mu^{-p}) < \infty$. Brazitikos and Chasapis have shown in [5] that in the 1-dimensional case one has $\mathbb{E}(\varphi_\mu^{-p}) \leqslant 2^p/(1-p)$ for all $0 < p < 1$ and any probability measure μ on $\mathbb{R}$. On the other hand, simple examples show that $\mathbb{E}_\mu(\varphi_\mu^{-p})$ may be infinite when $p \geqslant 1$. We obtain the following general result on the range of values of $p > 0$ for which $\mathbb{E}_\mu(\varphi_\mu^{-p})$ is finite.

Theorem 1.4. *There exists an absolute constant $c > 0$ such that*

$$J_\mu(p) := \int_{\mathbb{R}^n} \frac{1}{\varphi_\mu^p(x)}\, d\mu(x) < \infty$$

for every log-concave probability measure μ on $\mathbb{R}^n$ and any $0 < p \leqslant c/n$.

Finally, we discuss the connection of the integrability properties of Λ_μ^* with the notion of affine surface area. In the language of this article, Schütt and Werner [25] have proved that for every convex body K of volume 1 in $\mathbb{R}^n$ one has that

$$\lim_{s\to\infty} e^{\frac{2s}{n+1}}(1 - \mu_K(T_s(\mu_K))) = \frac{1}{2}\left(\frac{n+1}{\omega_{n-1}}\right)^{\frac{2}{n+1}} \mathrm{as}(K)$$

where $\mathrm{as}(K)$ is the affine surface area of K and ω_n is the volume of the Euclidean unit ball B_2^n (see Subsection 4.3 for a discussion). In particular, $\sup\{e^{\frac{2s}{n+1}}(1 - \mu_K(T_s(\mu_K))) : s > 0\} < \infty$. We provide an analogue of this fact in the more general setting of log-concave probability measures.

Theorem 1.5. *There exists an absolute constant $c > 0$ such that*

$$\sup\{e^{cs/n}(1 - \mu(T_s(\mu))) : s > 0\} \leqslant c_n$$

for every log-concave probability measure μ on $\mathbb{R}^n$, where $c_n > 0$ is a constant depending only on n.

2 Families of convex bodies associated with log-concave measures

We work in $\mathbb{R}^n$ and use standard notation: $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^n$ and the Euclidean norm is denoted by $|\cdot|$. We write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Lebesgue measure in $\mathbb{R}^n$ is denoted by $|\cdot|$ and σ is the rotationally invariant probability measure on S^{n-1}. The volume $\omega_n := |B_2^n|$ of the Euclidean unit ball is equal to $\omega_n = \pi^{n/2}/\Gamma\left(\frac{n}{2}+1\right) = (c_n/n)^{n/2}$ for some constant $c_n \approx 1$. Whenever we write $a \approx b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. We use the letters c, c_1, c_2 etc. to denote absolute positive constants whose value may change from line to line.

We say that $K \subset \mathbb{R}^n$ is a convex body if it is compact, convex and has non-empty interior. We often consider bounded convex sets K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$; since the closure of such a set is a convex body, we shall call these sets convex bodies too. We say that K is centrally symmetric if $x \in K$ implies that $-x \in K$ and that K is centered if the barycenter $\mathrm{bar}(K) = \frac{1}{|K|}\int_K x\,dx$ of K is at the origin. The radial function ϱ_K of a convex body K with $0 \in \mathrm{int}(K)$ is the function $\varrho_K(x) = \sup\{\lambda > 0 : \lambda x \in K\}$ defined for all $x \neq 0$, and the support function of K is the function $h_K(x) = \sup\{\langle x, y\rangle : y \in K\}$, $x \in \mathbb{R}^n$. The polar body K° of a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ is the convex body

$$K^\circ := \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\big\}.$$

We say that a Borel measure μ on $\mathbb{R}^n$ is log-concave if $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$ and $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any pair of compact sets A, B in $\mathbb{R}^n$ and any $\lambda \in (0,1)$. Borell [3] has proved that, under these assumptions, μ has a log-concave density f_μ. Recall that a function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\ln f$ to it is concave. The Brunn-Minkowski inequality implies that if K is a convex body in $\mathbb{R}^n$ then the indicator function $\mathbb{1}_K$ of K is the density of a log-concave measure, the Lebesgue measure on K.

We say that μ is even if $\mu(-B) = \mu(B)$ for every Borel subset B of $\mathbb{R}^n$ and that μ is centered if

$$\mathrm{bar}(\mu) := \int_{\mathbb{R}^n} \langle x, \xi\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \xi\rangle f_\mu(x) dx = 0$$

for all $\xi \in S^{n-1}$. We shall use the following result of Fradelizi from [13]: if μ is a centered log-concave probability measure on $\mathbb{R}^n$ then

$$\|f_\mu\|_\infty \leqslant e^n f_\mu(0). \tag{2.1}$$

For any log-concave measure μ on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx}.$$

A log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if it is centered and $\mathrm{Cov}(\mu) = I_n$, where I_n is the identity $n \times n$ matrix. Note that if μ is isotropic then $L_\mu = \|f_\mu\|_\infty^{1/n}$.

Let μ and ν be two log-concave probability measures on $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function which is defined μ-almost everywhere and satisfies

$$\nu(B) = \mu(T^{-1}(B))$$

for every Borel subset B of $\mathbb{R}^n$. We then say that T pushes forward μ to ν and write $T_*\mu = \nu$. It is easy to see that $T_*\mu = \nu$ if and only if for every bounded Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(x) d\nu(x) = \int_{\mathbb{R}^n} g(T(y)) d\mu(y). \tag{2.2}$$

It is not hard to check that for every log-concave probability measure μ on $\mathbb{R}^n$ there exists an invertible affine transformation T such that the log-concave probability measure $T_*\mu$ is isotropic, and $L_{T_*\mu} = L_\mu$.

The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 2$. The classical estimates $L_n \leqslant c\sqrt[4]{n}\ln n$ by Bourgain [4] and, fifteen years later, $L_n \leqslant c\sqrt[4]{n}$ by Klartag [18] remained the best known until 2020. In a breakthrough work, Chen [10] proved that for any $\varepsilon > 0$ one has $L_n \leqslant n^{\varepsilon}$ for all large enough n. This development was the starting point for a series of important works, including a recent technical breakthrough by Guan [16], that culminated in the final affirmative answer to the problem. Very recently, Klartag and Lehec [20] showed that $L_n \leqslant C$.

In the next three subsections we introduce and compare the families of convex bodies that are associated with a centered log-concave probability measure μ on $\mathbb{R}^n$ and have been mentioned in the introduction. Some of these families can be defined for any probability measure and the assumption that μ is log-concave or centered is unnecessary for some of the results that we present. However, for simplicity, we chose to define and state everything in the setting of centered log-concave probability measures.

We refer to Schneider's book [24] for basic facts from the Brunn-Minkowski theory and to the book [1] for basic facts from asymptotic convex geometry. We also refer to [8] for more information on isotropic convex bodies and log-concave probability measures.

2.1 Level sets of the density and K. Ball's bodies

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For every $t \geqslant 1$ we consider the convex set

$$R_t(\mu) = \{x \in \mathbb{R}^n : f_\mu(x) \geqslant e^{-t} f_\mu(0)\}.$$

Using the log-concavity of f_μ we easily check that $R_t(\mu)$ is convex. Note also that $0 \in \mathrm{int}(R_t(\mu))$. To show that $R_t(\mu)$ is bounded, we recall that since f_μ is log-concave and has finite positive integral we have that there exist constants $A, B > 0$ such that

$$f_\mu(x) \leqslant A e^{-B|x|} \tag{2.3}$$

for all $x \in \mathbb{R}^n$ (see [8, Lemma 2.2.1]). Therefore, if $x \in R_t(\mu)$ we get that $|x| \leqslant \frac{1}{B}\big(\ln(A/f_\mu(0)) + t\big)$. Another consequence of (2.3) is that f_μ has finite moments of all orders.

The next proposition, which will be very useful for us, shows that the measure of $R_t(\mu)$ increases to 1 exponentially fast as $t \to \infty$.

Proposition 2.1. *For every $t \geqslant 5(n-1)$ we have that $\mu(R_t(\mu)) \geqslant 1 - e^{-t/4}$.*

The proof is based on some one-dimensional considerations. Let $g : [0, \infty) \to [0, \infty)$ be a log-concave function. It is proved in [19, Lemma 4.3] that for every $m > 0$ the equation $(\ln g)'(r) = -\frac{m}{r}$ has a unique solution $r_m > 0$. Moreover, for every $0 \leqslant r \leqslant r_m$ one has that

$$g(r) \geqslant e^{-m} g(0). \tag{2.4}$$

Since $(\ln g)'$ is decreasing on the interval where it is defined, we see that $(\ln g)'(r) \leqslant -m/r_m$ for every $r > r_m$ such that $g(r) > 0$. Therefore, for every $\alpha \geqslant 1$ and every $r \geqslant \alpha r_m$ we have that

$$g(r) \leqslant e^{-(\alpha-1)m} g(r_m). \tag{2.5}$$

The next lemma is essentially contained in Klartag's [19, Lemma 5.2].

Lemma 2.2. *Let $m > 0$, $\alpha \geqslant 5$ and let $g : [0, \infty) \to [0, \infty)$ be a log-concave function with positive finite integral. Let $\varrho_{\alpha m} := \sup\{r > 0 : g(r) \geqslant e^{-\alpha m} g(0)\}$. Then,*

$$\int_0^{\varrho_{\alpha m}} r^m g(r)\, dr \geqslant \left(1 - e^{-\alpha m/4}\right) \int_0^\infty r^m g(r)\, dr.$$

Proof. We may assume that $\int_0^\infty r^m g(r)\,dr = 1$. For $r > 0$ we define

$$\varphi(r) = r^m g(r) \quad \text{and} \quad \Phi(r) = \int_0^r \varphi(u)\,du.$$

Note that φ is a log-concave function with $\int_0^\infty \varphi(r)\,dr = 1$. Define r_m as the unique solution of $(\ln g)'(r) = -m/r$. Then, equivalently, we have that $\varphi'(r_m) = 0$ and since φ is log-concave and $\varphi(r_m) > 0$ we conclude that $\varphi(r_m) = \max(\varphi)$. From (2.4) we know that

$$g(r) \geqslant e^{-m} g(0)$$

for all $0 \leqslant r \leqslant r_m$. It follows that if $M := g(r_m)$ then, for every $r > 0$ that satisfies $g(r) \geqslant e^{-(\alpha-1)m} M$ we have that $g(r) \geqslant e^{-\alpha m} g(0)$, and hence

$$\varrho_{\alpha m} \geqslant r' := \sup\{r > 0 : g(r) \geqslant e^{-(\alpha-1)m} M\}.$$

From the definition of r' it is clear that $r_m \leqslant r'$. Also, since $g(r') = e^{-(\alpha-1)m} M$, from (2.5) we see that $r' \leqslant \alpha r_m$. It follows that

$$\varphi(r') = \varphi(r_m) \left(\frac{r'}{r_m}\right)^m \frac{g(r')}{M} \leqslant \varphi(r_m) \alpha^m e^{-(\alpha-1)m} \leqslant \varphi(r_m) e^{-\alpha m/4} = e^{-\alpha m/4} \max(\varphi).$$

The last inequality follows from the fact that $3\alpha \geqslant 4(\ln \alpha + 1)$ when $\alpha \geqslant 5$. It is not hard to check that $\psi(s) = \varphi(\Phi^{-1}(s))$ is concave on $(0,1)$. Since φ attains its maximum at r_m, we have that ψ attains its maximum at $\Phi(r_m)$. Then, for any $0 < \varepsilon \leqslant 1$, using the concavity of ψ we can check that if $s \geqslant \Phi(r_m)$ and $\psi(s) \leqslant \varepsilon \cdot \max(\psi)$ we must have that $s \geqslant 1 - \varepsilon$. It follows that if $r \geqslant r_m$ and $\varphi(r) \leqslant \varepsilon \cdot \max(\psi) = \varepsilon \cdot \max(\varphi)$ then $\Phi(r) \geqslant 1 - \varepsilon$. Since $r' \geqslant r_m$ and r' satisfies $\varphi(r') \leqslant e^{-\alpha m/4} \max(\varphi)$, we conclude that

$$\Phi(r') \geqslant 1 - e^{-\alpha m/4}.$$

The result follows from the fact that $\varrho_{\alpha m} \geqslant r'$. $\square$

Lemma 2.2 immediately implies Proposition 2.1.

Proof of Proposition 2.1. Fix $\xi \in S^{n-1}$ and consider the log-concave integrable function $g : [0,\infty) \to [0,\infty)$ defined by $g(r) = f_\mu(r\xi)$. Applying Lemma 2.2 with $m = n-1$ and $\alpha = t/m$ we see that

$$\int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} f_\mu(r\xi)\,dr \geqslant \left(1 - e^{-t/4}\right) \int_0^\infty r^{n-1} f_\mu(r\xi)\,dr.$$

Then, integration in polar coordinates shows that

$$\begin{aligned}
\mu(R_t(\mu)) &= n\omega_n \int_{S^{n-1}} \int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} f_\mu(r\xi)\,dr\,d\sigma(\xi) \\
&\geqslant \left(1 - e^{-t/4}\right) n\omega_n \int_{S^{n-1}} \int_0^\infty r^{n-1} f_\mu(r\xi)\,dr\,d\sigma(\xi) = 1 - e^{-t/4},
\end{aligned}$$

which is the assertion of the proposition. $\square$

A second family of convex bodies associated with a centered log-concave probability measure μ on $\mathbb{R}^n$

was introduced by K. Ball, who also established their convexity in [2]: for every $t>0$, we define

$$K_t(\mu)=\left\{x\in\mathbb{R}^n:\int_0^\infty r^{t-1}f_\mu(rx)\,dr\geqslant\frac{f_\mu(0)}{t}\right\}.$$

From the definition it follows that the radial function of $K_t(\mu)$ is given by

$$\varrho_{K_t(\mu)}(x)=\left(\frac{1}{f_\mu(0)}\int_0^\infty tr^{t-1}f_\mu(rx)\,dr\right)^{1/t} \tag{2.6}$$

for $x\neq 0$. For every $0<t\leqslant s$ we have that

$$\frac{\Gamma(t+1)^{\frac{1}{t}}}{\Gamma(s+1)^{\frac{1}{s}}}K_s(\mu)\subseteq K_t(\mu)\subseteq e^{\frac{n}{t}-\frac{n}{s}}K_s(\mu). \tag{2.7}$$

A proof is given in [8, Proposition 2.5.7].

An immediate consequence of the definitions is the next inclusion between the bodies $K_t(\mu)$ and $R_t(\mu)$.

Proposition 2.3. *For every $s>t$ we have that*

$$R_t(\mu)\subseteq e^{t/s}K_s(\mu).$$

Proof. Let $\xi\in S^{n-1}$ and define $g:[0,\infty)\to[0,\infty)$ by $g(r)=f_\mu(r\xi)$. By the definition of $K_t(\mu)$ we have

$$[\varrho_{K_s(\mu)}(\xi)]^s=\frac{s}{f_\mu(0)}\int_0^\infty r^{s-1}g(r)dr.$$

Since $g(r)\geqslant e^{-t}f_\mu(0)$ for all $0\leqslant r\leqslant\varrho_{R_t(\mu)}(\xi)$ we see that

$$\int_0^\infty r^{s-1}g(r)dr\geqslant\int_0^{\varrho_{R_t(\mu)}(\xi)}r^{s-1}g(r)dr\geqslant e^{-t}f_\mu(0)\int_0^{\varrho_{R_t(\mu)}(\xi)}r^{s-1}dr=\frac{f_\mu(0)}{s}e^{-t}[\varrho_{R_t(\mu)}(\xi)]^s.$$

It follows that $\varrho_{K_s(\mu)}(\xi)\geqslant e^{-t/s}\varrho_{R_t(\mu)}(\xi)$. Since $\xi\in S^{n-1}$ was arbitrary, this shows that $R_t(\mu)\subseteq e^{t/s}K_s(\mu)$. □

In the opposite direction we have the next result.

Proposition 2.4. *For every $t\geqslant 2n$ and any $\alpha\geqslant 5$ we have that*

$$R_{\alpha t}(\mu)\supseteq\left(1-\frac{2n}{t}\right)K_t(\mu).$$

Under the additional assumption that μ is even, we have that

$$R_{\alpha t}(\mu)\supseteq\left(1-e^{-\alpha t/5}\right)K_t(\mu).$$

Proof. Given any $\xi\in S^{n-1}$ consider the log-concave function $g:[0,\infty)\to[0,\infty)$ defined by $g(r)=f_\mu(r\xi)$. Applying Lemma 2.2 with $m=t-1$ we see that

$$\int_0^{\varrho_{R_{\alpha t}(\mu)}(\xi)}r^{t-1}g(r)dr\geqslant(1-e^{-\alpha t/5})\int_0^\infty r^{t-1}g(r)dr.$$

By the definition of $K_t(\mu)$ we have

$$\int_0^\infty r^{t-1}g(r)dr = \frac{f_\mu(0)}{t}[\varrho_{K_t(\mu)}(\xi)]^t.$$

On the other hand,

$$\int_0^{\varrho_{R_{\alpha t}(\mu)}(\xi)} r^{t-1}g(r)dr \leqslant \|f_\mu\|_\infty \int_0^{\varrho_{R_{\alpha t}(\mu)}(\xi)} r^{t-1}dr = \frac{\|f_\mu\|_\infty}{t}[\varrho_{R_{\alpha t}(\mu)}(\xi)]^t.$$

Using also the fact that $\|f_\mu\|_\infty \leqslant e^n f_\mu(0)$ from (2.1), we get

$$e^n[\varrho_{R_{\alpha(t-1)}(\mu)}(\xi)]^t \geqslant (1-e^{-\alpha t/5})[\varrho_{K_t(\mu)}(\xi)]^t \tag{2.8}$$

and the result follows because

$$e^{-n/t}(1-e^{-\alpha t/5})^{1/t} \geqslant (1-n/t)(1-e^{-\alpha t/5}) \geqslant (1-n/t)^2 \geqslant 1-2n/t.$$

When μ is even, the term e^n does not appear in (2.8), and thus we obtain an improved estimate. □

2.2 Level sets of the Cramér transform and centroid bodies

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any $t \geqslant 1$ we define the L_t-centroid body $Z_t(\mu)$ of μ as the centrally symmetric convex body whose support function is

$$h_{Z_t(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x,y\rangle|^t f_\mu(x)dx\right)^{1/t}, \qquad y \in \mathbb{R}^n.$$

Note that $Z_t(\mu)$ is always centrally symmetric, and $Z_t(T_*\mu) = T(Z_t(\mu))$ for every $T \in GL(n)$ and $t \geqslant 1$. We shall also use the fact that if μ is isotropic then $Z_2(\mu) = B_2^n$. A variant of the L_t-centroid bodies of μ is defined as follows. For every $t \geqslant 1$ we consider the convex body $Z_t^+(\mu)$ with support function

$$h_{Z_t^+(\mu)}(y) = \left(\int_{\mathbb{R}^n} \langle x,y\rangle_+^t f_\mu(x)dx\right)^{1/t}, \qquad y \in \mathbb{R}^n,$$

where $a_+ = \max\{a,0\}$. When f_μ is even, we have that $Z_t^+(\mu) = 2^{-1/t}Z_t(\mu)$. In any case, it is clear that $Z_t^+(\mu) \subseteq Z_t(\mu)$. One can also check that if $1 \leqslant t < s$ then

$$\left(\frac{4}{e}\right)^{\frac{1}{t}-\frac{1}{s}} Z_t^+(\mu) \subseteq Z_s^+(\mu) \subseteq c_1\left(\frac{4(e-1)}{e}\right)^{\frac{1}{t}-\frac{1}{s}} \frac{s}{t} Z_t^+(\mu). \tag{2.9}$$

(for a proof see [17], where the family of bodies $\tilde{Z}_t^+(\mu) = 2^{1/t}Z_t^+(\mu)$ is considered).

For every $t \geqslant 1$ we define

$$M_t^+(\mu) := \left\{v \in \mathbb{R}^n : \int_{\mathbb{R}^n} \langle v,x\rangle_+^t d\mu(x) \leqslant 1\right\}.$$

Note that

$$Z_t^+(\mu) := (M_t^+(\mu))^\circ.$$

For every $t > 0$ we also define

$$B_t(\mu) := \{v \in \mathbb{R}^n : \Lambda_\mu^*(v) \leqslant t\}.$$

For all $s \geqslant t$ we have $M_s^+(\mu) \subseteq M_t^+(\mu)$ and $Z_t^+(\mu) \subseteq Z_s^+(\mu)$. Moreover, since μ is centered we have $\Lambda_\mu^*(0) = 0$ by Jensen's inequality, and the convexity of Λ_μ^* implies that $B_t(\mu) \subseteq B_s(\mu) \subseteq \frac{s}{t} B_t(\mu)$ for all $s \geqslant t > 0$. A proof of all these assertions can be found in [8].

The next proposition, which is a variant of [21, Proposition 3.2] of Latała and Wojtaszczyk, provides an inclusion relation between the bodies $Z_t^+(\mu)$ and $B_s(\mu)$.

Proposition 2.5. *Let μ be a centered log-concave probability measure μ on $\mathbb{R}^n$. For every $s \geqslant t \geqslant t_0$ we have that*

$$Z_t^+(\mu) \subseteq \left(1 + \frac{2\ln s}{s}\right) B_s(\mu).$$

Proof. Let $v \in Z_t^+(\mu)$. We shall show that $\Lambda_\mu^*(v/c_s) \leqslant s$, where $c_s \leqslant 1 + \frac{2\ln s}{s}$. To this end, we should check that

$$\frac{1}{c_s}\langle u, v\rangle - \Lambda_\mu(u) \leqslant s$$

for all $u \in \mathbb{R}^n$. For every $u \in \mathbb{R}^n$ we define β_u by the equation

$$\int_{\mathbb{R}^n} \langle u, x\rangle_+^t d\mu(x) = \beta_u^t.$$

Then, $u/\beta_u \in M_t^+(\mu)$ and since $Z_t^+(\mu) = (M_t^+(\mu))^\circ$ we have that $\langle u, v\rangle \leqslant \beta_u$. Note that, by Hölder's inequality, for all $s \geqslant t$ we have that

$$\int_{\mathbb{R}^n} \langle u, x\rangle_+^s d\mu(x) \geqslant \left(\int_{\mathbb{R}^n} \langle u, x\rangle_+^t d\mu(x)\right)^{s/t} = \beta_u^s. \tag{2.10}$$

For any $0 < \delta < \beta_u$, using (2.10) we see that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{\frac{\delta}{\beta_u}\langle u,x\rangle} d\mu(x) &\geqslant \int_{\mathbb{R}^n} e^{\frac{\delta}{\beta_u}\langle u,x\rangle} \mathbb{1}_{\{x:\langle u,x\rangle \geqslant 0\}}(x)\, d\mu(x) \geqslant \int_{\mathbb{R}^n} \frac{\delta^s \langle u, x\rangle_+^s}{s!\beta_u^s} \mathbb{1}_{\{x:\langle u,x\rangle \geqslant 0\}}(x)\, d\mu(x) \\ &= \int_{\mathbb{R}^n} \frac{\delta^s \langle u, x\rangle_+^s}{s!\beta_u^s}\, d\mu(x) \geqslant \frac{\delta^s}{s!}.\end{aligned}$$

So, $\Lambda_\mu(\delta u/\beta_u) \geqslant \ln\left(\frac{\delta^s}{s!}\right)$, and using the convexity of Λ_μ and the fact that $\Lambda_\mu(0) = 0$ we obtain

$$\Lambda_\mu(u) \geqslant \frac{\beta_u}{\delta}\Lambda_\mu(\delta u/\beta_u) \geqslant \beta_u \frac{1}{\delta}\ln\left(\frac{\delta^s}{s!}\right).$$

Therefore, for any $c > 0$ we get

$$\frac{1}{c}\langle u, v\rangle - \Lambda_\mu(u) \leqslant \beta_u\left[\frac{1}{c} - \frac{1}{\delta}\ln\left(\frac{\delta^s}{s!}\right)\right]. \tag{2.11}$$

The function $g_s(\delta) = \frac{1}{\delta}\ln\left(\frac{\delta^s}{s!}\right)$ attains its maximum value at δ_s where δ_s satisfies $s = \ln\left(\frac{\delta^s}{s!}\right)$, i.e. $\delta_s = e(s!)^{1/s}$. This maximum value is equal to $\max(g_s) = \frac{s}{\delta_s} = \frac{s}{e(s!)^{1/s}}$. So, if we choose $\delta = \delta_s$ and $c_s = \frac{1}{\max(g_s)}$, from (2.11) we see that

$$\frac{1}{c_s}\langle u, v\rangle - \Lambda_\mu(u) \leqslant 0 \tag{2.12}$$

for all $u \in \mathbb{R}^n$ that satisfy $\beta_u > \delta_s$. On the other hand, since μ is centered, we have that $\Lambda_\mu(u) \geqslant 0$ for all $u \in \mathbb{R}^n$, and hence, for all $u \in \mathbb{R}^n$ that satisfy $\beta_u \leqslant \delta_s$ we have that

$$\frac{1}{c_s}\langle u, v\rangle - \Lambda_\mu(u) \leqslant \frac{1}{c_s}\beta_u \leqslant \frac{\delta_s}{c_s} = s. \tag{2.13}$$

This shows that

$$\Lambda_\mu^*(v/c_s) = \sup\left\{\frac{1}{c_s}\langle u, v\rangle - \Lambda_\mu(u) : u \in \mathbb{R}^n\right\} \leqslant s,$$

or equivalently $v \in c_s B_s$. We have thus proved that $Z_t^+ \subseteq c_s B_s$ for all $s \geqslant t$ and it remains to estimate the constant c_s. By Stirling's formula,

$$c_s = \frac{e}{s}(s!)^{1/s} \sim \frac{e}{s}\left(\frac{s}{e}\right)(2\pi s)^{\frac{1}{2s}} = (2\pi s)^{\frac{1}{2s}} \sim 1 + \frac{\ln(2\pi s)}{2s}$$

as $s \to \infty$. It follows that if $s \geqslant t \geqslant t_0$ (where t_0 is a large enough constant) then $c_s \leqslant 1 + \frac{2\ln s}{s}$. $\square$

The next proposition establishes a reverse inclusion between the bodies $B_t(\mu)$ and $Z_s^+(\mu)$ (again, a variant of this result appears in [21, Proposition 3.5]).

Proposition 2.6. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any $t \geqslant 1$ and any $\delta \in (0,1)$ we have that*

$$B_t(\mu) \subseteq (1+\delta) Z^+_{c_1 t/\delta}(\mu)$$

where $c_1 > 0$ is an absolute constant.

Proof. Let $\delta \in (0,1)$ and $t \geqslant 1$. If $u \in M_s^+(\mu)$ then Hölder's inequality shows that $\|\langle u,\cdot\rangle_+\|_k \leqslant \|\langle u,\cdot\rangle_+\|_s \leqslant 1$ for all $k \leqslant s$, and (2.9) implies that $\|\langle u,\cdot\rangle_+\|_k \leqslant \frac{ck}{s}\|\langle u,\cdot\rangle_+\|_s \leqslant \frac{ck}{s}$ for all $k > s$, where $c > 0$ is an absolute constant. Since $\frac{k}{(k!)^{1/k}} \to e$, we may choose an absolute constant $\gamma > 0$ small enough so that $\frac{c\gamma k}{(k!)^{1/k}} \leqslant \frac{1}{2}$ for all $k \geqslant 1$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{\langle \gamma s u, x\rangle_+} d\mu(x) &= \sum_{k=0}^\infty \frac{1}{k!}\int_{\mathbb{R}^n} \langle \gamma s u, x\rangle_+^k d\mu(x) \leqslant \sum_{k\leqslant s}\frac{(\gamma s)^k}{k!} + \sum_{k>s}\frac{(\gamma s)^k}{k!}\left(\frac{ck}{s}\right)^k \\ &\leqslant e^{\gamma s} + \sum_{k>s}\frac{1}{2^k} \leqslant e^{\gamma s} + 1 \leqslant e^{\gamma s+1}\end{aligned}$$

if $s \geqslant s_0$. Therefore, for any $u \in M_s^+(\mu)$ we get $\Lambda_\mu(\gamma s u) \leqslant \gamma s + 1$.

Now, let $v \notin (1+\delta)Z_s^+(\mu)$. We can find $u \in M_s^+(\mu)$ such that $\langle v, u\rangle > 1+\delta$ and then

$$\Lambda_\mu^*(v) \geqslant \langle v, \gamma s u\rangle - \Lambda_\mu(\gamma s u) > (1+\delta)\gamma s - \gamma s - 1 = \delta\gamma s - 1 > t$$

if we assume that $s \geqslant \frac{2t}{\gamma\delta}$. Therefore, $v \notin B_t(\mu)$. This shows that $B_t(\mu) \subseteq (1+\delta)Z^+_{c_1 t/\delta}(\mu)$, where $c_1 = 2/\gamma$. $\square$

2.3 Level sets of the Cramér transform and floating bodies

Let μ be a probability measure on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all closed half-spaces H of $\mathbb{R}^n$ containing x. The function

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}$$

is called Tukey's half-space depth. This notion was introduced by Tukey in [26] for data sets as a measure of centrality for multivariate data; see the survey article of Nagy, Schütt and Werner [22] for an overview, connections with convex geometry and references. Note that the infimum in the definition of $\varphi_\mu(x)$ is determined by those closed half-spaces H for which x lies on the boundary $\partial(H)$ of H. For every $s \geqslant 1$ we define the set

$$T_s(\mu) = \{x \in \mathbb{R}^n : \varphi_\mu(x) \geqslant e^{-s}\}.$$

Note that $T_s(\mu)$ is convex: if $x, y \in T_s(\mu)$ then for any $z \in [x, y]$ and any $H \in \mathcal{H}(z)$ we have that either x or y belongs to H, and hence $\mu(H) \geqslant e^{-s}$, therefore $\varphi_\mu(z) \geqslant e^{-s}$.

Let $x \in \mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$ the half-space $\{z : \langle z - x, \xi\rangle \geqslant 0\}$ is in $\mathcal{H}(x)$, therefore

$$\varphi_\mu(x) \leqslant \mu(\{z : \langle z, \xi\rangle \geqslant \langle x, \xi\rangle\}) \leqslant e^{-\langle x,\xi\rangle}\mathbb{E}_\mu\big(e^{\langle z,\xi\rangle}\big) = \exp\big(-[\langle x, \xi\rangle - \Lambda_\mu(\xi)]\big),$$

and taking the infimum over all $\xi \in \mathbb{R}^n$ we see that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$. An immediate consequence of this inequality is the inclusion

$$T_s(\mu) \subseteq B_s(\mu). \tag{2.14}$$

Indeed, if $x \in T_s(\mu)$ then $e^{-s} \leqslant \varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$, which shows that $\Lambda_\mu^*(x) \leqslant s$, and hence $x \in B_s(\mu)$.

The next proposition shows that if μ is log-concave then a reverse inclusion holds in full generality, at least if s is large enough.

Proposition 2.7. *There exists $s_0 \geqslant 1$ such that, for every centered log-concave probability measure μ on $\mathbb{R}^n$ and any $s \geqslant s_0$,*

$$T_s(\mu) \subseteq B_s(\mu) \subseteq T_{s+3\ln s}(\mu).$$

Proof. We shall use a result of Brazitikos and Chasapis from [5]: For every $x \in \mathrm{supp}(\mu)$ and any $\varepsilon \in (0, 1)$ we have that

$$\Lambda_\mu^*(x) \geqslant (1 - \varepsilon)\ln\left(\frac{1}{\varphi_\mu(x)}\right) + \ln\left(\frac{\varepsilon}{2^{1-\varepsilon}}\right) = \ln\left(\frac{\varepsilon}{(2\varphi_\mu(x))^{1-\varepsilon}}\right).$$

Let $x \in B_s(\mu)$ and assume that $\varphi_\mu(x) < e^{-s-3\ln s}$. Then,

$$2\varphi_\mu(x) \leqslant e^{-s-3\ln s+\ln 2} < e^{-s-2\ln s}$$

provided that $s \geqslant 2$, and hence

$$s \geqslant \Lambda_\mu^*(x) \geqslant \ln\left(\varepsilon e^{(1-\varepsilon)(s+2\ln s)}\right) = \ln\varepsilon + (1 - \varepsilon)(s + 2\ln s)$$

for every $\varepsilon \in (0, 1)$ and any $s \geqslant 2$. The function $f(\varepsilon) = \ln\varepsilon + (1 - \varepsilon)(s + 2\ln s)$ attains its maximum at $\varepsilon = \frac{1}{s+2\ln s}$ and we must have

$$s \geqslant -\ln(s + 2\ln s) + \left(1 - \frac{1}{s + 2\ln s}\right)(s + 2\ln s) = s + 2\ln s - \ln(s + 2\ln s) - 1.$$

It follows that $\ln(s + 2\ln s) + 1 \geqslant 2\ln s$, which implies that $e(s + 2\ln s) \geqslant s^2$, a contradiction if $s \geqslant s_0$ for a large enough absolute constant $s_0 > 0$.

We have thus shown that if $s \geqslant s_0$ and $x \in B_s(\mu)$ then $\varphi_\mu(x) \geqslant e^{-s-3\ln s}$, i.e. $B_s(\mu) \subseteq T_{s+3\ln s}(\mu)$. The inclusion $T_s(\mu) \subseteq B_s(\mu)$ is (2.14) above. □

The next simple lemma compares the families $\{T_s(\mu)\}_{t>0}$ and $\{Z_t^+(\mu)\}_{t>0}$ of floating bodies and L_t-

centroid bodies.

Lemma 2.8. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any $\delta > 0$ and any $t \geqslant 1$ we have that*

$$T_{t\ln(1+\delta)}(\mu) \subseteq (1+\delta)Z_t^+(\mu).$$

Proof. Assume that $x \notin (1+\delta)Z_t^+(\mu)$. Then, we may find $\delta' > \delta$ and $\xi \in S^{n-1}$ such that $x \in H = \{y \in \mathbb{R}^n : \langle y, \xi\rangle \geqslant (1+\delta')h_{Z_t^+}(\xi)\}$. From Markov's inequality we see that

$$\mu(H) \leqslant (1+\delta')^{-t} < (1+\delta)^{-t} = e^{-t\ln(1+\delta)},$$

therefore, $\varphi_\mu(x) \leqslant \mu(H) < e^{-t\ln(1+\delta)}$, which shows that $x \notin T_{t\ln(1+\delta)}(\mu)$. The lemma follows. □

3 Moments of the Cramér transform

In this section we prove Theorem 1.1 and Theorem 1.2. We start with the observation that, without loss of generality, we can restrict our attention to isotropic log-concave probability measures. Indeed, a simple computation shows that if μ and ν are two centered log-concave probability measures and $\nu = T_*\mu$ for some $T \in GL(n)$ then $\Lambda_\nu(\xi) = \Lambda_\mu(T^t\xi)$ for all $\xi \in \mathbb{R}^n$, and hence, by the definition of the Legendre transform, we have that

$$\Lambda_\nu^*(x) = \Lambda_\mu^*(T^{-1}x)$$

for all $x \in \mathbb{R}^n$. Then, from (2.2) we get

$$\int_{\mathbb{R}^n} g(\Lambda_\nu^*(x))d\nu(x) = \int_{\mathbb{R}^n} g(\Lambda_\mu^*(y))d\mu(y) \tag{3.1}$$

for every bounded Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$. Since every centered log-concave probability measure μ on $\mathbb{R}^n$ has an isotropic push forward $\nu = T_*\mu$, where $T \in GL(n)$, we may check the assertion of both theorems for ν, and then it also holds true fror μ.

We shall also use the next simple but useful lemma.

Lemma 3.1. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any $\delta > 0$ and any Borel subset A of $\mathbb{R}^n$ we have that*

$$\mu((1+\delta)A) \leqslant e^{2n\delta}\mu(A).$$

Proof. Note that if $A \subset \mathbb{R}^n$ is a Borel set, then

$$\mu((1+\delta)A) = \int_{(1+\delta)A} f_\mu(x)\,dx = (1+\delta)^n \int_A f_\mu((1+\delta)x)\,dx.$$

Since f_μ is log-concave, we see that

$$f_\mu((1+\delta)x) \leqslant f_\mu(x)\left(\frac{f_\mu(x)}{f_\mu(0)}\right)^\delta \leqslant e^{n\delta}f_\mu(x)$$

for every $x \in \mathbb{R}^n$, because $f_\mu(x) \leqslant e^n f_\mu(0)$ by (2.1). It follows that

$$\mu((1+\delta)A) \leqslant (1+\delta)^n e^{n\delta}\mu(A) \leqslant e^{2n\delta}\mu(A)$$

as claimed. □

A weak integrability result can be obtained if we combine Proposition 2.5 with the next technical proposition (see [6, Proposition 5.6] for a proof).

Proposition 3.2. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $\delta \in (0,1)$ and any $t \geqslant C_\delta n \ln n$ we have that*

$$\mu((1+\delta)Z_t^+(\mu)) \geqslant 1 - e^{-c\delta t}$$

where $C_\delta = C\delta^{-1}\ln(2/\delta)$ and $C, c > 0$ are absolute positive constants.

Theorem 3.3. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For every $0 < p < 1$ we have that*

$$\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^p \, d\mu(x) < \infty.$$

Proof. We may assume that μ is isotropic. Let $0 < p < q < 1$. Let $t > 1$ that will be chosen appropriately large and apply Proposition 3.2 with $\delta = \frac{\ln t}{ct^q}$. Then, we get

$$\mu\left(\left(1 + \frac{\ln t}{ct^q}\right) Z_t^+(\mu)\right) \geqslant 1 - e^{-t^{1-q}\ln t} \tag{3.2}$$

provided that $t \geqslant C_1(q)t^q n \ln n$, or equivalently $t \geqslant t_n = C_2(q)(n\ln n)^{\frac{1}{1-q}}$. Now, let $t \geqslant t_n$. From Proposition 2.5 we see that

$$\left(1 + \frac{\ln t}{ct^q}\right) Z_t^+(\mu) \subseteq \left(1 + \frac{\ln t}{ct^q}\right)\left(1 + \frac{\ln t}{t}\right) B_t(\mu)$$

and applying Lemma 3.1 twice we obtain

$$\begin{aligned} \mu\left(\left(1 + \frac{\ln t}{ct^q}\right) Z_t^+(\mu)\right) &\leqslant \exp\left(\frac{2n\ln t}{ct^q} + \frac{2n\ln t}{t}\right)\mu(B_t(\mu)) \leqslant \exp\left(\frac{4n\ln t}{ct^q}\right)\mu(B_t(\mu)) \\ &\leqslant \left(1 - \frac{4n\ln t}{ct^q}\right)^{-1}\mu(B_t(\mu)). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we get

$$\mu(B_t(\mu)) \geqslant \left(1 - e^{-t^{1-q}\ln t}\right)\left(1 - \frac{4n\ln t}{ct^q}\right) \geqslant 1 - \frac{8n\ln t}{ct^q}$$

if $t \geqslant s_n$ where $s_n \geqslant t_n$ is large enough and depends only on n (note that $\exp(t^{1-q}\ln t) \geqslant ct^q/(4n\ln t)$ for large enough t, independently from n). Now, we write

$$\begin{aligned} \int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^p d\mu(x) &= \int_0^\infty pt^{p-1}\mu(\{x : \Lambda_\mu^*(x) > t\})\, dt = \int_0^\infty pt^{p-1}(1 - \mu(B_t(\mu)))\, dt \\ &= \int_0^{s_n} pt^{p-1}(1 - \mu(B_t(\mu)))\, dt + \int_{s_n}^\infty pt^{p-1}(1 - \mu(B_t(\mu)))\, dt \\ &\leqslant \int_0^{s_n} pt^{p-1}\, dt + \frac{8np}{c}\int_{s_n}^\infty t^{p-1}\frac{\ln t}{t^q}\, dt \\ &= s_n^p + \frac{8np}{c}\int_{s_n}^\infty \frac{\ln t}{t^{1+q-p}} dt < +\infty \end{aligned}$$

where the last integral converges because $1 + q - p > 1$. □

We have already described in the introduction that the main ingredient for the stronger integrability estimate of Theorem 1.1 is a lemma which compares the families $\{R_t(\mu)\}_{t>0}$ and $\{B_t(\mu)\}_{t>0}$. This is the content of Lemma 3.5 below, which is then combined with Proposition 2.1. For the proof of Lemma 3.5 we need a technical fact; if μ is isotropic and t is large enough then $R_t(\mu)$ contains a constant multiple of the Euclidean unit ball.

Lemma 3.4. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $t \geqslant 20n$ we have that*

$$R_t(\mu) \supseteq c_0 B_2^n$$

where $c_0 > 0$ is an absolute constant.

Proof. Since μ is isotropic, we have that $Z_2(\mu) = B_2^n$. We shall two facts about the bodies $K_t(\mu)$ and their centroid bodies. From [8, Proposition 2.5.8] we know that

$$e^{-1} \leqslant \big(f_\mu(0)|K_{2n}(\mu)|\big)^{2/n} \leqslant 2e.$$

Also, if we set $\overline{K_{2n}}(\mu) = |K_{2n}(\mu)|^{-\frac{1}{n}} K_{2n}(\mu)$ then [8, Proposition 5.1.6] shows that

$$f_\mu(0)^{\frac{1}{n}} Z_n(\mu) \subseteq 2e Z_n(\overline{K_{2n}}(\mu)) \subseteq 2e\overline{K_{2n}}(\mu).$$

Combining the above we get

$$K_{2n}(\mu) \supseteq \frac{1}{2e}\big(f_\mu(0)|K_{2n}(\mu)|\big)^{1/n} Z_n(\mu) \supseteq \frac{1}{2e^{3/2}} Z_n(\mu) \supseteq \frac{1}{2e^{3/2}} Z_2(\mu) = \frac{1}{2e^{3/2}} B_2^n.$$

From Proposition 2.4 (with $\alpha = 5$) we know that if $t \geqslant 20n$ then

$$R_t(\mu) \supseteq \frac{1}{2} K_{t/5}(\mu) \supseteq \frac{1}{2\sqrt{e}} K_{2n}(\mu)$$

because $K_{2n}(\mu) \subseteq \sqrt{e} K_{t/5}(\mu)$ by (2.7), and the lemma follows with $c_0 = 1/(2e)^2$. □

We are now ready to compare $\{R_t(\mu)\}_{t>0}$ and $\{B_t(\mu)\}_{t>0}$.

Lemma 3.5. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $\delta \in (0,1)$ and $t \geqslant n \ln n$ we have that*

$$(1-\delta) R_t(\mu) \subseteq B_{g(t,\delta)}$$

where $g(t,\delta) \leqslant 2t + n\ln(1/\delta)$. In particular, if $\delta = e^{-t/2}$ we get

$$(1 - e^{-t/2}) R_t(\mu) \subseteq B_{nt}(\mu).$$

Proof. From Lemma 3.4 we know that $R_t(\mu) \supseteq c_0 B_2^n$. Let $x \in (1-\delta) R_t(\mu)$. Since $(c_0\delta) B_2^n \subseteq \delta R_t(\mu)$ we have that

$$B(x, c_0\delta) := x + (c_0\delta) B_2^n \subseteq (1-\delta) R_t(\mu) + \delta R_t(\mu) = R_t(\mu).$$

Then, if H is a closed half-space such that $x \in \partial(H)$ we have that $H \cap B(x, c_0\delta) \subseteq R_t(\mu)$, and hence $f_\mu(y) \geqslant e^{-t} f_\mu(0)$ for all $y \in H \cap B(x, c_0\delta)$. Since $H \cap B(x, c_0\delta)$ is half of a ball of radius $c_0\delta$, it follows that

$$\mu(H) \geqslant \int_{H\cap B(x,c_0\delta)} f_\mu(y)\, d\mu(y) \geqslant e^{-t} f_\mu(0)\, |H \cap B(x, c_0\delta)| = e^{-t} f_\mu(0)\, \frac{\omega_n}{2} (c_0\delta)^n.$$

Now, recall that $f_\mu(0) \geqslant e^{-n}\|f_\mu\|_\infty = e^{-n} L_\mu^n \geqslant (c_1/e)^n$ because $L_\mu \geqslant c_1$ for some absolute constant $c_1 > 0$, and $\frac{\omega_n}{2} \geqslant \left(\frac{c_2}{n}\right)^{n/2}$ for some absolute constant $c_2 > 0$. Since H was arbitrary, by the definition of φ_μ we get

$$\varphi_\mu(x) \geqslant e^{-t}\left(\frac{c_3}{n}\right)^{n/2}\delta^n = e^{-g(t,\delta)},$$

with $c_3 = c_0^2 c_1^2 c_2/e^2$, where

$$g(t,\delta) = t + \frac{n}{2}\ln(c_4 n) + n\ln(1/\delta) \tag{3.4}$$

with $c_4 = 1/c_3$. Since $t \geqslant n\ln n$, we see that $\frac{n}{2}\ln(c_4 n) \leqslant t$ and the first assertion of the lemma follows. For the second assertion, note that $g(t, e^{-t/2}) \leqslant 2t + \frac{n}{2}t = \frac{n+4}{2}t \leqslant nt$ if $n \geqslant 4$. □

Lemma 3.5 shows that if $t \geqslant n\ln n$ then $R_t(\mu) \subseteq \frac{1}{1-\delta}B_{g(t,\delta)}(\mu) \subseteq (1+2\delta)B_{g(t,\delta)}(\mu)$ provided that $0 < \delta \leqslant 1/2$. Using also Lemma 3.1 we can prove Theorem 1.1.

Proof of Theorem 1.1. As we observed in the beginning of this section, we may assume that μ is isotropic. Let $t_n := n\ln n$. From Lemma 3.5 we know that if $t \geqslant t_n$ then

$$R_t(\mu) \subseteq (1 + 2e^{-t/2})B_{nt}(\mu).$$

Moreover, Proposition 2.1 shows that $\mu(R_t(\mu)) \geqslant 1 - e^{-t/4}$. Combining these facts with Lemma 3.1 we get

$$1 - e^{-t/4} \leqslant \mu(R_t(\mu)) \leqslant \exp\left(4n\, e^{-t/2}\right)\mu(B_{nt}(\mu)) \leqslant (1 + 8ne^{-t/2})\mu(B_{nt}(\mu)),$$

which finally gives

$$\mu(B_{nt}(\mu)) \geqslant 1 - e^{-t/8}$$

if $n \geqslant n_0$ for some fixed $n_0 \in \mathbb{N}$. Now, we write

$$\begin{aligned}\int_{\mathbb{R}^n} \exp\left(\Lambda_\mu^*(x)/(16n)\right)d\mu(x) &= 1 + \frac{1}{16}\int_0^\infty e^{\frac{t}{16}}\mu(\{x \in \mathbb{R}^n : \Lambda_\mu^*(x) > nt\})\,dt \\ &\leqslant 1 + \frac{1}{16}\int_0^{t_n} e^{\frac{t}{16}}\,dt + \frac{1}{16}\int_{t_n}^\infty e^{\frac{t}{16}}(1 - \mu(B_{nt}(\mu)))\,dt \\ &\leqslant e^{\frac{t_n}{16}} + \frac{1}{16}\int_{t_n}^\infty e^{\frac{t}{16}}e^{-\frac{t}{8}}dt = e^{\frac{t_n}{16}} + \frac{1}{16}\int_{t_n}^\infty e^{-\frac{t}{16}}dt < +\infty.\end{aligned}$$

Therefore, we have the assertion of the theorem with $c = \frac{1}{16}$. □

For the proof of Theorem 1.2 we use Lemma 3.5 again, with a different choice of δ depending on t. In fact we can estimate $\|\Lambda_\mu^*\|_{L^p(\mu)}$ for all $1 \leqslant p \leqslant cn$.

Theorem 3.6. *For every centered log-concave probability measure μ on $\mathbb{R}^n$ and any $1 \leqslant p \leqslant c_1 n$ we have that*

$$\|\Lambda_\mu^*\|_{L^p(\mu)} \leqslant c_2 pn\ln n$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. As in the previous proof, we may assume that μ is isotropic. Let $t_n := n\ln n$. Applying Lemma 3.5 with $\delta = t^{-p-2}$, or more precisely using (3.4), we see that

$$(1 - t^{-p-2})R_t(\mu) \subseteq B_{g(t,t^{-p-2})}(\mu)$$

where

$$g(t, t^{-(p+2)}) = t + \frac{n}{2}\ln(c_4 n) + (p+2)n\ln t \leqslant 2(p+3)t$$

because the function $m(t) = \frac{t}{\ln t}$ is increasing on $[e, \infty)$ and hence $\frac{n}{2} \leqslant \frac{n\ln n}{\ln(n\ln n)} \leqslant \frac{t}{\ln t}$ for all $t \geqslant t_n$, which implies that $n\ln t \leqslant 2t$ for all $t \geqslant t_n$. From Lemma 3.1 it follows that

$$\mu(B_{2(p+3)t})(\mu) \geqslant e^{-4n/t^{p+2}}\mu(R_t(\mu)) \geqslant e^{-4n/t^{p+2}}(1 - e^{-t/4}) \geqslant 1 - \frac{8n}{t^{p+2}} > 1 - \frac{1}{t^{p+1}}$$

for all $t \geqslant t_n$ and all $0 < p \leqslant n/8$ (here we use the fact that $\frac{t}{4} > (p+2)\ln t$ if $t \geqslant t_n$ and $p \leqslant n/8$, which implies that $e^{-t/4} < 2n/t^{p+2}$). Then, we write

$$\begin{aligned}
\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)/2(p+2)|^p d\mu(x) &= \int_0^\infty pt^{p-1}\mu(\{x \in \mathbb{R}^n : \Lambda_\mu^*(x) > 2(p+2)t\})\, dt \\
&= \int_0^\infty pt^{p-1}(1 - \mu(B_{2(p+2)t}(\mu)))\, dt \\
&\leqslant t_n^p + \int_{t_n}^\infty pt^{p-1}\frac{1}{t^{p+1}}\, dt \leqslant t_n^p + \frac{p}{t_n}.
\end{aligned}$$

It follows that $\|\Lambda_\mu^*\|_{L^p(\mu)} \leqslant cpt_n = cpn\ln n$, for some absolute constant $c > 0$. □

Remark 3.7. The upper bound of Theorem 1.2 is sharp: a computation in [7] shows that

$$\|\Lambda_{\mu_{D_n}}^*\|_1 \approx \|\Lambda_{\mu_{D_n}}^*\|_2 \approx n\ln n$$

where D_n is the centered Euclidean ball of volume 1 in $\mathbb{R}^n$ and μ_{D_n} is the uniform measure on D_n. On the other hand, it was proved in [6] that if μ is a log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$, then

$$\int_{\mathbb{R}^n} e^{-\Lambda_\mu^*(x)}\, d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right).$$

From Jensen's inequality we immediately get

$$\|\Lambda_\mu^*\|_1 = \int_{\mathbb{R}^n} \Lambda_\mu^*(x)\, d\mu(x) \geqslant cn/L_\mu^2 \geqslant c_1 n.$$

This lower bound is also optimal, as one can check from the example of the uniform measure on the cube $C_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Summarizing, for any centered log-concave probability measure μ on $\mathbb{R}^n$ we have that

$$c_1 n \leqslant \|\Lambda_\mu^*\|_{L^1(\mu)} \leqslant \|\Lambda_\mu^*\|_{L^2(\mu)} \leqslant c_2 n\ln n,$$

where $c_1, c_2 > 0$ are absolute constants.

4 Applications and further remarks

In this section we provide a number of applications of our approach. In particular, Theorem 4.2 establishes an asymptotically best possible uniform lower threshold for the expect measure of random polytopes with vertices that have a log-concave distribution.

4.1 Uniform thresholds for the measure of random polytopes

In this subsection we prove Theorem 1.3. We start with a short overview of related results that should be compared with our new bound. Uniform upper and lower thresholds were established by Chakraborti, Tkocz and Vritsiou in [9] in the case where μ is an even log-concave or κ-concave probability measure supported on a convex body K in $\mathbb{R}^n$. Assuming that μ is log-concave and $X_1, X_2, \dots$ are independent random points distributed according to μ, for any $n < N \leqslant \exp(c_1 n/L_\mu^2)$ we have that

$$\mathbb{E}_{\mu^N}\left(|K_N|/|K|\right) \leqslant \exp\left(-c_2 n/L_\mu^2\right), \tag{4.1}$$

where $K_N = \mathrm{conv}\{X_1, \dots, X_N\}$ and $c_1, c_2 > 0$ are absolute constants. In the same work it is shown that if μ is assumed κ-concave then for any $M \geqslant C$ and any $N \geqslant \exp\left(\frac{1}{\kappa}(\ln n + 2\ln M)\right)$ we have that

$$\mathbb{E}_{\mu^N}\left(|K_N|/|K|\right) \geqslant 1 - 1/M, \tag{4.2}$$

where $C > 0$ is an absolute constant. Afterwards, the same question was studied in [6] for 0-concave, i.e. log-concave, probability measures. The upper threshold in [6] states that there exists an absolute constant $c > 0$ such that if $N_1(n) = \exp(cn/L_n^2)$ then

$$\sup_\mu \left(\sup\left\{ \mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant N_1(n) \right\}\right) \longrightarrow 0 \tag{4.3}$$

as $n \to \infty$, where the first supremum is over all log-concave probability measures μ on $\mathbb{R}^n$. Regarding the lower threshold, it was first proved in [6] that, for any $\delta \in (0,1)$,

$$\inf_\mu \left(\inf\left\{ \mathbb{E}_{\mu^N}\left[\mu((1+\delta)K_N)\right] : N \geqslant \exp\left(C\delta^{-1}\ln(2/\delta)\, n\ln n\right) \right\}\right) \longrightarrow 1 \tag{4.4}$$

as $n \to \infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $C > 0$ is an absolute constant. Using Lemma 3.1, from (4.4) one can deduce that

$$\inf_\mu \left(\inf\left\{ \mathbb{E}\left[\mu(K_N)\right] : N \geqslant \exp(C(n\ln n)^2 u(n)) \right\}\right) \longrightarrow 1 \tag{4.5}$$

as $n \to \infty$, where $C > 0$ is an absolute constant, the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $u(n)$ is any function with $u(n) \to \infty$ as $n \to \infty$.

The proof of (4.4) is based on Proposition 3.2. We can obtain a variant of this fact, with a different proof and a slightly better dependence on δ.

Proposition 4.1. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any $\delta \in \left(\frac{3}{n}, 1\right)$ and any $t \geqslant \frac{c_1}{\delta} n\ln n$ we have that*

$$\mu((1+\delta)Z_t^+(\mu)) \geqslant 1 - e^{-c_2\delta t}$$

where $c_1, c_2 > 0$ are absolute positive constants.

Proof. Proposition 2.6 shows that for any $m \geqslant 1$ and any $\eta \in (0,1)$ we have that

$$B_m(\mu) \subseteq (1+\eta) Z_{cm/\eta}^+(\mu)$$

where $c > 0$ is an absolute constant. Now, from Lemma 3.5 we know that if $s \geqslant n\ln n$ then

$$(1-\eta)R_s(\mu) \subseteq B_{g(s,\eta)}$$

where $g(s,\eta) \leqslant 2s + n\ln(1/\eta)$. Assuming that $\frac{1}{n} < \eta < \frac{1}{3}$ we see that $g(s,\eta) \leqslant 3s$ and hence

$$R_s(\mu) \subseteq \frac{1}{1-\eta} B_{3s} \subseteq \frac{1+\eta}{1-\eta} Z^+_{3cs/\eta}(\mu) \subseteq (1+3\eta) Z^+_{3cs/\eta}(\mu)$$

for every $s \geqslant n\ln n$, because $\eta < \frac{1}{3}$ implies that $\frac{1+\eta}{1-\eta} \leqslant 1+3\eta$. Since $s \geqslant 5n$, we may also apply Proposition 2.1 to get $\mu(R_s(\mu)) \geqslant 1 - e^{-s/4}$. It follows that

$$\mu((1+3\eta) Z^+_{3cs/\eta}(\mu)) \geqslant 1 - e^{-s/4}$$

for every $s \geqslant n\ln n$ and any $\eta \in (1/n, 1/3)$. Setting $\delta = 3\eta$ and $t = 3cs/\eta = 9cs/\delta$ we get the assertion of the proposition. □

Having established Proposition 4.1 and following the proofs of Theorem 5.5 and Theorem 5.8 from [6] we can check that

$$\inf_{\mu} \Big(\inf \Big\{ \mathbb{E}_{\mu^N} \big[\mu((1+\delta)K_N) \big] : N \geqslant \exp\big(C\delta^{-1} n \ln n \big) \Big\} \Big) \longrightarrow 1 \tag{4.6}$$

and then

$$\inf_{\mu} \Big(\inf \Big\{ \mathbb{E} \big[\mu(K_N) \big] : N \geqslant \exp(Cn^2 (\ln n) u(n)) \Big\} \Big) \longrightarrow 1 \tag{4.7}$$

as $n \to \infty$. However, using directly the family $\{T_t(\mu)\}_{t>0}$ of floating bodies of μ instead of the family $\{Z_t^+(\mu)\}_{t>0}$ of centroid bodies of μ, as well as the comparison of the families $\{T_t(\mu)\}_{t>0}$ and $\{B_t(\mu)\}_{t>0}$ from Section 2, we can give an alternative proof of the uniform lower threshold with an optimal dependence on the dimension.

Theorem 4.2. *There exists an absolute constant $C > 0$ such that*

$$\inf_{\mu} \Big(\inf \Big\{ \mathbb{E}_{\mu^N} \big[\mu(K_N) \big] : N \geqslant \exp(Cn\ln n) \Big\} \Big) \longrightarrow 1$$

as $n \to \infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$.

Proof. Let μ be a log-concave probability measure μ on $\mathbb{R}^n$. Since the expectation $\mathbb{E}_{\mu^N}\big[\mu(K_N)\big]$ is a affinely invariant quantity, we may assume that μ is isotropic. According to Proposition 2.7, there exists $s_0 \geqslant 1$ such that, for any $s \geqslant s_0$,

$$B_s(\mu) \subseteq T_{s+3\ln s}(\mu).$$

Let $t_n := n\ln n$. In the proof of Theorem 1.2 we saw that $\mu(B_{10t}(\mu)) > 1 - 1/t^3$ for all $t \geqslant t_n$, and hence

$$\mu(T_{10t+3\ln(10t)}(\mu)) > 1 - 1/t^3$$

for all $t \geqslant t_n$. By the definition of the family $\{T_t(\mu)\}_{t>0}$, for any $x \in T_{10t+3\ln(10t)}(\mu)$ we have

$$\varphi_\mu(x) \geqslant e^{-10t - 3\ln(10t)} = (10t)^{-3} e^{-10t}.$$

We use the following standard lemma (which is stated in this form in [9, Lemma 3]; for a proof see [12] or [14, Lemma 4.1]): For every Borel subset A of $\mathbb{R}^n$ we have that

$$1 - \mu^N(K_N \supseteq A) \leqslant 2\binom{N}{n} \Big(1 - \inf_{x\in A} \varphi_\mu(x) \Big)^{N-n}.$$

Therefore,

$$\mathbb{E}_{\mu^N}[\mu(K_N)] \geqslant \mu(A)\left(1-2\binom{N}{n}\left(1-\inf_{x\in A}\varphi_\mu(x)\right)^{N-n}\right).$$

Setting $A = T_{10t+3\ln(10t)}(\mu)$ we get

$$\begin{aligned}\mu^N\Big(K_N \supseteq T_{10t+3\ln(10t)}(\mu)\Big) &\geqslant 1-2\binom{N}{n}\left[1-(10t)^{-3}e^{-10t}\right]^{N-n}\\ &\geqslant 1-\left(\frac{2eN}{n}\right)^n \exp\left(-(N-n)(10t)^{-3}e^{-10t}\right).\end{aligned}$$

This last quantity tends to 1 as $n\to\infty$ if

$$(10t)^3 n\ln(4eN/n) < (N-n)e^{-10t}, \tag{4.8}$$

and we easily check that (4.8) holds true if $t \geqslant t_n$ and $N \geqslant \exp(Ct)$ for a large enough absolute constant $C>0$. Therefore, if $N \geqslant \exp(Cn\ln n)$ we see that

$$\begin{aligned}\mathbb{E}_{\mu^N}\left[\mu(K_N)\right] &\geqslant \mu(T_{10t_n+3\ln(10t_n)}(\mu)) \times \mu^N\Big(K_N \supseteq T_{10t_n+3\ln(10t_n)}(\mu)\Big)\\ &\geqslant \mu(B_{10t_n}(\mu)) \times \mu^N\Big(K_N \supseteq T_{10t_n+3\ln(10t_n)}(\mu)\Big)\\ &\geqslant \left(1-t_n^{-3}\right)\left[1-\left(\frac{2eN}{n}\right)^n \exp\left(-(N-n)(10t_n)^{-3}e^{-10t_n}\right)\right] \longrightarrow 1\end{aligned}$$

as $n\to\infty$. □

4.2 Distribution of the half-space depth

It was proved in [6] that if μ is a log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$, then

$$\exp(-c_1 n) \leqslant \mathbb{E}_\mu(\varphi_\mu) \leqslant \exp\left(-c_2 n/L_\mu^2\right) \leqslant \exp(-c_3 n)$$

where L_μ is the isotropic constant of μ and $c_i>0$, $n_0\in\mathbb{N}$ are absolute constants. In this subsection we discuss the question to determine the values of $p>0$ for which $\mathbb{E}_\mu(\varphi_\mu^{-p})$ is finite. Brazitikos and Chasapis have shown in [5, Proposition 3.2] that in the 1-dimensional case one has $\mathbb{E}(\varphi_\mu^{-p}) \leqslant 2^p/(1-p) < \infty$ for all $0<p<1$ and any probability measure μ on $\mathbb{R}$.

A simple computation with the standard Gaussian measure γ_n on $\mathbb{R}^n$ shows that some restriction on p cannot be avoided. Using the rotational invariance of γ_n we easily check that

$$\varphi_{\gamma_n}(x) = 1-\Phi(|x|) = \frac{1}{\sqrt{2\pi}}\int_{|x|}^\infty e^{-t^2/2}dt$$

where $|x|$ denotes Euclidean norm. This implies that $\varphi_{\gamma_n}(x) \leqslant \frac{1}{\sqrt{2\pi}|x|}e^{-|x|^2/2}$, and hence

$$J_{\gamma_n}(p) := \int_{\mathbb{R}^n}\frac{1}{\varphi_{\gamma_n}^p(x)}\,d\gamma_n(x) \geqslant (2\pi)^{p/2}\int_{\mathbb{R}^n}|x|^p e^{-(1-p)|x|^2/2}\,dx.$$

It follows that $J_{\gamma_n}(p)<\infty$ for all $0<p<1$ but $J_{\gamma_n}(1)=\infty$.

Our results allow us to show that there exists an absolute constant $c > 0$ such that

$$J_\mu(p) := \int_{\mathbb{R}^n} \frac{1}{\varphi_\mu^p(x)}\, d\mu(x) < \infty$$

for any $0 < p \leqslant c/n$ and every log-concave probability measure μ on $\mathbb{R}^n$.

Proof of Theorem 1.4. We may assume that μ is centered. The theorem follows immediately if we combine Theorem 1.1 with the inequality

$$\Lambda_\mu^*(x) \geqslant \ln\left(\frac{\varepsilon}{(2\varphi_\mu(x))^{1-\varepsilon}}\right).$$

for every $x \in \mathrm{supp}(\mu)$ and any $\varepsilon \in (0,1)$ (this is a result of Brazitikos and Chasapis from [5] that we have already used in the proof of Proposition 2.7). Choosing $\varepsilon = 1/2$ we get

$$\frac{1}{2^{3/2}\varphi_\mu(x)^{1/2}} \leqslant e^{\Lambda_\mu^*(x)}$$

and hence

$$J_\mu(p) := \int_{\mathbb{R}^n} \frac{1}{\varphi_\mu^p(x)}\, d\mu(x) \leqslant 2^{3p}\mathbb{E}\left[e^{2p\Lambda_\mu^*(x)}\right] < \infty$$

if $2p \leqslant c/n$ where $c > 0$ is the absolute constant from Theorem 1.1. □

4.3 Affine surface area

We close this article with some remarks on the connection of the integrability properties of Λ_μ^* with the notion of affine surface area. Let us first consider a convex body K in $\mathbb{R}^n$. The affine surface area of K is defined by

$$\mathrm{as}(K) = \int_{\partial(K)} \kappa(x)^{\frac{1}{n+1}}\, d\mu_{\partial(K)}(x),$$

where $\kappa(x)$ is the generalized Gauss-Kronecker curvature at x and $\mu_{\partial(K)}$ is the surface measure on $\partial(K)$ (see [22] and the references therein). The affine isoperimetric inequality states that

$$\left(\frac{\mathrm{as}(K)}{\mathrm{as}(B_2^n)}\right)^{n+1} \leqslant \left(\frac{|K|}{|B_2^n|}\right)^{n-1}$$

with equality if and only if K is an ellipsoid (see [24, Section 10.5]). Using the fact that $\mathrm{as}(B_2^n) = n|B_2^n|$ we see that if $|K| = 1$ then $\mathrm{as}(K) \leqslant c_1$, where $c_1 > 0$ is an absolute constant. It is not hard to check that, for every $\delta \in (0,1/2)$, the floating body

$$K_\delta = \bigcap\{H^+ : H^+ \text{ is a closed half-space with } |K \cap H^-| = \delta\},$$

where H^- is the complementary half-space of H^+ satisfies

$$K_\delta = \{x \in \mathbb{R}^n : \varphi_{\mu_K}(x) \geqslant \delta\} = T_{\ln(1/\delta)}(\mu_K).$$

Schütt and Werner proved in [25] that for every convex body K in $\mathbb{R}^n$ one has that

$$\lim_{\delta\to 0} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}} = \frac{1}{2}\left(\frac{n+1}{\omega_{n-1}}\right)^{\frac{2}{n+1}} \mathrm{as}(K).$$

In particular, if $|K|=1$ then there exists $\delta_0>0$ such that if $0<\delta<\delta_0$ then

$$1-|K_\delta|\leqslant c_2 n\delta^{\frac{2}{n+1}}$$

or, equivalently, there exists $s_0>0$ such that

$$\sup\{e^{\frac{2s}{n+1}}(1-\mu_K(T_s(\mu_K))):s\geqslant s_0\}\leqslant c_2 n. \tag{4.9}$$

Taking also into account the fact that $T_s(\mu_K)\subseteq B_s(\mu_K)$ for every $s>0$, applying (4.9) one can give an alternative proof of the fact that

$$\int_K \exp(\kappa\Lambda^*_{\mu_K}(x))\,dx<\infty$$

for all $\kappa<\frac{2}{n+1}$. The details appear in [15, Theorem 6.5]. Theorem 1.5 is an analogue of (4.9) in the more general setting of log-concave probability measures.

Proof of Theorem 1.5. Let μ be a log-concave probability measure on $\mathbb{R}^n$. We may also assume that μ is centered. From Proposition 2.7 we know that there exists $s_0\geqslant 1$ such that $B_s(\mu)\subseteq T_{s+3\ln s}(\mu)\subseteq T_{2s}(\mu)$ for all $s\geqslant s_0$, In the proof of Theorem 1.1 we saw that if $t\geqslant n\ln n$ then $\mu(B_{nt}(\mu))\geqslant 1-e^{-t/8}$. Combining the above we see that if $s\geqslant 2n^2\ln n$ then

$$\mu(T_s(\mu))\geqslant\mu(B_{s/2}(\mu))\geqslant 1-e^{-s/(16n)},$$

which shows that $e^{s/(16n)}(1-\mu(T_s(\mu)))\leqslant 1$. It follows that

$$\sup\{e^{s/(16n)}(1-\mu(T_s(\mu))):s>0\}\leqslant\exp(2n^2\ln n/(16n))$$

and the theorem follows with $c=1/16$ and $c_n=\exp(n\ln n/8)$. $\square$

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