Threshold for the expected measure of random polytopes

Silouanos Brazitikos, Apostolos Giannopoulos and Minas Pafis

Abstract

Let μ be a log-concave probability measure on $\mathbb{R}^n$ and for any $N > n$ consider the random polytope $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$, where $X_1, X_2, \ldots$ are independent random points in $\mathbb{R}^n$ distributed according to μ. We study the question if there exists a threshold for the expected measure of K_N. Our approach is based on the Cramer transform Λ_μ^* of μ. We examine the existence of moments of all orders for Λ_μ^* and establish, under some conditions, a sharp threshold for the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N: it is close to 0 if $\ln N \ll \mathbb{E}_\mu(\Lambda_\mu^*)$ and close to 1 if $\ln N \gg \mathbb{E}_\mu(\Lambda_\mu^*)$. The main condition is that the parameter $\beta(\mu) = \mathrm{Var}_\mu(\Lambda_\mu^*)/(\mathbb{E}_\mu(\Lambda_\mu^*))^2$ should be small.

1 Introduction

We study the question how to obtain a threshold for the expected measure of a random polytope defined as the convex hull of independent random points with a log-concave distribution. The general formulation of the problem is the following. Given a log-concave probability measure μ on $\mathbb{R}^n$, let $X_1, X_2, \ldots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ define the random polytope

$$K_N = \mathrm{conv}\{X_1, \ldots, X_N\}.$$

Then, consider the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N, where $\mu^N = \mu \otimes \cdots \otimes \mu$ (N times). This is an affinely invariant quantity, so we may assume that μ is centered, i.e. the barycenter of μ is at the origin.

Given $\delta \in (0, 1)$ we say that μ satisfies a "δ-upper threshold" with constant ϱ_1 if

$$(1.1) \qquad \sup\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant \exp(\varrho_1 n)\} \leqslant \delta$$

and that μ satisfies a "δ-lower threshold" with constant ϱ_2 if

$$(1.2) \qquad \inf\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \geqslant \exp(\varrho_2 n)\} \geqslant 1 - \delta.$$

Then, we define $\varrho_1(\mu, \delta) = \sup\{\varrho_1 : (1.1) \text{ holds true}\}$ and $\varrho_2(\mu, \delta) = \inf\{\varrho_2 : (1.2) \text{ holds true}\}$. Our main goal is to obtain upper bounds for the difference

$$\varrho(\mu, \delta) := \varrho_2(\mu, \delta) - \varrho_1(\mu, \delta)$$

for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

One may also consider a sequence $\{\mu_n\}_{n=1}^\infty$ of log-concave probability measures μ_n on $\mathbb{R}^n$. Then, we say that $\{\mu_n\}_{n=1}^\infty$ exhibits a "sharp threshold" if there exists a sequence $\{\delta_n\}_{n=1}^\infty$ of positive reals such that $\delta_n \to 0$ and $\varrho(\mu_n, \delta_n) \to 0$ as $n \to \infty$. This terminology may be used to describe a variety of results that have been obtained for specific sequences of measures (in most cases, product measures or rotationally invariant measures). In Section 2 we provide a non-exhaustive list of contributions to this topic; starting with the classical work [14] of Dyer, Füredi and McDiarmid, which concerns the uniform measure on the cube, most of them establish a sharp threshold.

1

Our aim is to describe a general approach to the problem, working with an arbitrary log-concave probability measure μ on $\mathbb{R}^n$. Our approach is based on the Cramer transform of μ. Recall that the logarithmic Laplace transform of μ is defined by

$$\Lambda_\mu(\xi) = \ln \left(\int_{\mathbb{R}^n} e^{\langle \xi, z \rangle} d\mu(z) \right), \qquad \xi \in \mathbb{R}^n$$

and the Cramer transform of μ is the Legendre transform of Λ_μ, defined by

$$\Lambda_\mu^*(x) = \sup_{\xi \in \mathbb{R}^n} \left\{ \langle x, \xi \rangle - \Lambda_\mu(\xi) \right\}, \qquad x \in \mathbb{R}^n.$$

For every $t > 0$ we set

$$B_t(\mu) := \{x \in \mathbb{R}^n : \Lambda_\mu^*(x) \leqslant t\}$$

and for any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x. Then we consider the function φ_μ, called Tukey's half-space depth, defined by

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}.$$

We refer the reader to the survey article of Nagy, Schütt and Werner [23] for an extensive and comprehensive survey on Tukey's half-space depth, with an emphasis on its connections with convex geometry, and many references. From the definition of Λ_μ^* one can easily check that for every $x \in \mathbb{R}^n$ we have $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$ (see Lemma 3.1 in Section 3 below). In particular, for any $t > 0$ and for all $x \notin B_t(\mu)$ we have that $\varphi_\mu(x) \leqslant \exp(-t)$. A main idea, which appears in all the previous works on this topic, is to show that φ_μ is almost constant on the boundary $\partial(B_t(\mu))$ of $B_t(\mu)$. Our first main result shows that this is true, in general, if $\mu = \mu_K$ is the uniform measure on a centered convex body of volume 1 in $\mathbb{R}^n$.

Theorem 1.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t > 0$ we have that*

$$\inf\{\varphi_{\mu_K}(x) : x \in B_t(\mu_K)\} \geqslant \frac{1}{10} \exp(-t - 2\sqrt{n}).$$

This implies that

$$\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda^*(x) \leqslant \omega_{\mu_K}(x)$$

for every $x \in \mathbb{R}^n$, where $\omega_{\mu_K}(x) = \ln \left(\frac{1}{\varphi_{\mu_K}(x)} \right)$.

Theorem 1.1 may be viewed as a version of Cramér's theorem (see [13]) for random vectors uniformly distributed in convex bodies. We present the proof in Section 3. It exploits techniques from the theory of large deviations and a theorem of Nguyen [25] (proved independently by Wang [31]; see also [16]) which is exactly the ingredient that forces us to consider only uniform measures on convex bodies. It seems harder to prove, if true, an analogous estimate for any centered log-concave probability measure μ on $\mathbb{R}^n$; this is a basic question that our work leaves open.

The second step in our approach is to consider, for any centered log-concave probability measure μ on $\mathbb{R}^n$, the parameter

(1.3)
$$\beta(\mu) = \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2}$$

provided that

$$\|\Lambda_\mu^*\|_{L^2(\mu)} = \left(\mathbb{E}_\mu\big((\Lambda_\mu^*)^2 \big) \right)^{1/2} < \infty.$$

Roughly speaking, the plan is the following: provided that φ_μ is "almost constant" on $\partial(B_t(\mu))$ for all $t > 0$ and that $\beta(\mu) = o_n(1)$, one can establish a "sharp threshold" for the expected measure of K_N with

$$\varrho_2 \approx \varrho_1 \approx \|\Lambda_\mu^*\|_{L^1(\mu)} = \mathbb{E}_\mu(\Lambda_\mu^*).$$

2

We make these ideas more precise in Section 5 where we also illustrate them with a number of examples. Note that it is not clear in advance that Λ_μ^* has bounded second or higher order moments, which is necessary so that $\beta(\mu)$ would be well-defined. We study this question in Section 4 where we obtain an affirmative answer in the case of the uniform measure on a convex body. In fact we cover the more general case of κ-concave probability measures, $\kappa \in (0, 1/n]$, which are supported on a centered convex body.

Theorem 1.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\operatorname{supp}(\mu) = K$. Then,*

$$\int_{\mathbb{R}^n} e^{\frac{\kappa \Lambda_\mu^*(x)}{2}} d\mu(x) < \infty.$$

In particular, for all $p \geqslant 1$ we have that $\mathbb{E}_\mu\big((\Lambda_\mu^*(x))^p\big) < \infty$.

The method of proof of Theorem 1.2 gives in fact reasonable upper bounds for $\|\Lambda_\mu^*\|_{L^p(\mu)}$. In particular, if we assume that $\mu = \mu_K$ is the uniform measure on a centered convex body then we obtain a sharp two sided estimate for the most interesting case where $p = 1$ or 2.

Theorem 1.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n \geqslant 2$. Then,*

$$c_1 n/L_{\mu_K}^2 \leqslant \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)} \leqslant \|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)} \leqslant c_2 n \ln n,$$

where L_{μ_K} is the isotropic constant of the uniform measure μ_K on K and $c_1, c_2 > 0$ are absolute constants.

The left-hand side inequality of Theorem 1.3 follows easily from one of the main results in [9]. Both the lower and the upper bound are of optimal order with respect to the dimension. This can be seen e.g. from the example of the uniform measure on the cube or the Euclidean ball (see Section 5), respectively.

Besides Theorem 1.2, we show in Section 4 that Λ_μ^* has finite moments of all orders in the following cases:

(i) If μ is a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure or a product of such measures.

(ii) If μ is a centered log-concave probability measure on $\mathbb{R}^n$ and there exists a function $g : [1, \infty) \to [1, \infty)$ with $\lim_{t \to \infty} g(t)/\ln(t+1) = +\infty$ such that $Z_t^+(\mu) \supseteq g(t) Z_2^+(\mu)$ for all $t \geqslant 2$, where $\{Z_t^+(\mu)\}_{t \geqslant 1}$ is the family of one-sided L_t-centroid bodies of μ.

Again, it seems harder to prove, if true, an analogous result for any centered log-concave probability measure μ on $\mathbb{R}^n$; this is a second basic question that our work leaves open.

In Section 5 we describe the approach to the main problem and show how one can use the previous results to obtain bounds for $\varrho(\mu, \delta)$. We also clarify the role of the parameter $\beta(\mu)$. One would hope that $\beta(\mu)$ is small as the dimension increases, e.g. $\beta(\mu) \leqslant c/\sqrt{n}$. If so, then the next general result provides satisfactory lower bounds for $\varrho_1(\mu, \delta)$.

Theorem 1.4. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Assume that $\beta(\mu) < 1/8$ and $8\beta(\mu) < \delta < 1$. If $n/L_\mu^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu)}$ where L_μ is the isotropic constant of μ, then*

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt{8\beta(\mu)/\delta}\right) \frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

We are able to give satisfactory upper bounds for $\varrho_2(\mu, \delta)$ in the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$.

Theorem 1.5. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\beta(\mu_K) < 1/2$ and $2\beta(\mu_K) < \delta < 1$. If $n/L_{\mu_K}^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu_K)}$ then*

$$\varrho_2(\mu_K, \delta) \leqslant \left(1 + \sqrt{8\beta(\mu_K)/\delta}\right) \frac{\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*)}{n}.$$

3

Combining these two results we see that, provided that $\beta(\mu_K)$ is small compared to a fixed $\delta \in (0, 1)$, we have a threshold of the order

$$\varrho(\mu_K, \delta) \leqslant \frac{c}{n} \sqrt{\frac{\mathrm{Var}_{\mu_K}(\Lambda^*_{\mu_K})}{\delta}}.$$

The above discussion leaves open a third basic question: to estimate

$$\beta_n := \sup\{\beta(\mu) : \mu \text{ is a centered log-concave probability measure on } \mathbb{R}^n\}.$$

We illustrate the method that we develop in this work with a number of examples. We consider first the standard examples of the uniform measure on the unit cube and the Gaussian measure. As a direct consequence of our results, in both cases we obtain a bound $\varrho(\mu, \delta) \leqslant c(\delta)/\sqrt{n}$ for the threshold, where $c(\delta) > 0$ is a constant depending on δ. Finally, we examine the case of the uniform measure on the Euclidean ball D_n of volume 1 in $\mathbb{R}^n$ and obtain the following sharp threshold.

Theorem 1.6. *Let D_n be the centered Euclidean ball of volume 1 in $\mathbb{R}^n$. Then, the sequence $\mu_n := \mu_{D_n}$ exhibits a sharp threshold with $\varrho(\mu_n, \delta) \leqslant \frac{c}{\sqrt{\delta n}}$ and e.g. in the case where n is even we have that*

$$\mathbb{E}_{\mu_n}(\Lambda^*_{\mu_n}) = \frac{(n+1)}{2} H_{\frac{n}{2}} + O(\sqrt{n})$$

as $n \to \infty$, where $H_m = \sum_{k=1}^{m} \frac{1}{k}$.

2 Notation, background information and related literature

In this section we introduce notation and terminology that we use throughout this work, and provide background information on convex bodies and log-concave probability measures. We write $\langle \cdot, \cdot \rangle$ for the standard inner product in $\mathbb{R}^n$ and denote the Euclidean norm by $|\cdot|$. In what follows, B_2^n is the Euclidean unit ball, S^{n-1} is the unit sphere, and σ is the rotationally invariant probability measure on S^{n-1}. Lebesgue measure in $\mathbb{R}^n$ is also denoted by $|\cdot|$. The letters c, c', c_j, c'_j etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \approx b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$.

We refer to Schneider's book [29] for basic facts from the Brunn-Minkowski theory and to the book [2] for basic facts from asymptotic convex geometry. We also refer to [10] for more information on isotropic convex bodies and log-concave probability measures.

2.1 Log-concave probability measures

A convex body in $\mathbb{R}^n$ is a compact convex set $K \subset \mathbb{R}^n$ with non-empty interior. We say that K is centrally symmetric if $-K = K$ and that K is centered if the barycenter $\mathrm{bar}(K) = \frac{1}{|K|} \int_K x \, dx$ of K is at the origin. The Minkowski functional $\|\cdot\|_K$ of a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ is defined for all $x \in \mathbb{R}^n$ by $\|x\|_K = \inf\{s \geqslant 0 : x \in sK\}$ and the support function of K is the function

$$h_K(x) = \sup\{\langle x, y \rangle : y \in K\}, \qquad x \in \mathbb{R}^n.$$

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\ln f$ to it is concave. If f has finite positive integral then there exist constants $A, B > 0$ such that $f(x) \leqslant A e^{-B|x|}$ for all $x \in \mathbb{R}^n$ (see [10, Lemma 2.2.1]). In particular, f has finite moments of all orders. It is known (see [6]) that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$, then μ has a log-concave density f_μ. We say that μ is even if $\mu(-B) = \mu(B)$ for every Borel subset B of $\mathbb{R}^n$ and that μ is centered if

$$\mathrm{bar}(\mu) := \int_{\mathbb{R}^n} \langle x, \xi \rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \xi \rangle f_\mu(x) dx = 0$$

4

for all $\xi \in S^{n-1}$. We shall use the fact that if μ is a centered log-concave probability measure on $\mathbb{R}^k$ then

$$(2.1) \qquad \qquad \|f_\mu\|_\infty \leqslant e^k f_\mu(0).$$

This is a result of Fradelizi from [15]. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that the indicator function $\mathbf{1}_K$ of K is the density of a log-concave measure, the Lebesgue measure on K.

Given $\kappa \in [-\infty, 1/n]$ we say that a measure μ on $\mathbb{R}^n$ is κ-concave if

$$(2.2) \qquad \qquad \mu((1-\lambda)A + \lambda B) \geqslant ((1-\lambda)\mu^\kappa(A) + \lambda\mu^\kappa(B))^{1/\kappa}$$

for all compact subsets A, B of $\mathbb{R}^n$ with $\mu(A)\mu(B) > 0$ and all $\lambda \in (0,1)$. The limiting cases are defined appropriately. For $\kappa = 0$ the right hand side in (2.2) becomes $\mu(A)^{1-\lambda}\mu(B)^\lambda$ (therefore, 0-concave measures are the log-concave measures). In the case $\kappa = -\infty$ the right hand side in (2.2) becomes $\min\{\mu(A), \mu(B)\}$. Note that if μ is κ-concave and $\kappa_1 \leqslant \kappa$ then μ is κ_1-concave.

Next, let $\gamma \in [-\infty, \infty]$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called γ-concave if

$$f((1-\lambda)x + \lambda y) \geqslant ((1-\lambda)f^\gamma(x) + \lambda f^\gamma(y))^{1/\gamma}$$

for all $x, y \in \mathbb{R}^n$ with $f(x)f(y) > 0$ and all $\lambda \in (0,1)$. Again, we define the cases $\gamma = 0, +\infty$ appropriately. Borell [7] studied the relation between κ-concave probability measures and γ-concave functions and showed that if μ is a measure on $\mathbb{R}^n$ and the affine subspace F spanned by the support $\mathrm{supp}(\mu)$ of μ has dimension $\dim(F) = n$ then for every $-\infty \leqslant \kappa < 1/n$ we have that μ is κ-concave if and only if it has a non-negative density $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ and ψ is γ-concave, where $\gamma = \frac{\kappa}{1-\kappa n} \in [-1/n, +\infty)$.

Let μ and ν be two log-concave probability measures on $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function which is defined ν-almost everywhere and satisfies

$$\mu(B) = \nu(T^{-1}(B))$$

for every Borel subset B of $\mathbb{R}^n$. We then say that T pushes forward ν to μ and write $T_*\nu = \mu$. It is easy to see that $T_*\nu = \mu$ if and only if for every bounded Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(x)d\mu(x) = \int_{\mathbb{R}^n} g(T(y))d\nu(y).$$

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x \in \mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}.$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered and $\mathrm{Cov}(\mu) = I_n$, where I_n is the identity $n \times n$ matrix. In this case, $L_\mu = \|f_\mu\|_\infty^{1/n}$. For every μ there exists an affine transformation T such that $T_*\mu$ is isotropic. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 1$. Bourgain [8] established the upper bound $L_n \leqslant c\sqrt[4]{n}\ln n$; later, Klartag, in [21], improved this estimate to $L_n \leqslant c\sqrt[4]{n}$. In a breakthrough work, Chen [12] proved that for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $L_n \leqslant n^\varepsilon$ for every $n \geqslant n_0(\varepsilon)$. Very recently, Klartag and Lehec [22] showed that the hyperplane conjecture and the stronger Kannan-Lovász-Simonovits isoperimetric conjecture hold true up to a factor that is polylogarithmic in the dimension; more precisely, they achieved the bound $L_n \leqslant c(\ln n)^4$, where $c > 0$ is an absolute constant.

2.2   Known results

Several variants of the threshold problem have been studied, starting with the work of Dyer, Füredi and McDiarmid who established in [14] a sharp threshold for the expected volume of random polytopes with vertices uniformly distributed in the discrete cube $E_2^n = \{-1,1\}^n$ or in the solid cube $B_\infty^n = [-1,1]^n$. For example, in the first case, if $\kappa = 2/\sqrt{e}$ then for every $\varepsilon \in (0,1)$ one has

$$\lim_{n\to\infty} \sup \left\{2^{-n}\mathbb{E}|K_N| : N \leqslant (\kappa - \varepsilon)^n\right\} = 0$$

and

$$\lim_{n\to\infty} \inf \left\{2^{-n}\mathbb{E}|K_N| : N \geqslant (\kappa + \varepsilon)^n\right\} = 1.$$

A similar result holds true for the expected volume of random polytopes with vertices uniformly distributed in the cube B_∞^n; the corresponding value of the constant κ is $\kappa = 2\pi/e^{\gamma + 1/2}$, where γ is Euler's constant. In the terminology of the introduction, this last result is equivalent to $\varrho(\mu_{B_\infty^n}, \delta) = O_\delta(1)$ for any fixed value of $\delta \in \left(0, \frac{1}{2}\right)$.

   Further sharp thresholds for the volume of various classes of random polytopes have been given. In [18] a threshold for $\mathbb{E}_{\mu^N}|K_N|/(2\alpha)^n$ was established for the case where X_i have independent identically distributed coordinates supported on a bounded interval $[-\alpha, \alpha]$ under some mild additional assumptions. The articles [27] and [3], [4] address the same question for a number of cases where X_i have rotationally invariant densities. Exponential in the dimension upper and lower thresholds are obtained in [17] for the case where X_i are uniformly distributed in a simplex.

   Upper and lower thresholds were obtained recently by Chakraborti, Tkocz and Vritsiou in [11] for some general families of distributions. If μ is an even log-concave probability measure supported on a convex body K in $\mathbb{R}^n$ and if $X_1, X_2, \ldots$ are independent random points distributed according to μ, then for any $n < N \leqslant \exp(c_1 n/L_\mu^2)$ we have that

$$(2.3) \qquad \frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|} \leqslant \exp\left(-c_2 n/L_\mu^2\right),$$

where $c_1, c_2 > 0$ are absolute constants. A lower threshold is also established in [11] for the case where μ is an even κ-concave measure on $\mathbb{R}^n$ with $0 < \kappa < 1/n$, supported on a convex body K in $\mathbb{R}^n$. If $X_1, X_2, \ldots$ are independent random points in $\mathbb{R}^n$ distributed according to μ and $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ as before, then for any $M \geqslant C$ and any $N \geqslant \exp\left(\frac{1}{\kappa}(\log n + 2\log M)\right)$ we have that

$$(2.4) \qquad \frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|} \geqslant 1 - \frac{1}{M},$$

where $C > 0$ is an absolute constant.

   Analogues of these results in the setting of the present work were obtained in [9] for 0-concave, i.e. log-concave, probability measures. There exists an absolute constant $c > 0$ such that if $N_1(n) = \exp(cn/L_n^2)$ then

$$\sup_\mu \left(\sup \left\{ \mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant N_1(n) \right\} \right) \longrightarrow 0$$

as $n \to \infty$, where the first supremum is over all log-concave probability measures μ on $\mathbb{R}^n$. On the other hand, if $\delta \in (0,1)$ then,

$$\inf_\mu \left(\inf \left\{ \mathbb{E}_{\mu^N}[\mu((1+\delta)K_N)] : N \geqslant \exp\left(C\delta^{-1}\ln(2/\delta)\, n\ln n\right) \right\} \right) \longrightarrow 1$$

as $n \to \infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $C > 0$ is an absolute constant.

   It should be noted that an exponential in the dimension lower threshold is not possible in full generality. For example, in the case where X_i are uniformly distributed in the Euclidean ball one needs $N \geqslant \exp(cn\ln n)$

points so that the volume of a random K_N will be significantly large. Thus, apart from the constants depending on δ, the lower threshold above is sharp. However, it provides a weak threshold in the sense that we estimate the expectation $\mathbb{E}_{\mu^N}\big(\mu(1+\delta)K_N\big)$ (for an arbitrarily small but positive value of δ) while we would like to have a similar result for $\mathbb{E}_{\mu^N}\big[\mu(K_N)\big]$. It is shown in [9] that there exists an absolute constant $C > 0$ such that

$$\inf_{\mu}\Big(\inf\big\{\mathbb{E}\left[\mu(K_N)\right] : N \geqslant \exp(C(n\ln n)^2 u(n))\big\}\Big) \longrightarrow 1$$

as $n \to \infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $u(n)$ is any function with $u(n) \to \infty$ as $n \to \infty$.

3 Estimates for the half-space depth

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ with density $f := f_\mu$. Recall that the logarithmic Laplace transform of μ on $\mathbb{R}^n$ is defined by

$$\Lambda_\mu(\xi) = \log\Big(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} f(z)dz\Big).$$

It is easily checked by means of Hölder's inequality that Λ_μ is convex and $\Lambda_\mu(0) = 0$. Since $\mathrm{bar}(\mu) = 0$, Jensen's inequality shows that

$$\Lambda_\mu(\xi) = \log\Big(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} f(z)dz\Big) \geqslant \int_{\mathbb{R}^n} \langle \xi, z\rangle f(z)dz = 0$$

for all ξ. Therefore, Λ_μ is a non-negative function. One can check that the set $A(\mu) = \{\Lambda_\mu < \infty\}$ is open and Λ_μ is C^∞ and strictly convex on $A(\mu)$.

We also define

$$\Lambda_\mu^*(x) = \sup_{\xi \in \mathbb{R}^n} \{\langle x, \xi\rangle - \Lambda_\mu(\xi)\}.$$

In other words, Λ_μ^* is the Legendre transform of Λ_μ: recall that given a convex function $g : \mathbb{R}^n \to (-\infty, \infty]$, the Legendre transform $\mathcal{L}(g)$ of g is defined by

$$\mathcal{L}(g)(x) := \sup_{\xi \in \mathbb{R}^n} \{\langle x, \xi\rangle - g(\xi)\}.$$

The function Λ_μ^* is called the Cramer transform of μ. For every $t > 0$ we define the convex set

$$B_t(\mu) := \{x \in \mathbb{R}^n : \Lambda_\mu^*(x) \leqslant t\}.$$

For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x. Then we define

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}.$$

The function φ_μ is called Tukey's half-space depth. Our aim is to give upper and lower bounds for $\varphi_\mu(x)$ when $x \in \partial(B_t(\mu))$, $t > 0$.

Lemma 3.1. *Let μ be a Borel probability measure on $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we have $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$. In particular, for any $t > 0$ and for all $x \notin B_t(\mu)$ we have that $\varphi_\mu(x) \leqslant \exp(-t)$.*

Proof. Let $x \in \mathbb{R}^n$. We start with the observation that for any $\xi \in \mathbb{R}^n$ the half-space $\{z : \langle z - x, \xi\rangle \geqslant 0\}$ is in $\mathcal{H}(x)$, therefore

$$\varphi_\mu(x) \leqslant \mu(\{z : \langle z, \xi\rangle \geqslant \langle x, \xi\rangle\}) \leqslant e^{-\langle x, \xi\rangle}\mathbb{E}_\mu\big(e^{\langle z, \xi\rangle}\big) = \exp\big(-[\langle x, \xi\rangle - \Lambda_\mu(\xi)]\big),$$

and taking the infimum over all $\xi \in \mathbb{R}^n$ we see that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$, as claimed. $\square$

Next, we would like to obtain a lower bound for $\varphi_\mu(x)$ when $x \in B_t(\mu)$. In the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$, our estimate is the following.

Theorem 3.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t > 0$ we have that*

$$\inf\{\varphi_{\mu_K}(x) : x \in B_t(\mu_K)\} \geqslant \frac{1}{10}\exp(-t - 2\sqrt{n}).$$

The first part of the argument works for any centered log-concave probability measure μ with density f on $\mathbb{R}^n$. For every $\xi \in \mathbb{R}^n$ we define the probability measure μ_ξ with density

$$f_\xi(z) = e^{-\Lambda_\mu(\xi) + \langle \xi, z \rangle} f(z).$$

In the next lemma (see [10, Proposition 7.2.1]) we recall some basic facts for μ_ξ.

Lemma 3.3. *The barycenter of μ_ξ is $x = \nabla\Lambda_\mu(\xi)$ and $\mathrm{Cov}(\mu_\xi) = \mathrm{Hess}\,(\Lambda_\mu)(\xi)$.*

Next, we set

$$\sigma_\xi^2 = \int_{\mathbb{R}^n} \langle z - x, \xi \rangle^2 d\mu_\xi(z).$$

Let $t > 0$. Since $B_t(\mu)$ is convex, in order to give a lower bound for $\inf\{\varphi_\mu(x) : x \in B_t(\mu)\}$ it suffices to give a lower bound for $\mu(H)$, where H is any closed half-space whose bounding hyperplane supports $B_t(\mu)$. In that case,

$$(3.1) \qquad\qquad \mu(H) = \mu(\{z : \langle z - x, \xi \rangle \geqslant 0\})$$

for some $x \in \partial(B_t(\mu))$, with $\xi = \nabla\Lambda_\mu^*(x)$, or equivalently $x = \nabla\Lambda_\mu(\xi)$ (see e.g. Theorem 23.5 and Corollary 23.5.1 in [28]). Note that

$$(3.2) \qquad \mu(\{z : \langle z - x, \xi \rangle \geqslant 0\}) = \int_{\mathbb{R}^n} \mathbf{1}_{[0,\infty)}(\langle z - x, \xi \rangle) f(z)\, dz$$

$$= e^{\Lambda_\mu(\xi)} \int_{\mathbb{R}^n} \mathbf{1}_{[0,\infty)}(\langle z - x, \xi \rangle) e^{-\langle z, \xi \rangle}\, d\mu_\xi(z)$$

$$= e^{\Lambda_\mu(\xi)} e^{-\langle x, \xi \rangle} \int_{\mathbb{R}^n} \mathbf{1}_{[0,\infty)}(\langle z - x, \xi \rangle) e^{-\langle z - x, \xi \rangle}\, d\mu_\xi(z)$$

$$\geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z - x, \xi \rangle \leqslant \sigma_\xi t\})\, dt.$$

From Markov's inequality we see that

$$\mu_\xi(\{z : \langle z - x, \xi \rangle \geqslant 2\sigma_\xi\}) \leqslant \frac{1}{4}.$$

Moreover, since x is the barycenter of μ_ξ, Grünbaum's lemma (see [10, Lemma 2.2.6]) implies that

$$\mu_\xi(\{z : \langle z - x, \xi \rangle \geqslant 0\}) \geqslant \frac{1}{e}.$$

Therefore,

$$(3.3) \qquad \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z - x, \xi \rangle \leqslant \sigma_\xi t\})\, dt \geqslant \int_2^\infty \sigma_\xi e^{-\sigma_\xi t}\left(\frac{1}{e} - \frac{1}{4}\right) dt \geqslant \frac{4 - e}{4e} e^{-2\sigma_\xi}.$$

We would like now an upper bound for $\sup_\xi \sigma_\xi$. We can have this when $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 on $\mathbb{R}^n$, using a theorem of Nguyen [25] (proved independently by Wang [31]; see also [16]).

8

Theorem 3.4. *Let ν be a log-concave probability measure on $\mathbb{R}^n$ with density $g = \exp(-p)$, where p is a convex function. Then,*

$$\mathrm{Var}_\nu(p) \leqslant n.$$

Proof of Theorem 3.2. Set $\mu := \mu_K$. Since $f(z) = \mathbf{1}_K(z)$, the density f_ξ of μ_ξ is proportional to $e^{\langle \xi, z \rangle} \mathbf{1}_K(z)$. Using the fact that

$$\mathbb{E}_{\mu_\xi}(\langle \xi, z \rangle) = \langle \nabla \Lambda_\mu(\xi), \xi \rangle = \langle x, \xi \rangle,$$

from Theorem 3.4 we get that

$$\sigma_\xi^2 = \mathbb{E}_{\mu_\xi}(\langle z - x, \xi \rangle)^2 = \mathrm{Var}_{\mu_\xi}(\langle \xi, z \rangle) \leqslant n.$$

Then, combining (3.1), (3.2) and (3.3), for any bounding hyperplane H of $B_t(\mu)$ we have

$$\mu(H) \geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(0 \leqslant \langle z - x, \xi \rangle \leqslant \sigma_\xi t)\, dt$$

$$\geqslant \frac{4-e}{4e} e^{-\Lambda_\mu^*(x) - 2\sigma_\xi} \geqslant \frac{1}{10} \exp(-t - 2\sqrt{n}),$$

as claimed. $\qquad\square$

Theorem 3.2 shows that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$10\varphi_{\mu_K}(x) \geqslant \exp(-\Lambda_{\mu_K}^*(x) - 2\sqrt{n})$$

for all $x \in \mathbb{R}^n$. Setting

$$(3.4) \qquad \omega_{\mu_K}(x) = \ln\left(\frac{1}{\varphi_{\mu_K}(x)}\right)$$

and taking into account Lemma 3.1 we have the next two-sided estimate.

Corollary 3.5. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $x \in \mathrm{int}(K)$ we have that*

$$(3.5) \qquad \omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda_{\mu_K}^*(x) \leqslant \omega_{\mu_K}(x).$$

Note. A basic question that arises from the results of this section is whether an analogue of (3.5) holds true for any centered log-concave probability measure μ on $\mathbb{R}^n$. This would allow us to apply the next steps of the procedure that our approach suggests to all log-concave probability measures.

4 Moments of the Cramer transform

As explained in the introduction, we would like to know for which centered log-concave probability measures μ on $\mathbb{R}^n$ we have that Λ_μ^* has finite moments of all orders. Our first result provides an affirmative answer in the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$. In fact, the next theorem covers a more general case.

Theorem 4.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,*

$$\int_{\mathbb{R}^n} e^{\frac{\kappa \Lambda_\mu^*(x)}{2}}\, d\mu(x) < \infty.$$

In particular, for all $p \geqslant 1$ we have that $\mathbb{E}_\mu\big((\Lambda_\mu^*(x))^p\big) < \infty$.

9

The proof of Theorem 4.1 is based on the next lemma, which is proved in [11, Lemma 7] in the symmetric case.

Lemma 4.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,*

$$(4.1) \qquad \varphi_\mu(x) \geqslant e^{-2}\kappa(1 - \|x\|_K)^{1/\kappa}$$

for every $x \in K$, where $\|x\|_K$ is the Minkowski functional of K.

Sketch of the proof. We modify the argument from [11, Lemma 7] to cover the not necessarily symmetric case. First, consider the case $0 < \kappa < 1/n$. Let X be a random vector distributed according to μ. Given $\theta \in S^{n-1}$ let $b = h_K(\theta)$ and $a = h_K(-\theta)$. If g_θ is the density of $\langle X, \theta \rangle$ then $g_\theta^{\frac{\kappa}{1-\kappa}}$ is concave on $[-a, b]$, therefore

$$g_\theta(t) \geqslant g_\theta(0) \left(1 - \frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}}$$

for all $t \in [0, b]$. It follows that, for every $0 < s < b$,

$$\mathbb{P}(\langle X, \theta \rangle \geqslant s) = \int_s^b g_\theta(t)\, dt \geqslant g_\theta(0) \int_s^b \left(1 - \frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}} dt = \kappa g_\theta(0) b \left(1 - \frac{s}{b}\right)^{\frac{1}{\kappa}}.$$

Note that g_θ is a centered log-concave density. Therefore, $g_\theta(0) \geqslant e^{-1}\|g_\theta\|_\infty$ by (2.1) and $\|g_\theta\|_\infty b \geqslant \mathbb{P}(\langle X, \theta \rangle \geqslant 0) \geqslant e^{-1}$ by Grünbaum's lemma [10, Lemma 2.2.6], which implies that $g_\theta(0)b \geqslant e^{-2}$. It follows that

$$\mathbb{P}(\langle X, \theta \rangle \geqslant s) = \int_s^b g_\theta(t)\, dt \geqslant e^{-2}\kappa \left(1 - \frac{s}{b}\right)^{\frac{1}{\kappa}}.$$

Now, let $x \in K$. Then $\langle x, \theta \rangle \leqslant \|x\|_K h_K(\theta) = \|x\|_K b$, therefore

$$\mathbb{P}(\langle X, \theta \rangle \geqslant \langle x, \theta \rangle) \geqslant \mathbb{P}(\langle X, \theta \rangle \geqslant \|x\|_K b) \geqslant e^{-2}\kappa \left(1 - \|x\|_K\right)^{\frac{1}{\kappa}}.$$

For the case $\kappa = 1/n$ recall that a $1/n$-concave measure is κ-concave for every $\kappa \in (0, 1/n)$. This means that (4.1) holds true for all $\kappa \in (0, 1/n)$ and letting $\kappa \to 1/n$ we obtain the result. $\qquad\square$

Proof of Theorem 4.1. From Lemma 3.1 we know that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$, or equivalently,

$$e^{\frac{\kappa \Lambda_\mu^*(x)}{2}} \leqslant \frac{1}{\varphi_\mu(x)^{\kappa/2}}$$

for all $x \in K$. From Lemma 4.2 we know that

$$\varphi_\mu(x) \geqslant e^{-2}\kappa(1 - \|x\|_K)^{1/\kappa}$$

for every $x \in K$. It follows that

$$\int_K e^{\frac{\kappa \Lambda_\mu^*(x)}{2}}\, d\mu(x) \leqslant (e^2/\kappa)^{\kappa/2} \int_K \frac{1}{(1 - \|x\|_K)^{1/2}}\, d\mu(x).$$

Recall that the cone probability measure ν_K on the boundary $\partial(K)$ of a convex body K with $0 \in \mathrm{int}(K)$ is defined by

$$\nu_K(B) = \frac{|\{rx : x \in B, 0 \leqslant r \leqslant 1\}|}{|K|}$$

for all Borel subsets B of $\partial(K)$. We shall use the identity

$$\int_{\mathbb{R}^n} g(x)\,dx = n|K| \int_0^\infty r^{n-1} \int_{\partial(K)} g(rx)\,d\nu_K(x)\,dr$$

which holds for every integrable function $g : \mathbb{R}^n \to \mathbb{R}$ (see [24, Proposition 1]). Let f denote the density of μ on K. We write

$$\begin{aligned}
\int_K \frac{1}{(1 - \|x\|_K)^{1/2}} d\mu(x) &= \int_{\mathbb{R}^n} \frac{f(x)}{(1 - \|x\|_K)^{1/2}} \mathbf{1}_K(x)\,dx \\
&= n|K| \int_0^\infty r^{n-1} \int_{\partial(K)} \frac{f(ry)}{(1 - \|ry\|_K)^{1/2}} \mathbf{1}_K(ry)\,d\nu_K(y)\,dr \\
&= n|K| \int_0^1 \frac{r^{n-1}}{\sqrt{1 - r}} \int_{\partial(K)} f(ry)\,d\nu_K(y)\,dr \\
&\leqslant n|K|\|f\|_\infty \int_0^1 \frac{r^{n-1}}{\sqrt{1 - r}}\,dr = n|K|B(n, 1/2)\|f\|_\infty \leqslant c\sqrt{n}\|f\|_\infty < +\infty,
\end{aligned}$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the case of the uniform measure $\mu = \mu_K$ on a centered convex body K of volume 1 in $\mathbb{R}^n$ we see that

$$\int_K \left(\Lambda_{\mu_K}^*(x)/2n\right)^p dx \leqslant (c_1 p)^p \int_K e^{\frac{\Lambda_{\mu_K}^*(x)}{2n}}\,dx \leqslant (c_2 p)^p \sqrt{n},$$

where $c_1, c_2 > 0$ are absolute constants. This gives the following estimate for the moments of $\Lambda_{\mu_K}^*$:

$$\|\Lambda_{\mu_K}^*\|_{L^p(\mu_K)} \leqslant cpn^{1 + \frac{1}{2p}}$$

for all $p \geqslant 1$. However, essentially repeating the argument that we used for Theorem 4.1 we may obtain sharp estimates in the most interesting case $p = 1$ or 2. We need the next lemma.

Lemma 4.3. Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then,

$$\int_0^1 r^{n-1} \ln(1 - r)\,dr = -\frac{1}{n} H_n$$

and

$$\int_0^1 r^{n-1} \ln^2(1 - r)\,dr = \frac{1}{n} H_n^2 - \frac{1}{n} \sum_{k=1}^n \frac{1}{k^2}.$$

Proof. We consider the beta integral

$$B(x, y) = \int_0^1 r^{x-1}(1 - r)^{y-1}\,dr$$

and differentiate it with respect to y. Then, the desired integrals are equal to

$$\frac{\partial}{\partial y} B(x, y)\Big|_{y=1} \qquad \text{and} \qquad \frac{\partial^2}{\partial^2 y} B(x, y)\Big|_{y=1}.$$

We have

$$\frac{\partial}{\partial y} B(x, y) = B(x, y) \left(\frac{\Gamma'(y)}{\Gamma(y)} - \frac{\Gamma'(x + y)}{\Gamma(x + y)} \right) = B(x, y)\big(\psi(y) - \psi(x + y)\big),$$

11

where $\psi(y) = \frac{\Gamma'(y)}{\Gamma(y)}$ is the digamma function. Moreover,

$$\frac{\partial^2}{\partial^2 y} B(x,y) = B(x,y)\big((\psi(y) - \psi(x+y))^2 - (\psi'(y) - \psi'(x+y))\big).$$

Recall (see e.g. [1]) that

$$\psi(n+1) - \psi(1) = H_n := \sum_{k=1}^{n} \frac{1}{k} \qquad \text{and} \qquad \psi'(n) = \sum_{k=n}^{\infty} \frac{1}{k^2}$$

for all $n \geqslant 1$. Therefore,

$$\int_0^1 r^{n-1} \ln(1-r)\, dr = B(n,1)\big(\psi(1) - \psi(n+1)\big) = -\frac{1}{n} H_n,$$

and

$$\int_0^1 r^{n-1} \ln^2(1-r)\, dr = B(n,1)\big((\psi(1) - \psi(n+1))^2 - (\psi'(1) - \psi'(n+1))\big) = \frac{1}{n} H_n^2 - \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k^2},$$

as claimed. $\qquad\qquad\square$

Theorem 4.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n \geqslant 2$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,*

$$\mathbb{E}_\mu(\Lambda_\mu^*) \leqslant \big(\mathbb{E}_\mu[(\Lambda_\mu^*)^2]\big)^{1/2} \leqslant \frac{c \ln n}{\kappa} \|f\|_\infty^{1/2},$$

where $c > 0$ is an absolute constant and f is the density of μ.

Proof. Following the proof of Theorem 4.1 we write

$$\int_K \big(\Lambda_\mu^*(x)\big)^2 d\mu(x) \leqslant \int_K \ln^2\left(\frac{e^2}{\kappa} \frac{1}{(1 - \|x\|_K)^{1/\kappa}}\right) d\mu(x).$$

If f is the density of μ on K and ν_K is the cone measure of K, using the inequality $\ln^2(ab) \leqslant 2(\ln^2 a + \ln^2 b)$ where $a, b > 0$, we may write

$$\begin{aligned}
\frac{1}{2} \int_K \ln^2\left(\frac{e^2}{\kappa} \frac{1}{(1 - \|x\|_K)^{1/\kappa}}\right) d\mu(x) &- \ln^2\left(\frac{e^2}{\kappa}\right) \\
&\leqslant \int_{\mathbb{R}^n} f(x) \ln^2\left(\frac{1}{(1 - \|x\|_K)^{1/\kappa}}\right) \mathbf{1}_K(x)\, dx \\
&= n|K| \int_0^\infty r^{n-1} \int_{\partial(K)} f(ry) \ln^2\left(\frac{1}{(1 - \|ry\|_K)^{1/\kappa}}\right) \mathbf{1}_K(ry)\, d\nu_K(y)\, dr \\
&= \frac{n}{\kappa^2} \int_0^1 r^{n-1} \ln^2(1-r) \int_{\partial(K)} f(ry)\, d\nu_K(y)\, dr \\
&\leqslant \frac{n}{\kappa^2} \|f\|_\infty \int_0^1 r^{n-1} \ln^2(1-r)\, dr.
\end{aligned}$$

Since $1 \leqslant \int_K f(x)\, dx \leqslant \|f\|_\infty$, using also Lemma 4.3 we get

$$\begin{aligned}
\int_K \big(\Lambda_\mu^*(x)\big)^2 d\mu(x) &\leqslant \frac{2n}{\kappa^2}\left(\frac{1}{n} H_n^2 - \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k^2}\right) \|f\|_\infty + 2\ln^2\left(\frac{e^2}{\kappa}\right) \\
&\leqslant \left(\frac{2H_n^2}{\kappa^2} + 2\ln^2(e^2/\kappa)\right) \|f\|_\infty \leqslant \frac{c_1 \ln^2 n}{\kappa^2} \|f\|_\infty,
\end{aligned}$$

where $c_1 > 0$ is an absolute constant. This completes the proof. $\qquad\qquad\square$

Our next result concerns the one-dimensional case. Let μ be a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure and consider a random variable X, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ, i.e., $\mu(B) := P(X \in B)$, $B \in \mathcal{B}(\mathbb{R})$. We define

$$\alpha_+ = \alpha_+(\mu) := \sup\{x \in \mathbb{R}\colon \mu([x, \infty)) > 0\}) \quad \text{and} \quad \alpha_- = \alpha_-(\mu) := \sup\{x \in \mathbb{R}\colon \mu((-\infty, -x])) > 0\}).$$

Thus, $-\alpha_-, \alpha_+$ are the endpoints of the support of μ. Note that we may have $\alpha_\pm = +\infty$. We define $I_\mu = (-\alpha_-, \alpha_+)$. Recall that

$$\Lambda_\mu^*(x) := \sup\{tx - \Lambda_\mu(t)\colon t \in \mathbb{R}\}, \qquad x \in \mathbb{R}.$$

In fact, since $tx - \Lambda_\mu(t) < 0$ for $t < 0$ when $x \in [0, \alpha_+)$, we have that $\Lambda_\mu^*(x) = \sup\{tx - \Lambda_\mu(t)\colon t \geqslant 0\}$ in this case, and similarly $\Lambda_\mu^*(x) := \sup\{tx - \Lambda_\mu(t)\colon t \leqslant 0\}$ when $x \in (-\alpha_-, 0]$. One can also check that $\Lambda_\mu^*(\alpha_\pm) = +\infty$. See [18, Lemma 2.8] for the case $\alpha_\pm < +\infty$. In the case $\alpha_\pm = \pm\infty$, the convexity and monotonicity properties of Λ_μ^* imply again that $\lim_{t \to \pm\infty} \Lambda_\mu^*(t) = +\infty$.

Proposition 4.5. *Let μ be a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure. Then,*

$$\int_{I_\mu} e^{\Lambda_\mu^*(x)/2} d\mu(x) \leqslant 4,$$

where $I_\mu = \mathrm{supp}(\mu)$. In particular, for all $p \geqslant 1$ we have that

$$\int_{I_\mu} (\Lambda_\mu^*(x))^p \, d\mu(x) < +\infty.$$

Proof. Let $F(x) = \mu(-\infty, x]$. For any $x \in [0, \alpha_+)$ and $t \geqslant 0$ we have

$$\min\{F(x), 1 - F(x)\} = \varphi_\mu(x) \leqslant e^{-\Lambda_\mu^*(x)}.$$

It follows that

$$(4.2) \qquad \int_{I_\mu} e^{\Lambda_\mu^*(x)/2} d\mu(x) \leqslant \int_{I_\mu} \frac{1}{\sqrt{\min\{F(x), 1 - F(x)\}}} f(x) \, dx$$

$$\leqslant \int_{I_\mu} \frac{1}{\sqrt{F(x)}} f(x) \, dx + \int_{I_\mu} \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx.$$

Write f for the density of μ with respect to Lebesgue measure. Then, $(1 - F)'(x) = -f(x)$, which implies that

$$\int_0^{\alpha_+} \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx \leqslant -\int_0^{\alpha_+} \frac{1}{\sqrt{1 - F(x)}} (1 - F)'(x) dx = -2\sqrt{1 - F(x)} \Big|_0^{\alpha_+} = 2\sqrt{1 - F(0)}$$

since $F(\alpha_+) = 1$. In the same way we check that

$$\int_{-\alpha_-}^0 \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx \leqslant -\int_{-\alpha_-}^0 \frac{1}{\sqrt{1 - F(x)}} (1 - F)'(x) dx = -2\sqrt{1 - F(x)} \Big|_{-\alpha_-}^0 = 2 - 2\sqrt{1 - F(0)}.$$

This shows that

$$\int_{I_\mu} \frac{1}{\sqrt{1 - F(x)}} f(x) \, dx \leqslant 2.$$

In a similar way we obtain the same upper bound for the second summand in (4.2) and the result follows. $\quad\square$

Proposition 4.5 can be extended to products. Let μ_i, $1 \leqslant i \leqslant n$ be centered probability measures on $\mathbb{R}$, all of them absolutely continuous with respect to Lebesgue measure. If $\overline{\mu} = \mu_1 \otimes \cdots \otimes \mu_n$ then $I_{\overline{\mu}} = \prod_{i=1}^n I_{\mu_i}$ and we can easily check that

$$\Lambda_{\overline{\mu}}^*(x) = \sum_{i=1}^n \Lambda_{\mu_i}^*(x_i)$$

for all $x = (x_1, \ldots, x_n) \in I_{\overline{\mu}}$, which implies that

$$\int_{I_{\overline{\mu}}} e^{\Lambda_{\overline{\mu}}^*(x)/2} d\overline{\mu}(x) = \prod_{i=1}^n \left(\int_{I_{\mu_i}} e^{\Lambda_{\mu_i}^*(x_i)/2} d\mu_i(x_i) \right) \leqslant 4^n.$$

In particular, for all $p \geqslant 1$ we have that

$$\int_{I_{\overline{\mu}}} (\Lambda_{\overline{\mu}}^*(x))^p \, d\overline{\mu}(x) < +\infty.$$

We close this section with one more case where we can establish that Λ_μ^* has finite moments of all orders. We consider an arbitrary centered log-concave probability measure on $\mathbb{R}^n$ but we have to impose some conditions on the growth of its one-sided L_t-centroid bodies $Z_t^+(\mu)$. Recall that for every $t \geqslant 1$, the one-sided L_t-centroid body $Z_t^+(\mu)$ of μ is the convex body with support function

$$h_{Z_t^+(\mu)}(y) = \left(2 \int_{\mathbb{R}^n} \langle x, y \rangle_+^t f_\mu(x) dx \right)^{1/t},$$

where $a_+ = \max\{a, 0\}$. If μ is isotropic then $Z_2^+(\mu) \supseteq cB_2^n$ for an absolute constant $c > 0$. One can also check that if $1 \leqslant t < s$ then

$$\left(\frac{2}{e} \right)^{\frac{1}{t} - \frac{1}{s}} Z_t^+(\mu) \subseteq Z_s^+(\mu) \subseteq c_1 \left(\frac{2e-2}{e} \right)^{\frac{1}{t} - \frac{1}{s}} \frac{s}{t} Z_t^+(\mu).$$

For a proof of these claims see [19]. The condition we need is that the family of the one-sided L_t-centroid bodies grows with some mild rate as $t \to \infty$ (note that the assumption in the next proposition can be satisfied only for log-concave probability measures μ with support $\text{supp}(\mu) = \mathbb{R}^n$).

Proposition 4.6. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Assume that there exists an increasing function $g : [1, \infty) \to [1, \infty)$ with $\lim_{t\to\infty} g(t)/\ln(t+1) = +\infty$ such that $Z_t^+(\mu) \supseteq g(t)Z_2^+(\mu)$ for all $t \geqslant 2$. Then,*

$$\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^p d\mu(x) < +\infty$$

for every $p \geqslant 1$.

Proof. We use the following fact, proved in [9, Lemma 4.3]: If $t \geqslant 1$ then for every $x \in \frac{1}{2} Z_t^+(\mu)$ we have

$$\varphi_\mu(x) \geqslant e^{-c_1 t},$$

where $c_1 > 1$ is an absolute constant. Since $\Lambda_\mu^*(x) \leqslant \ln \frac{1}{\varphi_\mu(x)}$, this shows that $\Lambda_\mu^*(x) \leqslant c_1 t$ for all $x \in \frac{1}{2} Z_t^+(\mu)$. In other words,

$$(4.3) \qquad \frac{1}{2} Z_{t/c_1}^+(\mu) \subseteq B_t(\mu), \qquad t \geqslant c_1.$$

Since $\lim_{t\to\infty} g(t) = +\infty$, there exists $t_0 \geqslant c_1$ such that $\mu\left(\frac{g(t_0/c_1)}{2} Z_2^+(\mu) \right) \geqslant 2/3$. From Borell's lemma [10, Lemma 2.4.5] we know that, for all $t \geqslant t_0$,

$$1 - \mu\left(\frac{g(t/c_1)}{2} Z_2^+(\mu) \right) \leqslant e^{-c_2 g(t/c_1)/g(t_0/c_1)},$$

14

where $c_2 > 0$ is an absolute constant. We write

$$\int_{\mathbb{R}^n} |\Lambda_\mu^*(x)|^p d\mu(x) = \int_0^\infty pt^{p-1}\mu(\{x : \Lambda_\mu^*(x) \geqslant t\})\,dt = p\int_0^\infty t^{p-1}(1 - \mu(B_t(\mu)))\,dt.$$

From (4.3) it follows that

$$1 - \mu(B_t(\mu)) \leqslant 1 - \mu\left(\frac{1}{2}Z_{t/c_1}^+(\mu)\right) \leqslant 1 - \mu\left(\frac{g(t/c_1)}{2}Z_2^+(\mu)\right) \leqslant e^{-c_2 g(t/c_1)/g(t_0/c_1)}$$

for all $t \geqslant t_0$. Since $\lim_{t\to\infty} g(t)/\ln(t+1) = +\infty$, there exists $t_p \geqslant t_0$ such that

$$(p-1)\ln(t) \leqslant \frac{c_2}{2g(t_0/c_1)}g(t/c_1)$$

for all $t \geqslant t_p$. Assume that $p > 2$. Then, from the previous observations we get

$$p\int_{t_p}^\infty t^{p-1}(1 - \mu(B_t(\mu)))\,dt \leqslant p\int_{t_p}^\infty t^{p-1}\left(1 - \mu\left(\frac{g(t/c_1)}{2}Z_2^+(\mu)\right)\right)\,dt$$

$$\leqslant p\int_{t_p}^\infty t^{p-1}t^{-2(p-1)}\,dt = p\int_{t_p}^\infty t^{-(p-1)}\,dt < \infty.$$

This proves the result for $p > 2$ and then from Hölder's inequality it is clear that the assertion of the proposition is also true for all $p \geqslant 1$. □

Note. It is not hard to construct examples of log-concave probability measures, even on the real line, for which $\mathrm{supp}(\mu) = \mathbb{R}^n$ but the assumption of Proposition 4.6 is not satisfied. Consider for example a measure μ on $\mathbb{R}$ with density $f(x) = c \cdot \exp(-p)$ where p is an even convex function rapidly increasing to infinity, e.g. $p(t) = e^{t^2}$.

However, this does not exclude the possibility that for every centered log-concave probability measure μ on $\mathbb{R}^n$ the function Λ_μ^* has finite second or higher moments.

5 Threshold for the measure: the approach and examples

For any log-concave probability measure μ on $\mathbb{R}^n$ we define the parameter

(5.1) $$\beta(\mu) = \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2}$$

provided that

$$\|\Lambda_\mu^*\|_{L^2(\mu)} = (\mathbb{E}(\Lambda_\mu^*)^2)^{1/2} < \infty.$$

One of the main results in [9] states that if μ is a log-concave probability measure on $\mathbb{R}^n$ then

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right),$$

where $c > 0$ is an absolute constant. In fact, the proof of this estimate starts with Lemma 3.1 and follows from the next stronger result: If $n \geqslant n_0$ then

$$\int_{\mathbb{R}^n} \exp(-\Lambda_\mu^*(x))\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right)$$

where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants. Then, Jensen's inequality implies that

$$e^{-\mathbb{E}_\mu(\Lambda_\mu^*)} \leqslant \int_{\mathbb{R}^n} \exp(-\Lambda_\mu^*(x))\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right).$$

We will need this lower bound for $\mathbb{E}_\mu(\Lambda_\mu^*)$.

15

Lemma 5.1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$. Then,*

$$\mathbb{E}(\Lambda_\mu^*) \geqslant cn/L_\mu^2,$$

where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants.

We will also need a number of observations in the case $\mu = \mu_K$ where K is a centered convex body of volume 1 in $\mathbb{R}^n$. The next lemma provides a lower bound for $\mathrm{Var}(\Lambda_{\mu_K}^*)$.

Lemma 5.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\mathrm{Var}(\Lambda_{\mu_K}^*) \geqslant c/L_{\mu_K}^4,$$

where $c > 0$ is an absolute constant.

Proof. Borell has proved in [5, Theorem 1] that if T is a convex body in $\mathbb{R}^n$ and f is a non-negative, bounded and convex function on T, not identically zero and with $\min(f) = 0$, then the function

$$\Phi_f(p) = \ln \left[(n+p) \|f\|_p^p \right]$$

is convex on $[0, \infty)$. Consider a centered convex body K of volume 1 in $\mathbb{R}^n$. Applying Borell's theorem for the function $\Lambda_{\mu_K}^*$ on rK, $r \in (0,1)$ and the triple $p = 0, 1$ and 2, and finally letting $r \to 1^-$, we see that

$$(n+1)^2 \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)}^2 \leqslant n(n+2) \|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)}^2,$$

which implies that

$$\mathrm{Var}(\Lambda_{\mu_K}^*) \geqslant \frac{1}{n(n+2)} \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)}^2.$$

Then, taking into account Lemma 5.1 we obtain the result. $\qquad\square$

Recall the definition of $\omega_{\mu_K} = \ln(1/\varphi_{\mu_K})$ in (3.4) and consider the parameter

(5.2) $$\tau(\mu_K) = \frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K})}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2}.$$

The next lemma shows that we can estimate $\beta(\mu_K)$ if we can compute $\tau(\mu_K)$.

Lemma 5.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\beta(\mu_K) = \left(\tau(\mu_K) + O(L_{\mu_K}^2/\sqrt{n}) \right) \left(1 + O(L_{\mu_K}^2/\sqrt{n}) \right).$$

Proof. From Corollary 3.5 we know that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then for every $x \in \mathrm{int}(K)$ we have that $\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda_{\mu_K}^*(x) \leqslant \omega_{\mu_K}(x)$. Writing $\Lambda_{\mu_K}^* = \omega_{\mu_K} + h$ where $\|h\|_\infty \leqslant 5\sqrt{n}$ we easily see that

$$\mathrm{Var}_{\mu_K}(\Lambda_{\mu_K}^*) = \mathrm{Var}_{\mu_K}(\omega_{\mu_K}) + O(\sqrt{n}\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*))$$

where $X = O(Y)$ means that $|X| \leqslant cY$ for an absolute constant $c > 0$. Lemma 5.1 and the fact that $\mathbb{E}_{\mu_K}(\omega_{\mu_K}) = \mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*) + O(\sqrt{n})$ imply that

$$\frac{\mathbb{E}_{\mu_K}(\omega_{\mu_K})}{\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*)} = 1 + O(L_{\mu_K}^2/\sqrt{n}).$$

Taking also into account the fact that $L_{\mu_K}^2/\sqrt{n} = O((\ln n)^8/\sqrt{n}) = o(1)$ we get

$$\mathbb{E}_{\mu_K}(\omega_{\mu_K}) \approx \mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*).$$

Combining the above we see that

$$\beta(\mu_K) = \frac{\mathrm{Var}_{\mu_K}(\Lambda^*_{\mu_K})}{(\mathbb{E}_{\mu_K}(\Lambda^*_{\mu_K}))^2} = \frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K}) + O(\sqrt{n}\mathbb{E}_{\mu_K}(\Lambda^*_{\mu_K}))}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2}\left(\frac{\mathbb{E}_{\mu_K}(\omega_{\mu_K})}{\mathbb{E}_{\mu_K}(\Lambda^*_{\mu_K})}\right)^2$$

$$= \left(\frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K})}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2} + O\left(L^2_{\mu_K}/\sqrt{n}\right)\right)\left(1 + O\left(L^2_{\mu_K}/\sqrt{n}\right)\right)^2$$

$$= \left(\tau(\mu_K) + O\left(L^2_{\mu_K}/\sqrt{n}\right)\right)\left(1 + O\left(L^2_{\mu_K}/\sqrt{n}\right)\right),$$

as claimed. $\qquad\square$

Recall that $B_t(\mu) = \{x \in \mathbb{R}^n : \Lambda^*_\mu(x) \leqslant t\}$, where Λ^*_μ is the Cramer transform of μ. A version of the next lemma appeared originally in [14].

Lemma 5.4. *Let $t > 0$. For every $N > n$ we have*

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(B_t(\mu)) + N\exp(-t).$$

We use the lemma in the following way. Let $m := \mathbb{E}_\mu(\Lambda^*_\mu)$. Then, for all $\varepsilon \in (0,1)$, from Chebyshev's inequality we have that

$$\mu(\{\Lambda^*_\mu \leqslant m - \varepsilon m\}) \leqslant \mu(\{|\Lambda^*_\mu - m| \geqslant \varepsilon m\}) \leqslant \frac{\mathbb{E}_\mu|\Lambda^*_\mu - m|^2}{\varepsilon^2 m^2} = \frac{\beta(\mu)}{\varepsilon^2}.$$

Equivalently,

$$\mu(B_{(1-\varepsilon)m}(\mu)) \leqslant \frac{\beta(\mu)}{\varepsilon^2}.$$

Let $\delta \in (\beta(\mu), 1)$. We distinguish two cases:

(i) If $\beta(\mu) < 1/8$ and $8\beta(\mu) < \delta < 1$ then, choosing $\varepsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1-\varepsilon)m}(\mu)) \leqslant \frac{\delta}{2}.$$

Then, from Lemma 5.4 we see that

$$\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)) : N \leqslant e^{(1-2\varepsilon)m}\} \leqslant \mu(B_{(1-\varepsilon)m}(\mu)) + e^{(1-2\varepsilon)m}e^{-(1-\varepsilon)m}$$

$$\leqslant \frac{\delta}{2} + e^{-\varepsilon m} \leqslant \delta,$$

provided that $\varepsilon m \geqslant \ln(2/\delta)$. Since $m \geqslant c_1 n/L^2_\mu$, this condition is satisfied if $n/L^2_\mu \geqslant c_2\ln(2/\delta)\sqrt{\delta/\beta(\mu)}$. By the choice of ε we conclude that

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt{8\beta(\mu)/\delta}\right)\frac{\mathbb{E}_\mu(\Lambda^*_\mu)}{n}.$$

(ii) If $1/8 \leqslant \beta(\mu) < 1$ and $\beta(\mu) < \delta < 1$ then, choosing $\varepsilon = \sqrt{\frac{2\beta(\mu)}{\beta(\mu)+\delta}}$ we have that

$$\mu(B_{(1-\varepsilon)m}(\mu)) \leqslant \frac{\beta(\mu) + \delta}{2}.$$

Then, exactly as in (i), we see that

$$\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)) : N \leqslant e^{(1-\sqrt{\varepsilon})m}\} \leqslant \frac{\beta(\mu) + \delta}{2} + e^{(1-\sqrt{\varepsilon})m}e^{-(1-\varepsilon)m} \leqslant \frac{\beta(\mu) + \delta}{2} + e^{-(\sqrt{\varepsilon}-\varepsilon)m} \leqslant \delta,$$

17

provided that

$$(5.3) \qquad (\sqrt{\varepsilon} - \varepsilon)m \geqslant \ln\left(\frac{2}{\delta - \beta(\mu)}\right).$$

Note that $1 > \varepsilon \geqslant \sqrt{\beta(\mu)} \geqslant \frac{1}{2\sqrt{2}}$, and hence

$$\sqrt{\varepsilon} - \varepsilon = \frac{\sqrt{\varepsilon}}{(1 + \sqrt{\varepsilon})(1 + \varepsilon)}(1 - \varepsilon^2) \geqslant c_1'(1 - \varepsilon^2) = c_1'\frac{\delta - \beta(\mu)}{\delta + \beta(\mu)} \geqslant c_2'(\delta - \beta(\mu)),$$

where $c_i' > 0$ are absolute constants. Since $m \geqslant c_1 n/L_\mu^2$, we see that if $n/L_\mu^2 \geqslant \frac{c_2}{\delta - \beta(\mu)} \ln\left(\frac{2}{\delta - \beta(\mu)}\right)$ then (5.3) is satisfied. Therefore, we conclude that

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt[4]{\frac{2\beta(\mu)}{\beta(\mu) + \delta}}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

We summarize the above in the next theorem.

Theorem 5.5. *Let μ be a log-concave probability measure on $\mathbb{R}^n$.*

(i) *Let $\beta(\mu) < 1/8$ and $8\beta(\mu) < \delta < 1$. If $n/L_\mu^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu)}$ then*

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt{8\beta(\mu)/\delta}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

(ii) *Let $1/8 \leqslant \beta(\mu) < 1$ and $\beta(\mu) < \delta < 1$. If $n/L_\mu^2 \geqslant \frac{c_2}{\delta - \beta(\mu)} \ln\left(\frac{2}{\delta - \beta(\mu)}\right)$ then*

$$\varrho_1(\mu, \delta) \geqslant \left(1 - \sqrt[4]{\frac{2\beta(\mu)}{\beta(\mu) + \delta}}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

Remark 5.6. Paouris and Valettas have proved in [26, Theorem 5.6] that if μ is a log-concave probability measure on $\mathbb{R}^n$ and p is a convex function on $\mathbb{R}^n$ then

$$(5.4) \qquad \mu\left(\{x : p(x) < M(p) - t\|p - M(p)\|_{L^1(\mu)}\}\right) \leqslant \frac{1}{2}\exp(-t/16)$$

for all $t > 0$, where $M(p)$ is a median of p with respect to μ. Consider the parameter

$$\tilde{\beta}(\mu) = \frac{\|\Lambda_\mu^* - \mathbb{E}(\Lambda_\mu^*)\|_{L^2(\mu)}}{M(\Lambda_\mu^*)}.$$

Recall that Λ_μ^* is convex and set $M = M(\Lambda_\mu^*)$. Since

$$\|\Lambda_\mu^* - M(\Lambda_\mu^*)\|_{L^1(\mu)} \leqslant \|\Lambda_\mu^* - \mathbb{E}(\Lambda_\mu^*)\|_{L^1(\mu)} \leqslant \|\Lambda_\mu^* - \mathbb{E}(\Lambda_\mu^*)\|_{L^2(\mu)},$$

from (5.4) we see that

$$(5.5) \qquad \mu\left(\left\{x : \Lambda_\mu^*(x) < (1 - t\tilde{\beta}(\mu))M\right\}\right) \leqslant \frac{1}{2}\exp(-t/16)$$

for every $t > 0$. This shows that

$$\mu(B_{(1 - t\tilde{\beta}(\mu))M}(\mu)) \leqslant \frac{\delta}{4}$$

if $t \geqslant 16\ln(2/\delta)$. Then, from Lemma 5.4 we see that

$$\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)) : N \leqslant e^{(1-2t\tilde{\beta}(\mu))M}\} \leqslant \mu(B_{(1-t\tilde{\beta}(\mu))M}(\mu)) + e^{(1-2t\tilde{\beta}(\mu))M}e^{-(1-t\tilde{\beta}(\mu))M}$$

$$= \mu(B_{(1-t\tilde{\beta}(\mu))M}(\mu)) + e^{-t\tilde{\beta}(\mu)M} \leqslant \frac{\delta}{4} + e^{-t\tilde{\beta}(\mu)M} \leqslant \delta,$$

provided that $t\tilde{\beta}(\mu)M \geqslant \ln(2/\delta)$. Now, we restrict our attention to the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$. Then, Lemma 5.2 shows that

$$\tilde{\beta}(\mu_K)M(\Lambda^*_{\mu_K}) = \|\Lambda^*_{\mu_K} - \mathbb{E}(\Lambda^*_{\mu_K})\|_{L^2(\mu)} \geqslant c_1/L^2_{\mu_K}$$

where $c_1 > 0$ is an absolute constant. Choosing $t = c_2 L^2_{\mu_K} \ln(2/\delta)$ we get

(5.6) $$\varrho_1(\mu_K, \delta) \geqslant \left(1 - c_3 L^2_{\mu_K} \ln(2/\delta)\tilde{\beta}(\mu_K)\right) \frac{M(\Lambda^*_{\mu_K})}{n}.$$

A natural question is to examine how close $\mathbb{E}(\Lambda^*_\mu)$ and $M(\Lambda^*_\mu)$ are. This would allow us to compare (5.6) with Theorem 5.5 at least in the case of the uniform measure on a convex body. From [10, Lemma 2.4.10] we know that

$$\frac{1}{2}\|\Lambda^*_\mu - \mathbb{E}(\Lambda^*_\mu)\|_{L^2(\mu)} \leqslant \|\Lambda^*_\mu - M(\Lambda^*_\mu)\|_{L^2(\mu)} \leqslant 3\|\Lambda^*_\mu - \mathbb{E}(\Lambda^*_\mu)\|_{L^2(\mu)}$$

for any Borel probability measure μ on $\mathbb{R}^n$. Therefore, if we assume that $\beta(\mu_K) \leqslant \eta$ for some small enough $\eta \in (0, 1)$, we see that

$$M(\Lambda^*_\mu) \geqslant \|\Lambda^*_\mu\|_2 - 3\sqrt{\eta}\|\Lambda^*_\mu\|_1 \geqslant (1 - 3\sqrt{\eta})\mathbb{E}(\Lambda^*_\mu).$$

This gives a variant of Theorem 5.5 with a much better dependence on δ.

Theorem 5.7. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$ and let $\delta \in (0,1)$ and $\eta \in (0,1/9)$. If $\beta(\mu_K) \leqslant \eta$ and $c_3 L^2_{\mu_K} \ln(2/\delta)\tilde{\beta}(\mu_K) < 1$ then*

$$\varrho_1(\mu, \delta) \geqslant \left(1 - c_3 L^2_{\mu_K} \ln(2/\delta)\tilde{\beta}(\mu_K)\right)(1 - 3\sqrt{\eta})\frac{\mathbb{E}(\Lambda^*_{\mu_K})}{n}.$$

For the proof of the lower threshold we need a basic fact that plays a main role in the proof of all the lower thresholds that have been obtained so far. It is stated in the form below in [11, Lemma 3]. For a proof see [14] or [18, Lemma 4.1].

Lemma 5.8. *For every Borel subset A of $\mathbb{R}^n$ we have that*

$$1 - \mu^N(K_N \supseteq A) \leqslant 2\binom{N}{n}\left(1 - \inf_{x \in A}\varphi_\mu(x)\right)^{N-n}.$$

Therefore,

$$\mathbb{E}[\mu(K_N)] \geqslant \mu(A)\left(1 - 2\binom{N}{n}\left(1 - \inf_{x \in A}\varphi_\mu(x)\right)^{N-n}\right).$$

In order to apply Lemma 5.8 we note that if $m := \mathbb{E}_\mu(\Lambda^*_\mu)$ then as before, for all $\varepsilon \in (0,1)$, from Chebyshev's inequality we have that

$$\mu(\{\Lambda^*_\mu \geqslant m + \varepsilon m\}) \leqslant \mu(\{|\Lambda^*_\mu - m| \geqslant \varepsilon m\}) \leqslant \frac{\beta(\mu)}{\varepsilon^2}.$$

If $\beta(\mu) < 1/2$ and $2\beta(\mu) < \delta < 1$ then, choosing $\varepsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1+\varepsilon)m}(\mu)) \geqslant 1 - \frac{\delta}{2}.$$

19

Therefore, we will have that

$$\varrho_2(\mu,\delta) \leqslant (1+2\varepsilon)m/n$$

if our lower bound for $\inf_{x\in B_{(1+\varepsilon)m}(\mu)}\varphi_\mu(x)$ gives

(5.7)
$$2\binom{N}{n}\left(1 - \inf_{x\in B_{(1+\varepsilon)m}(\mu)}\varphi_\mu(x)\right)^{N-n} \leqslant \frac{\delta}{2}$$

for all $N \geqslant N_0 := \exp((1+2\varepsilon)m)$. Recall that in the case of the uniform measure on a centered convex body of volume 1, Theorem 3.2 shows that

$$\inf_{x\in B_{(1+\varepsilon)m}(\mu_K)}\varphi_{\mu_K}(x) \geqslant \frac{1}{10}\exp(-(1+\varepsilon)m - 2\sqrt{n}).$$

We require that n and m are large enough so that $1/2^n < \delta/2$ and $2\sqrt{n} \leqslant \frac{\varepsilon m}{2}$. Using also the fact that $\binom{N}{n} \leqslant e^{-1}\left(\frac{eN}{n}\right)^n$ we see that (5.7) will be satisfied if we also have

$$\left(\frac{2eN}{n}\right)^n e^{-\frac{N-n}{10}e^{-(1+3\varepsilon/2)m}} < 1.$$

Setting $x := N/n$ we see that this last is equivalent to

$$e^{(1+3\varepsilon/2)m} < \frac{x-1}{10\ln(2ex)}.$$

One can now check that if $N \geqslant \exp((1+2\varepsilon)m)$ then all the restrictions are satisfied if we assume that $n/L_{\mu_K}^2 \geqslant c_2\ln(2/\delta)\sqrt{\delta/\beta(\mu_K)}$. In this way we get the following.

Theorem 5.9. *Let $\beta,\delta > 0$ with $2\beta < \delta < 1$. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ with $\beta(\mu_K) = \beta$ and $n/L_{\mu_K}^2 \geqslant c_2\ln(2/\delta)\sqrt{\delta/\beta}$ then*

$$\varrho_2(\mu_K,\delta) \leqslant \left(1 + \sqrt{8\beta/\delta}\right)\frac{\mathbb{E}_{\mu_K}(\Lambda_{\mu_K}^*)}{n}.$$

An estimate analogous to the one in Theorem 5.5 (ii) is also possible but we shall not go through the details. From the discussion in this section it is clear that our approach is able to provide good bounds for the threshold $\varrho(\mu,\delta)$ if the parameter $\beta(\mu)$ is small, especially if $\beta(\mu) = o_n(1)$ as the dimension increases. We illustrate this with a number of examples.

Example 5.10 (Uniform measure on the cube). Let μ_{C_n} be the uniform measure on the unit cube $C_n = \left[-\frac{1}{2},\frac{1}{2}\right]^n$. Since $\mu_{C_n} = \mu_{C_1} \otimes \cdots \otimes \mu_{C_1}$ we have

$$\mathrm{Var}_{\mu_{C_n}}(\Lambda_{\mu_{C_n}}^*) = n\mathrm{Var}_{\mu_{C_1}}(\Lambda_{\mu_{C_1}}^*) \qquad \text{and} \qquad \mathbb{E}_{\mu_{C_n}}(\Lambda_{\mu_{C_n}}^*) = n\mathbb{E}_{\mu_{C_1}}(\Lambda_{\mu_{C_1}}^*).$$

Therefore,

$$\beta(\mu_{C_n}) = \frac{\mathrm{Var}_{\mu_{C_n}}(\Lambda_{\mu_{C_n}}^*)}{(\mathbb{E}_{\mu_{C_n}}(\Lambda_{\mu_{C_n}}^*))^2} = \frac{\beta(\mu_{C_1})}{n} \longrightarrow 0.$$

as $n \to \infty$. Then, Theorem 5.5 and Theorem 5.9 show that for any $\delta \in (0,1)$ there exists $n_0(\delta)$ such that, for any $n \geqslant n_0$,

$$\varrho_1(\mu_{C_n},\delta) \geqslant \left(1 - \sqrt{\frac{8\beta_{\mu_{C_n}}}{\delta}}\right)\frac{\mathbb{E}(\Lambda_{\mu_{C_n}}^*)}{n} \geqslant \left(1 - \frac{c_1}{\sqrt{\delta n}}\right)\mathbb{E}(\Lambda_{\mu_{C_1}}^*)$$

and

$$\varrho_2(\mu_{C_n},\delta) \leqslant \left(1 + \sqrt{\frac{8\beta_{\mu_{C_n}}}{\delta}}\right)\frac{\mathbb{E}(\Lambda_{\mu_{C_n}}^*)}{n} \leqslant \left(1 + \frac{c_2}{\sqrt{\delta n}}\right)\mathbb{E}(\Lambda_{\mu_{C_1}}^*),$$

20

which shows that

$$\varrho(\mu_{C_n}, \delta) \leqslant \frac{c}{\sqrt{\delta n}},$$

where $c > 0$ is an absolute constant. This estimate provides a sharp threshold for the measure of a random polytope K_N with independent vertices uniformly distributed in C_n. It provides a direct proof of the result of Dyer, Füredi and McDiarmid in [14] with a stronger estimate for the "width of the threshold".

Example 5.11 (Gaussian measure). Let γ_n denote the standard n-dimensional Gaussian measure with density $f_{\gamma_n}(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. Note that $\gamma_n = \gamma_1 \otimes \cdots \otimes \gamma_1$, and hence we may argue as in the previous example. We may also use direct computation to see that

$$\Lambda_{\gamma_n}(\xi) = \log \left(\int_{\mathbb{R}^n} e^{\langle \xi, z \rangle} f_{\gamma_n}(z) dz \right) = \frac{1}{2} |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and

$$\Lambda_{\gamma_n}^*(x) = \sup_{\xi \in \mathbb{R}^n} \{ \langle x, \xi \rangle - \Lambda_{\gamma_n}(\xi) \} = \frac{1}{2} |x|^2$$

for all $x \in \mathbb{R}^n$. It follows that

$$B_t(\gamma_n) = \{ x \in \mathbb{R}^n : \Lambda_{\gamma_n}^*(x) \leqslant t \} = \{ x \in \mathbb{R}^n : |x| \leqslant \sqrt{2t} \} = \sqrt{2t} B_2^n.$$

Note that if $x \in \partial(B_t(\gamma_n))$ then

$$\varphi_{\gamma_n}(x) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2t}}^{\infty} e^{-u^2/2} du \geqslant \frac{c}{\sqrt{t}} e^{-t}$$

for all $t \geqslant 1$ (see [20, p. 17] for a refined form of the lower bound that we use). By the standard concentration estimate for the Euclidean norm with respect to γ_n (see [30, Theorem 3.1.1]), we have $\| |x| - \sqrt{n} \|_{\psi_2} \leqslant C$, where $C > 0$ is an absolute constant, or equivalently, for any $s > 0$,

$$\gamma_n(\{ x \in \mathbb{R}^n : | |x| - \sqrt{n} | \geqslant s\sqrt{n} \}) \leqslant 2 \exp(-cs^2 n),$$

where $c > 0$ is an absolute constant. This shows that

$$\max\{ \gamma_n((1-s)\sqrt{n} B_2^n), 1 - \gamma_n((1+s)\sqrt{n} B_2^n) \} \leqslant 2 \exp(-cs^2 n)$$

for every $s \in (0, 1)$. Let $\varepsilon \in (0, 1/2)$. Applying Lemma 5.4 with $t = (1 - \varepsilon) n/2$ and $N \leqslant \exp((1/2 - \varepsilon)n)$ we see that

$$\mathbb{E}_{\gamma_n^N}(\gamma_n(K_N)) \leqslant \gamma_n(\sqrt{(1-\varepsilon)n} B_2^n) + \exp(-\varepsilon n/2)$$
$$\leqslant 2 \exp(-c\varepsilon^2 n) + \exp(-\varepsilon n/2),$$

using the fact that $\sqrt{1 - \varepsilon} \leqslant 1 - \varepsilon/2$. It follows that, for any $\delta \in (0, 1)$, if we choose $\varepsilon = c_1 \sqrt{\ln(4/\delta)}/\sqrt{n}$ we have

$$\sup \left\{ \mathbb{E}_{\gamma_n^N}(\gamma_n(K_N)) : N \leqslant e^{(\frac{1}{2} - \varepsilon)n} \right\} \leqslant \delta,$$

and hence

$$\varrho_1(\gamma_n, \delta) \geqslant \frac{1}{2} - \frac{c_1 \sqrt{\ln(4/\delta)}}{\sqrt{n}}.$$

Now, let $N \geqslant \exp((1/2 + \varepsilon)n)$. Applying Lemma 5.8 with $A = B_t(\gamma_n)$ where $t = (1 + \varepsilon)n/2$, we see that $\gamma_n(B_t(\gamma_n)) = \gamma_n(\sqrt{(1+\varepsilon)n} B_2^n) \geqslant 1 - 2 \exp(-c\varepsilon^2 n)$, because $\sqrt{1 + \varepsilon} \geqslant 1 + \varepsilon/3$. We also have

$$2 \binom{N}{n} \left(1 - \inf_{x \in B_t(\gamma_n)} \varphi_{\gamma_n}(x) \right)^{N-n} \leqslant \left(\frac{2eN}{n} \right)^n \exp \left(-\frac{c(N-n)}{\sqrt{n}} e^{-(1+\varepsilon)n/2} \right).$$

21

Let $\delta \in (0,1)$. We choose $\varepsilon = c_2\sqrt{\ln(4/\delta)}/\sqrt{n}$ and insert these estimates into Lemma 5.8. Arguing as in the proof of (5.7) we see that if $n \geqslant n_0(\delta)$ then

$$\inf\left\{\mathbb{E}_{\gamma_n^N}\left(\gamma_n(K_N)\right) : N \geqslant e^{\left(\frac{1}{2}+\varepsilon\right)n}\right\} \geqslant 1 - \delta,$$

and hence

$$\varrho_2(\gamma_n, \delta) \leqslant \frac{1}{2} + \frac{c_2\sqrt{\ln(4/\delta)}}{\sqrt{n}}.$$

Combining the above we get

$$\varrho(\gamma_n, \delta) \leqslant \frac{C\sqrt{\ln(4/\delta)}}{\sqrt{n}},$$

where $C > 0$ is an absolute constant.

We close this article with the example of the uniform measure on the Euclidean ball. It was proved in [3] that if $\varepsilon \in (0,1)$ and $K_N = \mathrm{conv}\{x_1, \ldots, x_N\}$ where $x_1, \ldots, x_N$ are random points independently and uniformly chosen from B_2^n then

$$\lim_{n\to\infty} \sup\left\{\frac{\mathbb{E}|K_N|}{|B_2^n|} : N \leqslant \exp\left((1-\varepsilon)\left(\frac{n+1}{2}\right)\ln n\right)\right\} = 0$$

and

$$\lim_{n\to\infty} \inf\left\{\frac{\mathbb{E}|K_N|}{|B_2^n|} : N \geqslant \exp\left((1+\varepsilon)\left(\frac{n+1}{2}\right)\ln n\right)\right\} = 1.$$

We shall obtain a similar conclusion with the approach of this work (the estimate below is in fact stronger since it sharpens the width of the threshold from $O(1)$ to $O(1/\sqrt{n})$).

Theorem 5.12. *Let D_n be the centered Euclidean ball of volume 1 in $\mathbb{R}^n$. Then, the sequence $\mu_n := \mu_{D_n}$ exhibits a sharp threshold with $\varrho(\mu_n, \delta) \leqslant \frac{c}{\sqrt{\delta n}}$ and e.g. if n is even then we have that*

$$\mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*) = \frac{(n+1)}{2}H_{\frac{n}{2}} + O(\sqrt{n})$$

as $n \to \infty$, where $H_m = \sum_{k=1}^m \frac{1}{k}$.

Proof. Note that if K is a centered convex body in $\mathbb{R}^n$ and $r > 0$ then $\Lambda_{\mu_{rK}}^*(x) = \Lambda_{\mu_K}^*(x/r)$ for all $x \in \mathbb{R}^n$, where μ_{rK} is the uniform measure on rK. It follows that

$$\frac{1}{|rK|}\int_{rK}[\Lambda_{\mu_{rK}}^*(x)]^p dx = \frac{1}{|K|}\int_K[\Lambda_{\mu_K}^*(x)]^p dx$$

for every $p > 0$ and $r > 0$. This shows that in order to compute $\beta(\mu_{D_n})$ it suffices to compute the ratio

$$\beta(\mu_{D_n}) + 1 = \frac{\frac{1}{|B_2^n|}\int_{B_2^n}\Lambda^*(x)^2 dx}{\left(\frac{1}{|B_2^n|}\int_{B_2^n}\Lambda^*(x)dx\right)^2}$$

where $\Lambda^* := \Lambda_{\mu_{B_2^n}}^*$. Having in mind Lemma 5.3 we start by computing $\tau(\mu_{B_2^n})$. Set $\omega := \omega_{\mu_{B_2^n}}$. Then, $\omega(x) = \ln(1/\varphi(x))$ where $\varphi(x) = F(|x|)$,

$$F(r) = c_n\int_r^1(1-t^2)^{\frac{n-1}{2}}dt, \qquad r \in [0,1]$$

and $c_n = \pi^{-1/2}\Gamma(\frac{n}{2}+1)/\Gamma(\frac{n+1}{2})$. From [3, Lemma 2.2] we know that

$$F(r) = (1-r^2)^{\frac{n+1}{2}} h(r,n),$$

where

$$\frac{1}{\sqrt{2\pi(n+2)}} \leqslant h(r,n) \leqslant \frac{1}{r\sqrt{2\pi n}}$$

for all $r \in (0,1]$. We assume that n is even (the case where n is odd can be treated in a similar way). Using polar coordinates we compute

$$\frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\,dx = -n\int_0^1 r^{n-1}\ln(F(r))\,dr$$

$$= -n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\,dr - n\int_0^1 r^{n-1}\ln(h(r,n))\,dr.$$

The leading term is the first one and one can compute it explicitly. After making the change of variables $r^2 = u$, we get

(5.9) $$-n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\,dr = -\frac{n(n+1)}{2}\int_0^1 r^{n-1}\ln(1-r^2)\,dr$$

$$= -\frac{n(n+1)}{4}\int_0^1 u^{\frac{n-2}{2}}\ln(1-u)\,du = \frac{n(n+1)}{2n}H_{\frac{n}{2}},$$

using also Lemma 4.3. For the second term we recall from (5.8) that $0 \leqslant -\ln(h(r,n)) \leqslant \frac{1}{2}\ln(2\pi(n+2)) \leqslant c_1\ln n$, and hence

$$-n\int_0^1 r^{n-1}\ln(h(r,n))\,dr \leqslant c_1\ln n\int_0^1 nr^{n-1}\,dr = c_1\ln n.$$

Therefore,

(5.10) $$\frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\,dx = \frac{n+1}{2}H_{\frac{n}{2}} + O(\ln n).$$

Using again polar coordinates we write

$$\frac{1}{|B_2^n|}\int_{B_2^n}(\omega(x))^2\,dx = n\int_0^1 r^{n-1}\ln^2(F(r))\,dr$$

$$= n\int_0^1 r^{n-1}\ln^2((1-r^2)^{\frac{n+1}{2}})\,dr + n\int_0^1 r^{n-1}\ln^2(h(r,n))\,dr$$

$$+ 2n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\ln(h(r,n))\,dr,$$

As before, the leading term is the first one and we can compute it explicitly. After making the change of variables $r^2 = u$, we get

$$n\left(\frac{n+1}{2}\right)^2\int_0^1 r^{n-1}\ln^2(1-r^2)\,dr = n\frac{(n+1)^2}{8}\int_0^1 u^{\frac{n-2}{2}}\ln^2(1-u)\,du$$

$$= n\frac{(n+1)^2}{8}\left(\frac{2}{n}H_{\frac{n}{2}}^2 - \frac{2}{n}\sum_{k=1}^{n/2}\frac{1}{k^2}\right).$$

23

On the other hand, from (5.8) we see that if $h(r,n) \leqslant 1$ then $0 \leqslant -\ln(h(r,n)) \leqslant \frac{1}{2}\ln(2\pi(n+2)) \leqslant c_1 \ln n$, while if $h(r,n) > 1$ then $0 \leqslant \ln(h(r,n)) \leqslant \ln(1/r)$. Therefore, $\ln^2(h(r,n)) \leqslant c_2(\ln n)^2 + \ln^2(r)$ for all $r \in (0,1]$. It follows that

$$n\int_0^1 r^{n-1}\ln^2(h(r,n))\, dr \leqslant c_3(\ln n)^2 + n\int_0^1 r^{n-1}\ln^2 r\, dr \leqslant c_4(\ln n)^2.$$

Using again the fact that $\ln(h(r,n)^{-1}) \leqslant c_1 \ln n$ as well as (5.9), we check that

$$2n\int_0^1 r^{n-1}\ln((1-r^2)^{\frac{n+1}{2}})\ln(h(r,n))\, dr \leqslant \frac{n(n+1)}{2n}H_{\frac{n}{2}} \cdot c_1 \ln n \leqslant c_5 n(\ln n)^2.$$

From these estimates we have

(5.11) $$\frac{1}{|B_2^n|}\int_{B_2^n}(\omega(x))^2\, dx = \frac{(n+1)^2}{4}H_{\frac{n}{2}}^2 + O(n(\ln n)^2).$$

From (5.10) and (5.11) we finally get

$$\tau(\mu_{B_2^n}) = O\left(\frac{n(\ln n)^2}{n^2 H_{\frac{n}{2}}^2}\right) = O(1/n).$$

Then, Lemma 5.3 and a simple computation show that

$$\beta(\mu_{D_n}) = \left(\tau(\mu_{B_2^n}) + O(L_{\mu_{B_2^n}}^2/\sqrt{n})\right)\left(1 + O(L_{\mu_{B_2^n}}^2/\sqrt{n})\right) = O(1/\sqrt{n}),$$

because $L_{\mu_{B_2^n}} \approx 1$. Finally, note that by the estimate (3.5) in Corollary 3.5 we have

$$\mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*) = \frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\, dx + O(\sqrt{n}) = \frac{(n+1)}{2}H_{\frac{n}{2}} + O(\sqrt{n})$$

as $n \to \infty$. $\qquad\square$

Note. The above discussion leaves open the following basic question: to estimate

$$\beta_n^* := \sup\{\beta(\mu_K) : K \text{ is a centered convex body of volume 1 in } \mathbb{R}^n\}$$

or, more generally,

$$\beta_n := \sup\{\beta(\mu) : \mu \text{ is a centered log-concave probability measure on } \mathbb{R}^n\}.$$

References

[1] E. Artin, *The gamma function*, Athena Series: Selected Topics in Mathematics, Holt, Rinehart and Winston, New York-Toronto-London, 1964 vii+39 pp.

[2] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Vol. I*, Mathematical Surveys and Monographs **202**, American Mathematical Society, Providence, RI, 2015.

[3] G. Bonnet, G. Chasapis, J. Grote, D. Temesvari and N. Turchi, *Threshold phenomena for high-dimensional random polytopes*, Commun. Contemp. Math. **21** (2019), no. 5, 1850038, 30 pp.

[4] G. Bonnet, Z. Kabluchko and N. Turchi, *Phase transition for the volume of high-dimensional random polytopes*, Random Structures Algorithms **58** (2021), no. 4, 648–663.

[5] C. Borell, *Complements of Lyapunov's inequality*, Math. Ann. **205** (1973), 323–331.

[6] C. Borell, *Convex measures on locally convex spaces*, Ark. Mat. **12** (1974), 239–252.

[7] C. Borell, *Convex set functions in d-space*, Period. Math. Hungar. **6** (1975), 111–136.

[8] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Lecture Notes in Mathematics **1469**, Springer, Berlin (1991), 127–137.

[9] S. Brazitikos, A. Giannopoulos and M. Pafis, *Half-space depth of log-concave probability measures*, Preprint.

[10] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs **196**, American Mathematical Society, Providence, RI, 2014.

[11] D. Chakraborti, T. Tkocz and B-H. Vritsiou, *A note on volume thresholds for random polytopes*, Geom. Dedicata **213** (2021), 423–431.

[12] Y. Chen, *An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture*, Geom. Funct. Anal. **31** (2021), 34-61.

[13] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer, New York, 1998.

[14] M. E. Dyer, Z. Füredi and C. McDiarmid, *Volumes spanned by random points in the hypercube*, Random Structures Algorithms **3** (1992), 91–106.

[15] M. Fradelizi, *Sections of convex bodies through their centroid*, Arch. Math. (Basel) **69** (1997), no. 6, 515–522.

[16] M. Fradelizi, M. Madiman and L. Wang, *Optimal concentration of information content for log-concave densities*, High dimensional probability VII. Vol. 71. Progr. Probab. Springer (2016), 45–60.

[17] A. Frieze, W. Pegden and T. Tkocz, *Random volumes in d-dimensional polytopes*, Discrete Anal. 2020, Paper No. 15, 17 pp.

[18] D. Gatzouras and A. Giannopoulos, *Threshold for the volume spanned by random points with independent coordinates*, Israel J. Math. **169** (2009), 125–153.

[19] O. Guédon and E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, Geom. Funct. Anal. **21** (2011), 1043–1068.

[20] K. Ito and H. P. McKean, *Diffusion processes and their sample paths*, Springer-Verlag, 1965.

[21] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274–1290.

[22] B. Klartag and J. Lehec, *Bourgain's slicing problem and KLS isoperimetry up to polylog*, Preprint.

[23] S. Nagy, C. Schütt and E. M. Werner, *Halfspace depth and floating body*, Stat. Surv. **13** (2019), 52–118.

[24] A. Naor and D. Romik, *Projecting the surface measure of the sphere of ℓ_p^n*, Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 241–261.

[25] V. H. Nguyen, *Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa's theorem*, J. Funct. Anal. **266** (2014), 931–955.

[26] G. Paouris and P. Valettas, *Variance estimates and almost Euclidean structure*, Adv. Geom. **19** (2019), no. 2, 165–189.

[27] P. Pivovarov, *Volume thresholds for Gaussian and spherical random polytopes and their duals*, Studia Math. **183** (2007), no. 1, 15–34.

[28] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series **28**, Princeton University Press, Princeton, NJ (1970).

[29] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Second expanded edition. Encyclopedia of Mathematics and its Applications **151**, Cambridge University Press, Cambridge, 2014.

[30] R. Vershynin, *High-dimensional probability. An introduction with applications in data science*, Cambridge Series in Statistical and Probabilistic Mathematics, 47. Cambridge University Press, Cambridge, 2018.

[31] L. Wang, *Heat capacity bound, energy fluctuations and convexity*, Ph.D. Thesis, Yale University, 2014.

SILOUANOS BRAZITIKOS: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis 157-84, Athens, Greece.

E-mail: `silouanb@math.uoa.gr`

APOSTOLOS GIANNOPOULOS: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis 157-84, Athens, Greece.

E-mail: `apgiannop@math.uoa.gr`

MINAS PAFIS: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis 157-84, Athens, Greece.

E-mail: `mipafis@math.uoa.gr`