

A probabilistic take on isoperimetric-type inequalities

Grigoris Paouris *

Peter Pivovarov †

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Abstract

We extend a theorem of Groemer's on the expected volume of a random polytope in a convex body. The extension involves various ways of generating random convex sets. We also treat the case of absolutely continuous probability measures rather than convex bodies. As an application, we obtain a new proof of a recent result of Lutwak, Yang and Zhang on the volume of Orlicz-centroid bodies.

Keywords: isoperimetric inequalities, rearrangements, random convex sets

1 Introduction

The Euclidean ball is the extremal case in a host of isoperimetric problems in convex geometry. If \mathcal{K}^n denotes the class of convex bodies in \mathbb{R}^n , then various functionals $\Phi : \mathcal{K}^n \rightarrow \mathbb{R}^+$ are minimized (or maximized) on the Euclidean ball. A result of this type, and the main motivation for the present article, involves the functional

$$\Phi(K) := \frac{1}{\text{vol}_n(K)^N} \int_K \cdots \int_K \text{vol}_n(\text{conv}\{x_1, \dots, x_N\}) dx_1 \dots dx_N \quad (K \in \mathcal{K}^n).$$

Thus $\Phi(K)$ gives the expected volume of the convex hull of independent random points sampled in K . In [11], Groemer proved that

$$\Phi(K) \geq \Phi(B_2^n),$$

where B_2^n is the Euclidean ball; equality holds if and only if K is an ellipsoid. Similar results hold for various functionals Φ involving the volume of random sets associated with K (e.g., [7], [2], [22], [5], [13], [10], [9, Chapter 9], [6]).

We extend Groemer's theorem, and a number of related results, in two directions. Firstly, we work in the class $\mathcal{P}_{[n]}$ of all probability measures on \mathbb{R}^n that are absolutely continuous

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with respect to Lebesgue measure. Whereas Steiner symmetrization is typically used in \mathcal{K}^n , we make use of rearrangement inequalities; especially those related to the well-known theorem of Brascamp, Lieb and Luttinger [3]. The second difference is that we adopt an operator-theoretic viewpoint by considering random matrices applied to various convex sets. This is a natural, well-studied approach in the Local Theory of Banach spaces (see, e.g., [19] and the references therein). In our context, if $N \geq n$ and x_1, \dots, x_N are independent random points with x_i distributed according to $\mu_i \in \mathcal{P}_{[n]}$, we treat the $n \times N$ random matrix $[x_1 \dots x_N]$ as a linear operator from \mathbb{R}^N to \mathbb{R}^n ; applying $[x_1 \dots x_N]$ to a convex body $C \subset \mathbb{R}^N$ produces a random convex set in \mathbb{R}^n , i.e.,

$$[x_1 \dots x_N]C = \left\{ \sum_{i=1}^N c_i x_i : (c_i) \in C \right\}.$$

We seek the minimum of the expected volume of the latter set, subject to a uniform upper bound on the densities of the μ_i 's. Even in the class $\mathcal{P}_{[n]}$, the Euclidean ball plays a special role.

Theorem 1.1. *Let $N \geq n$ and $\mu_1, \dots, \mu_N \in \mathcal{P}_{[n]}$; denote the density of μ_i by f_i . Let C be a convex body in \mathbb{R}^N and set*

$$\mathcal{F}_C(f_1, \dots, f_N) = \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \text{vol}_n([x_1 \dots x_N]C) \prod_{i=1}^N f_i(x_i) dx_N \dots dx_1. \quad (1)$$

If $\|f_i\|_\infty \leq 1$ for $i = 1, \dots, N$, then

$$\mathcal{F}_C(f_1, \dots, f_N) \geq \mathcal{F}_C(\mathbb{1}_{D_n}, \dots, \mathbb{1}_{D_n}),$$

where $D_n \subset \mathbb{R}^n$ is the Euclidean ball of volume one.

If $K \subset \mathbb{R}^n$ is a convex body with $\text{vol}_n(K) = 1$, we can take $f_i = \mathbb{1}_K$. By choosing $C \subset \mathbb{R}^N$ suitably, we recover various known inequalities. If $C = \text{conv}\{\pm e_1, \dots, \pm e_N\}$, then

$$[x_1 \dots x_N]C = \text{conv}\{\pm x_1, \dots, \pm x_N\},$$

which corresponds to the symmetric analogue of Groemer's result mentioned above. For another example, take $C = [-1, 1]^N$. In this case,

$$[x_1 \dots x_N]C = \left\{ \sum_{i=1}^N \alpha_i x_i : |\alpha_i| \leq 1 \text{ for } i = 1, \dots, N \right\},$$

which is just the zonotope (i.e., Minkowski sum of line segments) generated by the line segments $[-x_i, x_i] = \{\alpha x_i : |\alpha| \leq 1\}$. Thus Theorem 1.1 also recovers a result due to

Bourgain, Meyer, Milman and Pajor [2]. For the class \mathcal{K}^n , a general framework for proofs of results of this type is discussed in [6]; the underlying principle goes back to a result of Shephard [24]. In addition to the extension to $\mathcal{P}_{[n]}$, a new insight provided by Theorem 1.1 is that rather than applying a particular method to a given functional, it applies to many functionals at once; one need only select $C \subset \mathbb{R}^N$.

Furthermore, one is not limited to choosing a single $C \subset \mathbb{R}^N$. By taking a sequence of convex bodies $C_N \subset \mathbb{R}^N$ for $N = n, n+1, \dots$ and applying a simple limiting argument, we get additional applications. We obtain a family of isoperimetric inequalities, not necessarily involving random sets. For instance, we retrieve, and extend to the class $\mathcal{P}_{[n]}$, the following theorem of Lutwak, Yang and Zhang [18] (here we deal only with the symmetric case; cf. Remark 5.5).

Theorem 1.2. *Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a Young function, i.e., convex, strictly increasing with $\psi(0) = 0$. Let $\mu \in \mathcal{P}_{[n]}$. Define the Orlicz-centroid body $Z_\psi(\mu)$ of μ corresponding to ψ by its support function*

$$h(Z_\psi(f), y) = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \psi \left(\frac{|\langle x, y \rangle|}{\lambda} \right) d\mu(x) \leq 1 \right\}.$$

If f denotes the density of μ and if $\|f\|_\infty \leq 1$, then

$$\text{vol}_n(Z_\psi(\mu)) \geq \text{vol}_n(Z_\psi(\lambda_{D_n})),$$

where λ_{D_n} is the restriction of Lebesgue measure to D_n .

Despite the fact that the latter theorem involves non-random sets, our proof shows that it can be seen as a Law of Large Numbers, which is the “probabilistic take” referred to in the title. In the present paper, we do not consider equality cases in Theorems 1.1 and 1.2. When $\mu = \mathbb{1}_K$ and $K \subset \mathbb{R}^n$ is a convex body (with the origin in its interior) equality holds in Theorem 1.2 if and only if K is a centered ellipsoid [18].

The paper is organized as follows. In Section 2, we collect definitions and basic facts about rearrangements and give an overview of inequalities related to [3]. In Section 3, we isolate a condition (which we call Groemer’s Convexity Condition (**GCC**)) under which one can conclude a minimization result such as Theorem 1.1. In the presence of (**GCC**), rearrangement inequalities allow us to pass to densities that are rotationally invariant; moving then to the Euclidean ball is done in §3.1. In Section 4, we verify that the particular integrand in $\mathcal{F}_C(f_1, \dots, f_N)$ satisfies (**GCC**). Section 5 concludes with applications; in particular, the proof of Theorem 1.2.

2 Preliminaries on rearrangements of functions

Let A be a Borel subset of \mathbb{R}^n with finite Lebesgue measure. The **symmetric rearrangement** A^* of A is the open ball with centre at the origin, whose volume is equal to the

measure of A . Since we choose A^* to be open, χ_{A^*} is lower semicontinuous. The **symmetric decreasing rearrangement** of χ_A is defined by

$$\chi_A^* = \chi_{A^*}.$$

We consider Borel measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ which satisfy the following condition: for every $t > 0$, the set $\{x \in \mathbb{R}^n : f(x) > t\}$ has finite Lebesgue measure. In this case, we say that f **vanishes at infinity**. For such f , the symmetric decreasing rearrangement f^* is defined by

$$f^*(x) = \int_0^\infty \chi_{\{f>t\}}^*(x) dt = \int_0^\infty \chi_{\{f>t\}}(x) dt.$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a measurable function vanishing at infinity. For $\theta \in S^{n-1}$, we fix a coordinate system such that $e_1 := \theta$. The **Steiner symmetrization** $f^*(\cdot|\theta)$ of f with respect to θ^\perp is defined as follows: for $x_2, \dots, x_n \in \mathbb{R}$, we set $h(t) = f(t, x_2, \dots, x_n)$ and define

$$f^*(t, x_2, \dots, x_n|\theta) := h^*(t). \quad (2)$$

We refer the reader to the book [16] or the introductory notes [4] for further background material on rearrangements of functions.

2.1 Brascamp, Lieb & Luttinger and consequences

In this section we give an overview of results related to the Brascamp, Lieb & Luttinger rearrangement inequality [3, Theorem 1.2] (for functions of one variable). The main consequence which we use here was observed by M. Christ [7, Theorem 4.2]. We prefer to explicitly state the ingredients used in the proof to point out connections to pertinent results in the literature.

Theorem 2.1 ([3]). *Let $f_1, \dots, f_M : \mathbb{R} \rightarrow \mathbb{R}^+$ be non-negative measurable functions. Let $u_1, \dots, u_M \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^M f_i(\langle x, u_i \rangle) dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^M f_i^*(\langle x, u_i \rangle) dx \quad (3)$$

Corollary 2.2. *Let K be a symmetric convex set in \mathbb{R}^n . Suppose that f_1, \dots, f_n are non-negative measurable functions defined on \mathbb{R} . Then*

$$\int_K \prod_{i=1}^n f_i(x_i) dx \leq \int_K \prod_{i=1}^n f_i^*(x_i) dx.$$

The corollary can be proved by approximating K by intersections of slabs of the form

$$K_m = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : |\langle x, u_i \rangle| \leq 1\}$$

for suitable $u_1, \dots, u_m \in \mathbb{R}^n$. In this case, $\mathbb{1}_{K_m} = \prod_{i=1}^m \mathbb{1}_{[-1,1]}(\langle \cdot, u_i \rangle)$ and one can apply (3) with $M = m + n$. For an extension of Corollary 2.2 to certain cases when K is non-convex, see [8]; see [22] for the case when f_i is the indicator of a compact subset of \mathbb{R} ; related results appear in [1].

We say that $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is **quasi-concave** if for all s the set $\{x : F(x) > s\}$ is convex. Similarly, $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is **quasi-convex** if for all s the set $\{x : F(x) < s\}$ is convex. An immediate consequence of Corollary 2.2 is the following.

Corollary 2.3. *Let $F : \mathbb{R}^N \rightarrow \mathbb{R}^+$ be an even quasi-concave function and g_i be real non-negative integrable functions. Then*

$$\int_{\mathbb{R}^N} F(t)g_1(t_1) \cdots g_N(t_N)dt \leq \int_{\mathbb{R}^N} F(t)g_1^*(t_1) \cdots g_N^*(t_N)dt.$$

If $F : \mathbb{R}^N \rightarrow \mathbb{R}^+$ is even and quasi-convex then

$$\int_{\mathbb{R}^N} F(t)g_1(t_1) \cdots g_N(t_N)dt \geq \int_{\mathbb{R}^N} F(t)g_1^*(t_1) \cdots g_N^*(t_N)dt.$$

Proof. For $s \geq 0$, let $K(s) := \{x : F(x) > s\}$. Then $K(s)$ is symmetric and convex. Using the layer-cake representation (cf. [16, Theorem 1.13]), Fubini's Theorem, and Proposition 2.1, we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(t)g_1(t_1) \cdots g_N(t_N)dt &= \int_0^\infty \int_{K(s)} g_1(t_1) \cdots g_N(t_N)dt ds \\ &\leq \int_0^\infty \int_{K(s)} g_1^*(t_1) \cdots g_N^*(t_N)dt ds \\ &= \int_{\mathbb{R}^N} F(t)g_1(t_1) \cdots g_N(t_N)dt. \end{aligned}$$

For the second assertion, one can use the fact that $\mathbb{1}_{\{F \leq s\}} + \mathbb{1}_{\{F > s\}} = 1$. □

3 Groemer's Convexity Condition

For a function $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}$, set

$$\mathcal{F}_F(f_1, \dots, f_N) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(x_1, \dots, x_N) f_1(x_1) \cdots f_N(x_N) dx_1, \dots, dx_N$$

In this section we isolate a condition on F from which one can conclude a minimization result such as Theorem 1.1. We will say that $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies **Groemer's Convexity Condition**, or simply **(GCC)** in short, if for every $z \in \mathbb{R}^n$ and for every $y_1, \dots, y_N \in z^\perp$ the function $F_Y : \mathbb{R}^N \rightarrow \mathbb{R}^+$ defined by

$$F_Y(t) = F(y_1 + t_1 z, \dots, y_N + t_N z)$$

is even and convex.

Proposition 3.1. *Let $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a function that satisfies **(GCC)**. Let f_1, \dots, f_n be non-negative integrable functions defined on \mathbb{R}^n and let $\theta \in S^{n-1}$. Then*

$$\mathcal{F}_F(f_1, \dots, f_n) \geq \mathcal{F}_F(f_1^*(\cdot|\theta), \dots, f_n^*(\cdot|\theta)),$$

where $f^*(\cdot|\theta)$ is the Steiner symmetrization of f about θ^\perp (cf. (2)).

Proof. Using the notation for the Steiner symmetrization of f with respect to θ , $\mathcal{F}_F(f_1, \dots, f_n)$ is equal to

$$\begin{aligned} &= \int_{\mathbb{R}^{n-1}} \dots \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} F(y_1 + t_1 e_1, \dots, y_N + t_N e_1) \prod_{i=1}^N f_i(y_i + t_i e_1) dt_1 \dots dt_N dy_1 \dots dy_N \\ &= \int_{\mathbb{R}^{n-1}} \dots \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^N} F_Y(t_1, \dots, t_N) h_1(t_1) \dots h_N(t_N) dt_1 \dots dt_N dy_1 \dots dy_N \\ &\geq \int_{\mathbb{R}^{n-1}} \dots \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^N} F_Y(t_1, \dots, t_N) h_1^*(t_1) \dots h_N(t_N) dt_1 \dots dt_N dy_1 \dots dy_N, \end{aligned}$$

which is simply equal to $\mathcal{F}_F(f_1^*(\cdot|\theta), \dots, f_n^*(\cdot|\theta))$ (cf. Corollary 2.3). \square

Successive symmetrizations with respect to $n - 1$ dimensional subspaces yield the symmetric rearrangement f_i^* for each f_i , $i \leq N$. In particular, we will make use of the following result, proved in [3].

Proposition 3.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a measurable function with compact support. Then there exists a sequence of functions f_n , where $f_0 = f$ and $f_{n+1} = f_n(\cdot|\theta)$ for some $\theta \in S^{n-1}$, such that*

$$\lim_{n \rightarrow \infty} \|f_n - f^*\|_{L_1} = 0.$$

By a standard approximation argument, we obtain the following proposition.

Proposition 3.3. *Suppose $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies **(GCC)** and f_1, \dots, f_n are non-negative integrable functions on \mathbb{R}^n . Then*

$$\mathcal{F}_F(f_1, \dots, f_n) \geq \mathcal{F}_F(f_1^*, \dots, f_n^*). \quad (4)$$

Remark 3.4. (1) As the proof shows, in Proposition 3.3, one can replace the **(GCC)** assumption on F by the following: for almost all $\theta \in S^{n-1}$ and almost all choices of $y_1, \dots, y_N \in \theta^\perp$, the sets $\{F_Y \leq s\}$ are centrally symmetric and convex.

(2) If F satisfies the quasi-concave analogue of (1) above, i.e., if for almost all $\theta \in S^{n-1}$ and almost all choices of $y_1, \dots, y_N \in \theta^\perp$, the level sets $\{F_Y > s\}$ are centrally symmetric and convex, then

$$\mathcal{F}_F(f_1, \dots, f_N) \leq \mathcal{F}_F(f_1^*, \dots, f_N^*).$$

The latter inequality was observed by M. Christ [7, Theorem 4.2]; such functions F are referred to there as “Steiner convex.”

3.1 From rotational invariance to the ball

Let f_1, \dots, f_N be bounded integrable functions with $\int_{\mathbb{R}^n} f(x) dx = 1$. We will say that f is rotationally invariant if $f(x) = f(y)$ whenever $\|x\|_2 = \|y\|_2$. As in the introduction, let $\mathcal{P}_{[n]}$ be the class of probability measures on \mathbb{R}^n that are absolutely continuous with respect to Lebesgue measure; let $\mathcal{RP}_{[n]} \subset \mathcal{P}_{[n]}$ be the subclass consisting of rotationally invariant measures. The previous proposition shows that if F satisfies **(GCC)**, then

$$\inf_{\mathcal{P}_{[n]}} \mathcal{F}_F(f_1, \dots, f_N) = \inf_{\mathcal{RP}_{[n]}} \mathcal{F}_F(f_1, \dots, f_N),$$

where the f_i 's are the densities of measures in $\mathcal{P}_{[n]}$ and $\mathcal{RP}_{[n]}$, respectively.

The remainder of this section is devoted to studying the quantity

$$\inf_{\mathcal{RP}_{[n]}} \mathcal{F}_F(f_1, \dots, f_N)$$

under the additional assumption that $\|f_i\|_\infty \leq 1$, for $1 \leq i \leq N$. The following lemma is standard; the proof given for completeness.

Lemma 3.5. *Let $f : \mathbb{R}^+ \rightarrow [0, 1]$ be a measurable function and assume that*

$$A := \int_0^\infty f(t)t^{n-1} dt < \infty.$$

Let $g = \mathbf{1}_{[0, (nA)^{1/n}]}$. Then for any increasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$\int_0^\infty \phi(t)f(t)t^{n-1} dt \geq \int_0^\infty \phi(t)g(t)t^{n-1} dt.$$

Proof. Note that

$$\int_0^\infty f(t)t^{n-1} dt = \int_0^\infty g(t)t^{n-1} dt.$$

By assumption, $\|f\|_\infty \leq 1$ and hence for any $0 \leq s \leq (nA)^{1/n}$,

$$\int_0^s f(t)t^{n-1}dt \leq \int_0^s g(t)t^{n-1}dt.$$

Consequently, for any $0 \leq s \leq \infty$,

$$\int_s^\infty f(t)t^{n-1}dt \geq \int_s^\infty g(t)t^{n-1}dt.$$

Without loss of generality, we may assume that $\phi(0) = 0$. Then

$$\begin{aligned} \int_0^\infty \phi(t)f(t)t^{n-1}dt &= \int_0^\infty \int_0^t \phi'(s)f(t)t^{n-1}dsdt \\ &= \int_0^\infty \phi'(s) \int_s^\infty f(t)t^{n-1}dtds \\ &\geq \int_0^\infty \phi'(s) \int_s^\infty g(t)t^{n-1}dtds \\ &= \int_0^\infty \int_0^t \phi'(s)g(t)t^{n-1}dsdt \\ &= \int_0^\infty \phi(t)g(t)t^{n-1}dt. \end{aligned}$$

□

Lemma 3.6. *Let $\mu \in \mathcal{RP}_{[n]}$ and assume that its density $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies $\|f\|_\infty \leq 1$. For $\phi \in S^{n-1}$ and $s \geq 0$, set*

$$H(\phi, s) = \{x \in \mathbb{R}^n : \langle x, \phi \rangle \geq s\}.$$

Then

$$\mu(H(\phi, s)) \geq \text{vol}_n(D_n \cap H(\phi, s))$$

Proof. Let $g = \mathbb{1}_{D_n}$. For each fixed $\theta \in S^{n-1}$, the function from \mathbb{R}^+ to \mathbb{R}^+ defined by

$$r \mapsto \mathbb{1}_{H(\phi, s)}(r\theta)$$

is increasing and hence so is

$$r \mapsto \int_{S^{n-1}} \mathbb{1}_{H(\phi, s)}(r\theta) d\sigma(\theta).$$

Using spherical coordinates and applying Lemma 3.5, we get

$$\begin{aligned} \int_{H(\phi,s)} f(x)dx &= n\omega_n \int_0^\infty \int_{S^{n-1}} \mathbb{1}_{H(\phi,s)}(r\theta) f(r\theta) r^{n-1} d\sigma(\theta) dr \\ &\geq n\omega_n \int_0^\infty \int_{S^{n-1}} \mathbb{1}_{H(\phi,s)}(r\theta) g(r\theta) r^{n-1} d\sigma(\theta) dr \\ &= \int_{H(\phi,s)} g(x)dx. \end{aligned}$$

□

Lemma 3.7. *Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that for any $x \in \mathbb{R}^n$, the function from \mathbb{R} to \mathbb{R} defined by*

$$s \mapsto \rho(sx)$$

is convex. Let X be a symmetric random vector with values in \mathbb{R}^n . Then the function from \mathbb{R}^+ to \mathbb{R}^+ defined by

$$s \mapsto \mathbb{E}\rho(sX)$$

is an increasing function.

Proof. It is sufficient to show that

$$\mathbb{E}\rho(aX) \leq \mathbb{E}\rho(X) \tag{5}$$

for any $0 \leq a \leq 1$. For such a , we can write $a = b(1) + (1-b)(-1)$ with $0 \leq b \leq 1$ and use the convexity assumption

$$\rho(aX) \leq b\rho(X) + (1-b)\rho(-X),$$

from which (5) follows on taking expectations. □

Lemma 3.8. *If $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies (GCC) then for any $x_1, \dots, x_N \in \mathbb{R}^n$ and any $1 \leq j \leq N$, the function from \mathbb{R} to \mathbb{R} defined by*

$$s \mapsto F(x_1, \dots, sx_j, \dots, x_N) \tag{6}$$

is convex.

Proof. For $1 \leq i \leq N$ with $i \neq j$, write $x_i = x'_i + s_i x_j$ with $s_i \in \mathbb{R}$ and $x'_i \perp x_j$. In the definition of (GCC), take $z = x_j$, $y_j = 0$ and $y_i = x'_i$ for all $i \neq j$. Then the map $G_Y : \mathbb{R}^N \rightarrow \mathbb{R}^+$ given by

$$G_Y(t) := F(y_1 + t_1 s_1 z, \dots, t_j z, \dots, y_N + s_N t_N z)$$

is convex since

$$G_Y(t) = F_Y(s_1 t_1, \dots, t_j, \dots, s_N t_N).$$

But the restriction of G_Y to the line $\{t \in \mathbb{R}^N : t_j \in \mathbb{R}, t_i = 1 \text{ for each } i \neq j\}$ is just the function in (6). \square

Proposition 3.9. *Let $f_i : \mathbb{R}^n \rightarrow [0, 1]$ be rotationally invariant probability densities. Suppose $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies (GCC). Then*

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(\mathbf{1}_{D_n}, \dots, \mathbf{1}_{D_n}).$$

Proof. Using spherical coordinates for each $x_i \in \mathbb{R}^n$, we will write

$$x_i := r_i \theta_i, \text{ with } 0 \leq r_i < \infty, \text{ and } \theta_i \in S^{n-1} \text{ for } i = 1, \dots, N.$$

Then $\mathcal{F}_F(f_1, \dots, f_N)$ is equal to

$$(n\omega_n)^N \int_0^\infty \dots \int_0^\infty \int_{S^{n-1}} \dots \int_{S^{n-1}} F(r_1 \theta_1, \dots, r_N \theta_N) \prod_{i=1}^N f_i(r_i \theta_i) r_i^{n-1} d\sigma(\theta_1) \dots d\sigma(\theta_N) dr_1 \dots dr_N.$$

Fix $1 \leq j \leq N$ and suppose $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_N$ are fixed non-negative scalars. Suppose momentarily that $\theta_1, \dots, \theta_N \in S^{n-1}$ are fixed vectors. By Lemma 3.8, the function from \mathbb{R}^+ to \mathbb{R}^+ defined by

$$r_j \mapsto F(r_1 \theta_1, \dots, r_j \theta_j, \dots, r_N \theta_N)$$

is convex. Averaging now in $\theta_j \in S^{n-1}$, Lemma 3.7 implies that the function

$$r_j \mapsto \int_{S^{n-1}} F(r_1 \theta_1, \dots, r_j \theta_j, \dots, r_N \theta_N) d\sigma(\theta_j)$$

is increasing. By assumption, we have

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} f_j(x) dx \\ &= n\omega_n \int_0^\infty \int_{S^{n-1}} f_j(r_j \theta_j) r_j^{n-1} d\sigma(\theta_j) dr_j. \end{aligned}$$

Since f_j depends only on the value of r_j , we have that for any $\theta_j \in S^{n-1}$,

$$\int_0^\infty f_j(r_j \theta_j) r_j^{n-1} dr_j = (n\omega_n)^{-1}.$$

Thus we apply Lemma 3.5 with $A = (n\omega_n)^{-1}$ to see that

$$\int_0^\infty \int_{S^{n-1}} F(r_1 \theta_1, \dots, r_j \theta_j, \dots, r_N \theta_N) f_j(r_j \theta_j) r_j^{n-1} d\sigma(\theta_j) dr_j$$

is at least as large as

$$\int_0^{\omega_n^{-1/n}} \int_{S^{n-1}} F(r_1\theta_1, \dots, r_j\theta_j, \dots, r_N\theta_N) r_j^{n-1} d\sigma(\theta_j) dr_j$$

Applying Fubini's theorem iteratively, we have that $\mathcal{F}_F(f_1, \dots, f_N)$ is larger than or equal to

$$(n\omega_n)^N \int_0^{\omega_n^{-1/n}} \dots \int_0^{\omega_n^{-1/n}} \int_{S^{n-1}} \dots \int_{S^{n-1}} F(r_1\theta_1, \dots, r_N\theta_N) \prod_{i=1}^N r_i^{n-1} d\sigma(\theta_1) \dots d\sigma(\theta_N) dr_1 \dots dr_N,$$

which is simply $\mathcal{F}_F(\mathbf{1}_{D_n}, \dots, \mathbf{1}_{D_n})$ in spherical coordinates. \square

We summarize the results of this section with the following theorem.

Theorem 3.10. *Let $\mu_1, \dots, \mu_N \in \mathcal{P}_{[n]}$; denote the density of μ_i by f_i . Suppose $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies **(GCC)** and set*

$$\mathcal{F}_F(f_1, \dots, f_N) := \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} F(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \dots dx_N. \quad (7)$$

Then

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(f_1^*, \dots, f_N^*).$$

Moreover, if $f_i = f_i^*$ and $\|f_i\|_\infty \leq 1$ for $i = 1, \dots, N$, we also have

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(\mathbf{1}_{D_n}, \dots, \mathbf{1}_{D_n}).$$

4 Verifying GCC

Let C be a symmetric convex body in \mathbb{R}^n . For $x_1, \dots, x_N \in \mathbb{R}^n$, let $T(x_1, \dots, x_N) = [x_1 \cdots x_N]$ be the $n \times N$ matrix with columns the x_i 's. Throughout this section, we let $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}^+$ be the function

$$F(x_1, \dots, x_N) := \text{vol}_n(T(x_1, \dots, x_N)C). \quad (8)$$

Note that for any $S \in SL_n$,

$$F(S(x_1), \dots, S(x_N)) = F(x_1, \dots, x_N). \quad (9)$$

Indeed, for any $n \times n$ matrix M , we have

$$\begin{aligned} F(M(x_1), \dots, M(x_N)) &= \text{vol}_n([M(x_1) \cdots M(x_N)]C) \\ &= \text{vol}_n(M[x_1 \cdots x_N]C) \\ &= |\det(M)| F(x_1, \dots, x_N). \end{aligned}$$

Our goal is to show that F satisfies **(GCC)** so that we can apply Theorem 3.10.

Proposition 4.1. *Let F be as defined in (8). Let $\theta \in S^{n-1}$ and $y_1, \dots, y_N \in \theta^\perp$. Set $Y := \{y_1, \dots, y_N\}$. Let $T_Y(t) := [y_i + t_i\theta]$ and define $F_Y : \mathbb{R}^N \rightarrow \mathbb{R}^+$ by*

$$F_Y(t) = \text{vol}_n(T_Y(t)C).$$

Then F_Y is (i) even and (ii) convex. In particular, F satisfies (GCC).

Proof. The proof is analogous to that of [11, Lemma 3]. Note that

$$[y_1 + t_1\theta \dots y_N + t_N\theta]C = \left\{ \sum_{i=1}^N c_i(y_i + t_i\theta) : (c_i) \in C \right\},$$

while

$$[y_1 - t_1\theta \dots y_N - t_N\theta]C = \left\{ \sum_{i=1}^N c_i(y_i - t_i\theta) : (c_i) \in C \right\}.$$

The latter two sets are reflections of each other about θ^\perp , hence $F_Y(t) = F_Y(-t)$.

For the second assertion, let us set $P := P_{\theta^\perp}$, the orthogonal projection onto θ^\perp . For any compact, convex set $A \subset \mathbb{R}^n$, define functions $f_A, g_A : PA \rightarrow \mathbb{R}$ by

$$f_A(y) := \sup\{\lambda : y + \lambda\theta \in A\} \tag{10}$$

and

$$g_A(y) := \inf\{\lambda : y + \lambda\theta \in A\}. \tag{11}$$

Then f_A is concave and g_A is convex.

Let $s, t \in \mathbb{R}^N$ and consider the functions

$$f_{T_Y(s)C}, g_{T_Y(s)C} : PT_Y(s)C \rightarrow \mathbb{R}$$

and

$$f_{T_Y(t)C}, g_{T_Y(t)C} : PT_Y(t)C \rightarrow \mathbb{R}$$

defined as in (10) and (11). For convenience of notation, set

$$f_s := f_{T_Y(s)C}, \quad g_s := g_{T_Y(s)C}$$

and

$$f_t := f_{T_Y(t)C}, \quad g_t := g_{T_Y(t)C}.$$

Since P is the orthogonal projection on θ^\perp , we have

$$PT_Y(s)C = P[y_i + s_i\theta]C = [y_i]C = P[y_i + t_i\theta]C = PT_Y(t)C.$$

Thus setting $D = PT_Y(s)C = PT_Y(t)C$, we can define $f, g : D \rightarrow \mathbb{R}$ by

$$f = (1/2)f_s + (1/2)f_t, \quad g = (1/2)g_s + (1/2)g_t.$$

Set

$$\widehat{C} := \{y + \lambda\theta : y \in D, g(y) \leq \lambda \leq f(y)\}.$$

We claim that

$$T_Y(s/2 + t/2)C \subset \widehat{C}. \quad (12)$$

Indeed, let $x \in T_Y(s/2 + t/2)C$ so that for some $c = (c_1, \dots, c_N) \in C$, we have

$$x = \sum_{i=1}^N c_i (y_i + (s_i/2 + t_i/2)\theta) = y + \sum_{i=1}^N c_i (s_i/2 + t_i/2)\theta,$$

with $y := \sum_{i=1}^N c_i y_i \in D$. Note that

$$y + \left(\sum_{i=1}^N c_i s_i \right) \theta = \sum_{i=1}^N c_i (y_i + s_i \theta) \in T_Y(s)C$$

and hence

$$g_s(y) \leq \sum_{i=1}^N c_i s_i \leq f_s(y).$$

Similarly,

$$g_t(y) \leq \sum_{i=1}^N c_i t_i \leq f_t(y).$$

Thus

$$\begin{aligned} g(y) &= (1/2)g_s(y) + (1/2)g_t(y) \\ &\leq (1/2) \sum_{i=1}^N c_i s_i + (1/2) \sum_{i=1}^N c_i t_i \\ &\leq (1/2)f_s(y) + (1/2)f_t(y) \\ &= f(y), \end{aligned}$$

which shows that $x = y + \sum_{i=1}^N c_i (s_i/2 + t_i/2)\theta \in \widehat{C}$ and establishes (12). Next, observe that

$$\begin{aligned} \text{vol}_d(\widehat{C}) &= \int_D f(y) - g(y) dy \\ &= (1/2) \int_D f_s(y) - g_s(y) dy + (1/2) \int_D f_t(y) - g_t(y) dy \\ &= (1/2) \text{vol}_d(T_Y(s)C) + (1/2) \text{vol}_d(T_Y(t)C). \end{aligned}$$

This shows that F_Y is convex. □

Proof of Theorem 1.1. The desired inequality follows from Theorem 3.10 and Proposition 4.1. \square

4.1 Further Extensions of Theorem 1.1

Before proceeding to applications, we briefly mention two natural extensions of Theorem 1.1.

Remark 4.2. In Theorem 1.1, one can replace $\text{vol}_n(\cdot)$ by intrinsic volumes (refer to e.g., [23] for background on intrinsic volumes) by using the argument in [13, Lemma 2.3]. We omit the details.

Remark 4.3. Let $g_1 : (0, \infty) \rightarrow (0, \infty)$ be an increasing function and $g_2 : (0, \infty) \rightarrow (0, \infty)$ be decreasing. If $F : \otimes_{i=1}^N \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies **(GCC)** then $g_1 \circ F$ satisfies the condition in Remark 3.4 (1); similarly, $g_2 \circ F$ satisfies the condition in Remark 3.4 (2). Thus if f_1, \dots, f_N are non-negative integrable functions on \mathbb{R}^n , then

$$\mathcal{F}_{g_1 \circ F}(f_1, \dots, f_N) \geq \mathcal{F}_{g_1 \circ F}(f_1^*, \dots, f_N^*). \quad (13)$$

and

$$\mathcal{F}_{g_2 \circ F}(f_1, \dots, f_N) \leq \mathcal{F}_{g_2 \circ F}(f_1^*, \dots, f_N^*). \quad (14)$$

For instance, if $g_2(t) = t^{-p}$ for $p > 0$, then (14) gives upper bounds for $\mathcal{F}_{F^{-p}}(f_1, \dots, f_N)$ provided that one can compute the corresponding quantity in the rotationally invariant case. This is possible in several cases but beyond our present scope.

5 Applications

In this section we prove a corollary of Theorem 1.1 and use it to derive various isoperimetric inequalities.

As in the introduction, let \mathcal{K}^n denote the collection of all convex bodies in \mathbb{R}^n . Denote the Hausdorff metric by δ^H , i.e., for $K_1, K_2 \in \mathcal{K}^n$,

$$\delta^H(K_1, K_2) := \inf\{\delta > 0 : K_1 \subset K_2 + \delta B_2^n, K_2 \subset K_1 + \delta B_2^n\}.$$

We assume that μ_1, μ_2, \dots are probability measures in $\mathcal{P}_{[n]}$; denote the density of μ_i by f_i . Let X_1, X_2, \dots be independent random vectors distributed according to densities f_1, f_2, \dots respectively. Let X_1^*, X_2^*, \dots be independent random vectors distributed according to f_1^*, f_2^*, \dots respectively. For each $N \geq n$, let $T_N = T_N(X_1, \dots, X_N) : \mathbb{R}^N \rightarrow \mathbb{R}^n$ be the operator represented by the $n \times N$ matrix

$$T_N = [X_1 \cdots X_N].$$

Similarly, for each $N \geq n$, let $T_N^{\text{sym}} : \mathbb{R}^N \rightarrow \mathbb{R}^n$ be the operator with matrix

$$T_N^{\text{sym}} = [X_1^* \cdots X_N^*].$$

For notational reasons, it is convenient to assume that all random vectors X_i and X_i^* are defined on an underlying probability space $(\Omega, \Sigma, \mathbb{P})$ and \mathbb{E} denotes expectation with respect to \mathbb{P} .

Corollary 5.1. *Suppose that $(C_N)_{N=n}^\infty$ is a sequence of convex bodies with $C_N \subset \mathbb{R}^N$. Let T_N and T_N^{sym} be the linear operators defined above. Let $M \in L_1(\Omega, \Sigma, \mathbb{P})$. Assume that*

$$\text{vol}_n(T_N C_N) \leq M \text{ (a.s.)}$$

and

$$\text{vol}_n(T_N^{\text{sym}} C_N) \leq M \text{ (a.s.)}.$$

Suppose that \mathcal{C} and \mathcal{C}^* are (random) convex bodies in \mathbb{R}^n defined by the following

$$\mathcal{C} := \lim_{N \rightarrow \infty} T_N C_N \text{ (a.s.)}$$

and

$$\mathcal{C}^* := \lim_{N \rightarrow \infty} T_N^{\text{sym}} C_N \text{ (a.s.)},$$

where the convergence is in the Hausdorff metric. Then

$$\mathbb{E} \text{vol}_n(\mathcal{C}) \geq \mathbb{E} \text{vol}_n(\mathcal{C}^*).$$

Proof. We use the following three facts: (1) $\text{vol}_n(\cdot)$ is continuous with respect to convergence of convex bodies in the Hausdorff metric, (2) the Lebesgue Dominated Convergence Theorem, and (3) Theorem 1.1.

$$\begin{aligned} \mathbb{E} \text{vol}_n(\mathcal{C}) &= \mathbb{E} \lim_{N \rightarrow \infty} \text{vol}_n(T_N C_N) \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \text{vol}_n(T_N C_N) \\ &\geq \lim_{N \rightarrow \infty} \mathbb{E} \text{vol}_n(T_N^{\text{sym}} C_N) \\ &= \mathbb{E} \lim_{N \rightarrow \infty} \text{vol}_n(T_N^{\text{sym}} C_N) \\ &= \mathbb{E} \text{vol}_n(\mathcal{C}^*) \end{aligned}$$

□

To use the corollary, it is convenient to have several basic facts from convexity at hand. We record them here for the reader's convenience. We refer to the introductory chapters of [23] or [9] for additional background material on convexity.

Verifying convergence in the Hausdorff metric is often done by using support functions. Recall that if $K \in \mathcal{K}^n$, its support function is defined by

$$h(K, y) = \sup\{\langle x, y \rangle : x \in K\}.$$

We will use the following standard lemma (see, e.g., [23, page 53]).

Lemma 5.2. *Let $K, L \in \mathcal{K}^N$. Then*

$$\delta^H(K, L) = \sup_{y \in S^{n-1}} |h(K, y) - h(L, y)|.$$

If $T : \mathbb{R}^N \rightarrow \mathbb{R}^n$ is any linear operator, denote its adjoint by $T^t : \mathbb{R}^n \rightarrow \mathbb{R}^N$. In particular, if $T_N = [x_1 \dots x_N]$, then $T_N^t : \mathbb{R}^n \rightarrow \mathbb{R}^N$ is given by

$$T_N^t y = (\langle x_1, y \rangle, \dots, \langle x_N, y \rangle) \quad (y \in \mathbb{R}^n).$$

Using this fact, we can write an explicit formula for the support function of $T_N C_N$.

Lemma 5.3. *Let $T : \mathbb{R}^N \rightarrow \mathbb{R}^n$ be a linear operator. Suppose $C \subset \mathbb{R}^N$ is a convex body. Then for any $y \in \mathbb{R}^n$,*

$$h(TC, y) = h(C, T^t y).$$

Proof.

$$h(TC, y) = \sup\{\langle Tx, y \rangle : x \in C\} = \sup\{\langle x, T^t y \rangle : x \in C\} = h(C, T^t y).$$

□

Before proving Theorem 1.2, we mention one special case.

5.1 L_p -centroid bodies

Let $K \subset \mathbb{R}^n$ be a bounded Borel measurable set with $\text{vol}_n(K) = 1$. Let $Z_p(K)$ denote the L_p -centroid body of K , i.e., the body with support function

$$h(Z_p(K), y) = \left(\int_K |\langle x, y \rangle|^p dx \right)^{1/p}.$$

L_p -centroid bodies were introduced by Lutwak, Yang and Zhang [17] (under a different normalization). L_p -centroid bodies play an important role in concentration of measure for convex bodies, e.g., [21], [15], [12], [14]. In this section we show how Corollary 5.1 gives a short proof of the following result.

Corollary 5.4. *Let $K \subset \mathbb{R}^n$ be a bounded Borel measurable set with $\text{vol}_n(K) = 1$. Then*

$$\text{vol}_n(Z_p(K)) \geq \text{vol}_n(Z_p(D_n)),$$

where D_n is the Euclidean ball of volume one.

For star-shaped bodies $K \subset \mathbb{R}^n$ the latter inequality, together with the equality conditions, is proved in [17]. In [20], the latter result is extended to measures $\mu \in \mathcal{P}_{[n]}$, although it makes use of the result for star-shaped bodies. In the next section, we prove the more general Orlicz version (also for measures $\mu \in \mathcal{P}_{[n]}$); the proof of this special case is given here to illustrate the direct connection to the Law of Large Numbers.

Proof. Taking $C_N = N^{-1/p}B_q^N$ in Lemma 5.3, we have

$$h(N^{-1/p}T_N B_q^N, y)^p = h(N^{-1/p}B_q^N, T_N^t y)^p = \frac{1}{N} \sum_{i=1}^N |\langle X_i, y \rangle|^p$$

for each $y \in S^{n-1}$. By the Strong Law of Large Numbers,

$$\lim_{N \rightarrow \infty} h(N^{-1/p}T_N B_q^N, y)^p = \int_K |\langle x, y \rangle|^p dx \text{ (a.s.)}.$$

Thus for any $y \in S^{n-1}$,

$$\lim_{N \rightarrow \infty} h(N^{-1/p}T_N B_q^N, y) = \left(\int_K |\langle x, y \rangle|^p dx \right)^{1/p} \text{ (a.s.)}.$$

Pointwise convergence of support functions in fact implies uniform convergence (see, e.g., [23, page 54]). Therefore, in the Hausdorff metric,

$$Z_p(K) = \lim_{N \rightarrow \infty} N^{-1/p}T_N B_q^N \text{ (a.s.)}.$$

Finally, let $R(K)$ denote the circumradius of K , i.e.,

$$R(K) = \inf\{R > 0 : K \subset RB_2^n\}.$$

Since $|\langle X_i, y \rangle| \leq R(K)$, we have $N^{-1/p}T_N B_q^N \subset R(K)B_2^n$ and hence Corollary 5.1 gives the desired result. \square

5.2 Orlicz centroid bodies

Here we use Corollary 5.1 to prove Theorem 1.2 stated in the introduction. As in the statement of said theorem, let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a Young function, i.e., convex, strictly increasing with $\psi(0) = 0$. Let $\mu \in \mathcal{P}_{[n]}$. Define the Orlicz-centroid body $Z_\psi(\mu)$ of μ corresponding to ψ by its support function

$$h(Z_\psi(\mu), y) = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \psi \left(\frac{|\langle x, y \rangle|}{\lambda} \right) d\mu(x) \leq 1 \right\}.$$

Remark 5.5. By our definition, $Z_\psi(\mu)$ is centrally-symmetric. In [18], Orlicz-centroid bodies are defined and studied for more general functions ψ .

The idea of the proof is the same as that of Corollary 5.4. Set

$$B_{\psi/N} := \left\{ t = (t_1, \dots, t_N) \in \mathbb{R}^N : \frac{1}{N} \sum_{i=1}^N \psi(|t_i|) \leq 1 \right\}.$$

One can check that $B_{\psi/N}$ is convex, symmetric, bounded and the origin is an interior point, hence

$$\|t\|_{B_{\psi/N}} := \inf\{\lambda > 0 : t \in \lambda B_{\psi/N}\}$$

defines a norm on \mathbb{R}^N , commonly called the Orlicz norm associated with ψ . In particular, $\|\cdot\|_{B_{\psi/N}}$ is the support function for $B_{\psi/N}^\circ$, the polar of $B_{\psi/N}$.

If $T : \mathbb{R}^N \rightarrow \mathbb{R}^n$ is a linear operator, the support function of $TB_{\psi/N}^\circ$ is

$$h(TB_{\psi/N}^\circ, y) = h(B_{\psi/N}^\circ, T^t y) = \|T^t y\|_{B_{\psi/N}} \quad (y \in S^{n-1});$$

cf. Lemma 5.3.

Lemma 5.6. *Let $\mu \in \mathcal{P}_{[n]}$. Let x_1, x_2, \dots be a sequence of vectors in \mathbb{R}^n and suppose that*

$$\text{span}\{x_1, \dots, x_n\} = \mathbb{R}^n. \quad (15)$$

Let ψ be a Young function. Assume that for each $y \in S^{n-1}$ and each $\lambda > 0$, we have

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{i=1}^N \psi \left(\frac{|\langle x_i, y \rangle|}{\lambda} \right) - \int_{\mathbb{R}^n} \psi \left(\frac{|\langle x, y \rangle|}{\lambda} \right) d\mu(x) \right| = 0. \quad (16)$$

Let $T_N = T_N(x_1, \dots, x_N)$ be the $n \times N$ matrix with columns x_1, \dots, x_N . Then

$$Z_\psi(\mu) = \lim_{N \rightarrow \infty} T_N B_{\psi/N}^\circ. \quad (17)$$

Proof. It will be shown that for each $y \in S^{n-1}$, we have pointwise convergence of support functions

$$\lim_{N \rightarrow \infty} h(T_N B_{\psi/N}^\circ, y) = h(Z_\psi(\mu), y). \quad (18)$$

This is sufficient as pointwise convergence implies uniform convergence (as noted in the proof of Corollary 5.4)

Fix $y \in S^{n-1}$. For simplicity of notation, for each $N \geq n$, let $g_N : (0, \infty) \rightarrow (0, \infty)$ be defined by

$$g_N(\lambda) := \frac{1}{N} \sum_{i=1}^N \psi \left(\frac{|\langle x_i, y \rangle|}{\lambda} \right).$$

By (15), there exists $i \in \{1, \dots, n\}$ such that $\langle x_i, y \rangle \neq 0$, hence g_N is strictly positive. Consider also $g : (0, \infty) \rightarrow (0, \infty)$ defined by

$$g(\lambda) := \int_{\mathbb{R}^n} \psi \left(\frac{|\langle x, y \rangle|}{\lambda} \right) d\mu(x).$$

Since ψ is convex and strictly increasing, g and g_N are continuous and strictly decreasing. Let us also set

$$\lambda(N) := h(T_N B_{\psi/N}^\circ, y) = \inf\{\lambda > 0 : g_N(\lambda) \leq 1\}$$

and

$$\lambda_0 := h(Z_\psi(\mu), y) = \inf\{\lambda > 0 : g(\lambda) \leq 1\}.$$

Suppose towards a contradiction that (18) is false. Then there exists $\varepsilon_0 > 0$ and a subsequence $(N(j))_{j=1}^\infty \subset \mathbb{N}$ such that either

- (i) $\lambda(N_j) \geq \lambda_0 + \varepsilon_0$ for each $j = 1, 2, \dots$, or
- (ii) $\lambda(N_j) \leq \lambda_0 - \varepsilon_0$ for each $j = 1, 2, \dots$

Suppose first that (i) holds. Set

$$\lambda_* := \inf_j \lambda(N(j))$$

so that

$$\lambda_* \geq \lambda_0 + \varepsilon_0. \tag{19}$$

Let $\eta > 0$. For each $j = 1, 2, \dots$, by definition of $\lambda(N(j))$ and the fact that $g_{N(j)}$ is decreasing, we have

$$1 < g_{N_j}(\lambda(N_j) - \eta) \leq g_{N_j}(\lambda_* - \eta).$$

Thus by (16),

$$1 \leq \lim_{j \rightarrow \infty} g_{N_j}(\lambda_* - \eta) = g(\lambda_* - \eta)$$

As $\eta > 0$ was arbitrary, and g is continuous, we have $1 \leq g(\lambda_*)$. If $1 < g(\lambda_*)$, then $\lambda_* < \lambda_0$, contradicting (19). On the other hand, if $1 = g(\lambda_*)$, then as g is a strictly decreasing continuous function, we have $\lambda_* = \lambda_0$, contradicting (19).

Suppose now that (ii) holds. Set

$$\lambda^* := \sup_j \lambda(N(j))$$

so that

$$\lambda^* \leq \lambda_0 - \varepsilon_0. \tag{20}$$

Let $\eta > 0$. For each $j = 1, 2, \dots$, by the definition of $\lambda(N(j))$ and the fact that g_{N_j} is decreasing, we have

$$g_{N_j}(\lambda^* + \eta) \leq g_{N_j}(\lambda(N_j) + \eta) \leq 1.$$

Thus by (16),

$$g(\lambda^* + \eta) = \lim_{j \rightarrow \infty} g_{N_j}(\lambda^* + \eta) \leq 1.$$

Thus $\lambda_0 \leq \lambda^* + \eta$. As $\eta > 0$ was arbitrary, we in fact have $\lambda_0 \leq \lambda^*$, contradicting (20). \square

Proof of Theorem 1.2. By standard approximation arguments, we can assume that μ is compactly supported, say

$$\text{supp}(\mu) \subset RB_2^n.$$

Let X_1, X_2, \dots be independent random vectors distributed according to μ . Let $T_N = T_N(X_1, \dots, X_N)$ be the matrix with columns X_1, \dots, X_N . Set

$$\bar{\lambda} := \frac{R}{\psi^{-1}(1)}$$

and observe that for any N and for any $y \in S^{n-1}$,

$$\frac{1}{N} \sum_{i=1}^N \psi \left(\frac{|\langle X_i, y \rangle|}{\bar{\lambda}} \right) \leq \frac{1}{N} \sum_{i=1}^N \psi(\psi^{-1}(1)) \leq 1,$$

hence

$$h(T_N B_{\psi/N}^\circ, y) = \|T_N^t y\|_{B_{\psi/N}} \leq \bar{\lambda}.$$

Thus for any N , we have

$$T_N B_{\psi/N}^\circ \subset \bar{\lambda} B_2^n. \tag{21}$$

This shows that (5.1) in Corollary 5.1 is satisfied. On the other hand, by the Strong Law of Large Numbers, the X_i 's satisfy the assumption (16) in Lemma 5.6 almost surely. Hence, in the Hausdorff metric,

$$Z_\psi(\mu) = \lim_{N \rightarrow \infty} T_N B_{\psi/N}^\circ \text{ (a.s.)},$$

and Corollary 5.1 applies. \square

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References

- [1] A. Baernstein, II and M. Loss, *Some conjectures about L^p norms of k -plane transforms*, Rend. Sem. Mat. Fis. Milano **67** (1997), 9–26 (2000). MR 1781031 (2001h:44005)
- [2] J. Bourgain, M. Meyer, V. Milman, and A. Pajor, *On a geometric inequality*, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math., vol. 1317, Springer, Berlin, 1988, pp. 271–282. MR 950987 (89h:46012)
- [3] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Functional Analysis **17** (1974), 227–237. MR 0346109 (49 #10835)
- [4] A. Burchard, *A short course on rearrangement inequalities*, available at <http://www.math.utoronto.ca/almut/rearrange.pdf>, 2009.
- [5] S. Campi, A. Colesanti, and P. Gronchi, *A note on Sylvester’s problem for random polytopes in a convex body*, Rend. Istit. Mat. Univ. Trieste **31** (1999), no. 1-2, 79–94. MR 1763244 (2001d:52014)
- [6] S. Campi and P. Gronchi, *Extremal convex sets for Sylvester-Busemann type functionals*, Appl. Anal. **85** (2006), no. 1-3, 129–141. MR 2198835 (2006i:52009)
- [7] M. Christ, *Estimates for the k -plane transform*, Indiana Univ. Math. J. **33** (1984), no. 6, 891–910. MR 763948 (86k:44004)
- [8] C. Draghici, *Inequalities for integral means over symmetric sets*, J. Math. Anal. Appl. **324** (2006), no. 1, 543–554. MR MR2262490 (2007i:26026)
- [9] R. J. Gardner, *Geometric tomography*, second ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006. MR 2251886 (2007i:52010)
- [10] A. Giannopoulos, M. Hartzoulaki, and A. Tsolomitis, *Random points in isotropic unconditional convex bodies*, J. London Math. Soc. (2) **72** (2005), no. 3, 779–798. MR 2190337 (2006i:60012)
- [11] H. Groemer, *On the mean value of the volume of a random polytope in a convex set*, Arch. Math. (Basel) **25** (1974), 86–90. MR 0341286 (49 #6036)
- [12] O. Guédon and E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, preprint.
- [13] M. Hartzoulaki and G. Paouris, *Quermassintegrals of a random polytope in a convex body*, Arch. Math. (Basel) **80** (2003), no. 4, 430–438. MR 1982842 (2004c:52009)

- [14] B. Klartag and E. Milman, *Centroid bodies and the logarithmic laplace transform - a unified approach*, preprint.
- [15] R. Latała and J. O. Wojtaszczyk, *On the infimum convolution inequality*, *Studia Math.* **189** (2008), no. 2, 147–187. MR 2449135 (2009k:52011)
- [16] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. MR MR1817225 (2001i:00001)
- [17] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, *J. Differential Geom.* **56** (2000), no. 1, 111–132. MR MR1863023 (2002h:52011)
- [18] ———, *Orlicz centroid bodies*, *J. Differential Geom.* **84** (2010), no. 2, 365–387. MR 2652465
- [19] P. Mankiewicz and N. Tomczak-Jaegermann, *Quotients of finite-dimensional Banach spaces; random phenomena*, *Handbook of the geometry of Banach spaces*, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1201–1246. MR 1999195 (2005f:46018)
- [20] G. Paouris, *On the existence of supergaussian directions on convex bodies*, preprint.
- [21] ———, *Concentration of mass on convex bodies*, *Geom. Funct. Anal.* **16** (2006), no. 5, 1021–1049. MR 2276533 (2007k:52009)
- [22] R. E. Pfiefer, *Maximum and minimum sets for some geometric mean values*, *J. Theoret. Probab.* **3** (1990), no. 2, 169–179. MR MR1046328 (91d:60028)
- [23] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, *Encyclopedia of Mathematics and its Applications*, vol. 44, Cambridge University Press, Cambridge, 1993. MR MR1216521 (94d:52007)
- [24] G. C. Shephard, *Shadow systems of convex sets*, *Israel J. Math.* **2** (1964), 229–236. MR 0179686 (31 #3931)

Grigoris Paouris
Department of Mathematics
Texas A & M University
College Station
TX 77843-3368
grigoris_paouris@yahoo.co.uk

Peter Pivovarov
Department of Mathematics
Texas A & M University
College Station
TX 77843-3368
ppivovarov@math.tamu.edu