Isotropic surface area measures

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Abstract

The purpose of this note is to bring into attention an apparently forgotten result of C.M. Petty: a convex body has minimal surface area among its affine transformations of the same volume if and only if its area measure is isotropic. We obtain sharp affine inequalities which demonstrate the fact that this "surface isotropic" position is a natural framework for the study of hyperplane projections of convex bodies.

1 Introduction

We shall work in $\mathbb{R}^n$ equipped with a fixed Euclidean structure and write $|\cdot|$ for the corresponding Euclidean norm. We denote the Euclidean unit ball and the unit sphere by D_n and S^{n-1} respectively, and we write σ for the rotationally invariant probability measure on S^{n-1}. The volume of appropriate dimension will be also denoted by $|\cdot|$. We shall write ω_n for the volume of the Euclidean unit ball in $\mathbb{R}^n$. Finally, $L(\mathbb{R}^n,\mathbb{R}^n)$ is the space of all linear transformations of $\mathbb{R}^n$.

Let K be a convex body in $\mathbb{R}^n$. The area measure σ_K is defined on S^{n-1} and corresponds to the usual surface measure on K via the Gauss map. If A is a Borel subset of S^{n-1}, then

$$\sigma_K(A)=\nu\{x\in \mathrm{bd}(K):\ \text{the outer normal to } K \text{ at } x \text{ belongs to } A\},$$

where ν is the $(n-1)$–dimensional surface measure on K. In the case where K is a polytope with facets $\{F_j\}_{j\leq m}$ and normals $\{u_j\}_{j\leq m}$, the measure σ_K is purely atomic with mass $|F_j|$ at u_j, $j=1,\dots,m$. The surface area of K is clearly given by

$$\partial(K)=\sigma_K(S^{n-1}).$$

For every convex body K we define the affinely invariant quantity

$$\partial_K=\min\{\partial(TK)/|K|^{\frac{n-1}{n}}:T\in L(\mathbb{R}^n,\mathbb{R}^n)\ ,\ |\det T|=1\},$$

the minimal surface area of an affine image of K with volume 1. It is not hard to check that this minimum is attained for some T.

Definition A Borel measure μ on S^{n-1} will be called *isotropic* if

$$\int_{S^{n-1}} |\langle u, \theta \rangle|^2 d\mu(u) = \frac{\mu(S^{n-1})}{n}$$

for every $\theta \in S^{n-1}$.

C.M. Petty [5] has proved the following characterization of the area measure of a convex body with minimal surface area:

Theorem 1 *Let K be a convex body in $\mathbb{R}^n$. Then, $\partial(K) = \partial_K |K|^{\frac{n-1}{n}}$ if and only if σ_K is isotropic.*

In Section 2 we give a short proof of this fact. Actually, we came to the same conclusion without being aware of Petty's result. The latter was brought to our attention later, when V.D. Milman pointed out to us that Petty had studied the minimal surface position. The argument shows that for fixed volume the position is unique up to an orthogonal transformation.

We shall say that a convex body K is *surface isotropic* if its area measure σ_K is isotropic. This class of bodies with minimal surface area is very useful for questions related to hyperplane projections of convex bodies (one may say that it plays the role isotropic bodies – see [4] – play for questions related to sections). In Section 3 we demonstrate this in several ways:

We show that if K is surface isotropic then

$$\frac{\partial(K)}{2n} \leq |P_\theta(K)| \leq \frac{\partial(K)}{2\sqrt{n}} \tag{1.1}$$

for every $\theta \in S^{n-1}$, where P_θ denotes the orthogonal projection onto the subspace $\theta^\perp$ perpendicular to θ. Both estimates are sharp in the case of the cube. However, one can see that most projections of K have area close to the upper bound. More precisely, there exists an absolute constant $c > 0$ such that

$$|P_\theta(K)| \geq c\frac{\partial(K)}{\sqrt{n}} \tag{1.2}$$

with probability greater than $1 - 2^{-n}$ with respect to the standard measure σ on S^{n-1}. This should be compared with the fact that all $(n-1)$–dimensional central sections of an isotropic body have the same area up to an absolute constant (see [4]).

The projection body ΠK of K is the symmetric convex body whose support function is defined by $h_{\Pi K}(\theta) = |P_\theta(K)|$, $\theta \in S^{n-1}$. We write $\Pi^* K$ for the polar projection body.

The volume radius of ΠK and $\Pi^* K$ are determined by the minimal surface area ∂_K: If $|K| = 1$, then

$$|\Pi^* K|^{1/n} \simeq \frac{1}{\partial_K} \quad \text{and} \quad |\Pi K|^{1/n} \simeq \frac{\partial_K}{n} \tag{1.3}$$

up to absolute constants. This fact together with a result of G. Zhang [11] show that $\partial_K \leq cn$ for every body K. K. Ball [2] has proved a reverse isoperimetric inequality of this type. His result is exact in the symmetric case: If K is symmetric, then $\partial_K \leq 2n$ and there is equality in the case of a cube.

K. Ball notes in [1] that every convex body K has an affine image $\tilde{K}$ such that

$$|P_\theta(\tilde{K})| \leq M\sqrt{n}|\tilde{K}|^{\frac{n-1}{n}} \tag{1.4}$$

for some absolute constant $M > 0$ and for every $\theta \in S^{n-1}$, and conjectures that the constant M could be taken to be 1 in the symmetric case. Ball's reverse isoperimetric inequality and (1.1) show that, in the symmetric case, the surface isotropic position satisfies (1.4) with $M = 1$. The cube shows that (1.4) is sharp.

K. Ball [1] has also proved that every convex body K has an affine image $\tilde{K}$ such that for every unit vector θ,

$$|P_\theta(\tilde{K})| \geq |\tilde{K}|^{\frac{n-1}{n}}. \tag{1.5}$$

The classical isoperimetric inequality and (1.2) show that with high probability the projections of a surface isotropic body satisfy

$$|P_\theta(K)| \geq c|K|^{\frac{n-1}{n}}. \tag{1.6}$$

The information given by (1.2) is actually much stronger. However, we do not know if (1.6) holds for *every* $\theta \in S^{n-1}$. We can show that if σ_K is isotropic then the Loomis–Whitney type inequality

$$|K|^{\frac{n-1}{n}} \leq \exp\left(\frac{1}{\partial(K)}\int_{S^{n-1}} \log|P_\theta(K)|\,\sigma_K(d\theta)\right) \tag{1.7}$$

is true.

Finally, we show that extremal counterexamples for Shephard's problem (in the sense of [1]) can be constructed inside the class of surface isotropic symmetric convex bodies. There exist surface isotropic K's satisfying $|P_\theta(K)| \simeq |K|^{\frac{n-1}{n}}$ and others satisfying $|P_\theta(K)| \simeq \sqrt{n}|K|^{\frac{n-1}{n}}$ for every $\theta \in S^{n-1}$.

For well–known facts from the classical and the asymptotic theory of convex bodies we refer the reader to the books of R. Schneider [9] and G. Pisier [7].

2 A Proof of Petty's theorem

Petty's theorem will be an immediate consequence of the following three claims:

Claim 1 *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. If $\partial(K) = \partial_K$, then σ_K is isotropic.*

Proof: Let R be a volume preserving transformation of $\mathbb{R}^n$. It is not hard to check that

$$\partial((R^{-1})^* K) = \int_{S^{n-1}} |R(u)| \sigma_K(du). \tag{2.1}$$

Consider any $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and let $\varepsilon > 0$ be small enough. Then, $(I+\varepsilon T)/[\det(I+\varepsilon T)]^{1/n}$ is volume preserving, so our minimality assumption for K and (2.1) give

$$\int_{S^{n-1}} |(I+\varepsilon T)(u)| \sigma_K(du) \geq [\det(I+\varepsilon T)]^{\frac{1}{n}} \partial_K. \tag{2.2}$$

Observe that $|u+\varepsilon Tu| = 1+\varepsilon\langle u, Tu\rangle + O(\varepsilon^2)$ and $[\det(I+\varepsilon T)]^{1/n} = 1+\varepsilon\frac{\mathrm{tr}T}{n} + O(\varepsilon^2)$. Letting $\varepsilon \to 0$ we get

$$\int_{S^{n-1}} \langle u, Tu\rangle \sigma_K(du) \geq \frac{\mathrm{tr}T}{n} \partial_K, \tag{2.3}$$

and by symmetry we conclude that

$$\int_{S^{n-1}} \langle u, Tu\rangle \sigma_K(du) = \frac{\mathrm{tr}T}{n} \partial_K \tag{2.4}$$

for every linear transformation T. This implies

$$\int_{S^{n-1}} u_k u_l \sigma_K(du) = \frac{\partial_K}{n} \delta_{kl} \quad , \quad k,l = 1, \ldots, n, \tag{2.5}$$

which completes the proof. □

Claim 2 *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. If σ_K is isotropic, then $\partial(K) = \partial_K$.*

Proof: Assume that σ_K is isotropic. This is equivalent to the condition (2.4). If T is any volume preserving transformation, then

$$\partial(TK) = \int_{S^{n-1}} |(T^{-1})^* u| \sigma_K(du) \geq \int_{S^{n-1}} \langle u, T^{-1}u\rangle \sigma_K(du) \tag{2.6}$$

$$= \frac{\mathrm{tr}(T^{-1})}{n} \partial(K) \geq \partial(K),$$

because $\mathrm{tr}(T^{-1})/n \geq [\det(T^{-1})]^{1/n} = 1$. This shows that K has minimal surface area. □

Claim 3 *The minimal surface position is unique up to orthogonal transformations.*

Proof: Assume that K has minimal surface area, and $\partial(RK) = \partial(K)$ for some volume preserving transformation R. We can write $R = UT$, where T^{-1} is symmetric positive–definite and U is an orthogonal transformation. Repeating the computation of (2.6) we have

$$\partial(K) = \partial(UT(K)) = \partial(TK) \geq \frac{\mathrm{tr}(T^{-1})}{n} \partial(K).$$

This means that $\mathrm{tr}(T^{-1}) = n$, and since T is symmetric positive–definite we must have $T = \mathrm{Id}$. This proves the claim. $\square$

We close this section with a lemma which is useful for approximation arguments:

Lemma 2.1 *Every surface isotropic convex body is the limit of a sequence of surface isotropic polytopes in the Hausdorff metric.*

Proof: Let K_i be a sequence of polytopes converging to K in the Hausdorff metric. There exists a second sequence T_i of volume preserving transformations such that $\tilde{K}_i = T_i K_i$ is surface isotropic. We may also assume that $T_i K_i \to L = TK$ for some T with $|\mathrm{det} T| = 1$. Now, $\sigma_{\tilde{K}_i} \to \sigma_L$ and one can easily check that σ_L is isotropic. From Claim 3 it follows that $T \in O(n)$, and the proof is complete. $\square$

3 Projections of a surface isotropic body

Let K be a convex body in $\mathbb{R}^n$. The relation between the area of the $(n-1)$–dimensional projections of K and the area measure σ_K is well–known: For every $\theta \in S^{n-1}$ we have

$$|P_\theta(K)| = \frac{1}{2} \int_{S^{n-1}} |\langle u, \theta \rangle| d\sigma_K(u). \tag{3.1}$$

Using (3.1) we can easily obtain lower and upper bounds for the projections of a surface isotropic body:

Proposition 3.1 *If σ_K is isotropic, then*

$$\frac{\partial(K)}{2n} \leq |P_\theta(K)| \leq \frac{\partial(K)}{2\sqrt{n}}$$

for every $\theta \in S^{n-1}$.

Proof: A direct application of the Cauchy–Schwarz inequality shows that

$$|P_\theta(K)| \leq \frac{1}{2} \left(\int_{S^{n-1}} |\langle u, \theta \rangle|^2 d\sigma_K(u) \right)^{1/2} \sqrt{\partial(K)} = \frac{\partial(K)}{2\sqrt{n}}. \tag{3.2}$$

On the other hand, it is obvious that

$$|P_\theta(K)| \geq \frac{1}{2} \int_{S^{n-1}} |\langle u, \theta \rangle|^2 d\sigma_K(u) = \frac{\partial(K)}{2n}. \tag{3.3}$$ $\square$

The example of the cube Q shows that both estimates in Proposition 3.1 are sharp. One can easily check that

$$1 \leq |P_\theta(Q)| \leq \sqrt{n}$$

for every $\theta \in S^{n-1}$ with equality on both sides for suitable θ. The cube is surface isotropic, and $\partial(Q) = 2n$. Therefore, the ratio between the maximal and the minimal projection of a surface isotropic body can be as large as $\sqrt{n}$. However, we are going to show that the area of the projections is close to the upper bound with high probability. To this end we are using the following result of K. Ball [2] whose proof is based on the Brascamp–Lieb inequality:

Lemma 3.2 *Let $\{u_j\}_{j\leq m}$ be unit vectors in $\mathbb{R}^n$ and $\{c_j\}_{j\leq m}$ be positive numbers satisfying*

$$\mathrm{Id} = \sum_{j=1}^{m} c_j u_j \otimes u_j.$$

Define a norm in $\mathbb{R}^n$ by $\|x\| = \sum_{j=1}^m \alpha_j |\langle x, u_j\rangle|$, where $\alpha_j > 0$. If L is the unit ball of $(\mathbb{R}^n, \|\cdot\|)$, then

$$|L| \leq \frac{2^n}{n!} \prod_{j=1}^{m} \left(\frac{c_j}{\alpha_j}\right)^{c_j}. \quad \square$$

Assume that K is a polytope with facets F_j and normals u_j, $j = 1, \ldots, m$, which has isotropic area measure. Then, (2.4) is equivalent to

$$\mathrm{Id} = \sum_{j=1}^{m} \frac{n|F_j|}{\partial(K)} u_j \otimes u_j \tag{3.4}$$

for every $x \in \mathbb{R}^n$. If $\Pi^* K$ is the polar projection body of K we have

$$\|\theta\|_{\Pi^* K} = |P_\theta(K)| = \frac{1}{2} \sum_{j=1}^{m} |F_j| |\langle \theta, u_j\rangle| \tag{3.5}$$

for every $\theta \in S^{n-1}$. We can therefore apply Lemma 3.2 to obtain

$$|\Pi^* K| \leq \frac{2^n}{n!} \left(\frac{2n}{\partial(K)}\right)^n. \tag{3.6}$$

Approximating we get the same estimate for a general surface isotropic convex body. On the other hand, an application of Hölder's inequality shows that

$$|\Pi^* K| = \omega_n \int_{S^{n-1}} \frac{1}{|P_\theta(K)|^n} \sigma(d\theta) \geq \omega_n \left(\int_{S^{n-1}} |P_\theta(K)| \sigma(d\theta)\right)^{-n}. \tag{3.7}$$

Combining this with Cauchy's surface area formula

$$\int_{S^{n-1}} |P_\theta(K)| \sigma(d\theta) = \frac{\omega_{n-1}}{n\omega_n} \partial(K), \tag{3.8}$$

we have a double–sided estimate for the volume of the polar projection body of a surface isotropic body K:

Proposition 3.3 *Let K be a convex body in $\mathbb{R}^n$ whose area measure σ_K is isotropic. Then,*

$$\omega_n \left(\frac{n\omega_n}{\omega_{n-1}} \right)^n \frac{1}{\partial(K)^n} \leq |\Pi^* K| \leq \frac{4^n n^n}{n!} \frac{1}{\partial(K)^n}. \quad \square$$

Using the Blaschke–Santaló inequality and its exact reverse for zonoids (see [3], [8]) we estimate the volume of ΠK in terms of $\partial(K)$:

Corollary 3.4 *Let K be a surface isotropic convex body in $\mathbb{R}^n$. Then,*

$$\left(\frac{\partial(K)}{n} \right)^n \leq |\Pi K| \leq \omega_n \left(\frac{\omega_{n-1}}{n\omega_n} \right)^n \partial(K)^n. \quad \square$$

Note that all the estimates in Proposition 3.3 and Corollary 3.4 are sharp: it is enough to consider the ball and the cube.

Proposition 3.3 has some immediate consequences:

Assume that $|K| = 1$ and $\partial(K) = \partial_K$. We have

$$\int_{S^{n-1}} \frac{1}{|P_\theta(K)|^n} \sigma(d\theta) = \frac{|\Pi^* K|}{\omega_n} \leq \left(\frac{c\sqrt{n}}{\partial_K} \right)^n, \tag{3.9}$$

hence Markov's inequality gives:

Theorem 3.5 *There exists an absolute constant $c > 0$ such that for every surface isotropic body K of volume 1*

$$c \frac{\partial_K}{\sqrt{n}} \leq |P_\theta(K)| \leq \frac{\partial_K}{2\sqrt{n}}$$

holds true for all θ in a subset of S^{n-1} with σ–measure exceeding $1 - 2^{-n}$. $\square$

It is an observation of C.M. Petty [6] that if T is a linear operator on $\mathbb{R}^n$ of determinant 1 then, for every K,

$$\Pi^*(TK) = T(\Pi^* K).$$

In particular, the volume of the polar projection body is invariant under volume preserving affine transformations. Then, Proposition 3.3 and Corollary 3.4 show that $|\Pi^* K|^{1/n}$ and $|\Pi K|^{1/n}$ are determined up to absolute constants by the minimal surface area ∂_K:

Theorem 3.6 *There exist absolute constants $c_1, c_2, c_3, c_4 > 0$ such that if $|K| = 1$ then*

$$\frac{c_1}{\partial_K} \leq |\Pi^* K|^{1/n} \leq \frac{c_2}{\partial_K} \quad \text{and} \quad c_3 \frac{\partial_K}{n} \leq |\Pi K|^{1/n} \leq c_4 \frac{\partial_K}{n}. \quad \square$$

G. Zhang [11] has proved that if $|K| = 1$, then

$$|\Pi^* K| \geq \binom{2n}{n} n^{-n} \tag{3.10}$$

with equality if and only if K is a simplex. This can be combined with Proposition 3.3 to give an upper estimate for ∂_K:

Theorem 3.7 *For every convex body K we have*

$$\partial_K \leq 4n^2/[n!\binom{2n}{n}]^{1/n} \leq cn,$$

where $c > 0$ is an absolute constant. □

The constant c may be (asymptotically) chosen to be equal to e. K. Ball [2] has proved an exact reverse isoperimetric inequality of this type in the symmetric case: Let Q be the cube of volume 1. Every symmetric convex body K has an affine image $\tilde{K}$ satisfying $|\tilde{K}| = 1$ and $\partial(\tilde{K}) \leq \partial(Q)$. In our notation this means that

$$\partial_K \leq 2n \tag{3.11}$$

for every symmetric convex body. Observe that this would follow from (3.6) if the cube was minimizing $|\Pi^* K|$ over all symmetric bodies of volume 1. Combining (3.11) with Proposition 3.1 and dropping the assumption that $|K| = 1$ we get:

Proposition 3.8 *Let K be a symmetric convex body in $\mathbb{R}^n$ whose area measure σ_K is isotropic. Then,*

$$|P_\theta(K)| \leq \sqrt{n}|K|^{\frac{n-1}{n}}$$

for every $\theta \in S^{n-1}$. □

The estimate given by Proposition 3.8 was conjectured by K. Ball and it is sharp for the cube. One can use Ball's non–symmetric reverse isoperimetric inequality (or Theorem 3.7) to state an analogous result for an arbitrary surface isotropic body K.

We now turn to lower bounds. We shall make use of an extension of the Loomis–Whitney inequality that K. Ball [1] proved and used for the proof of (1.5):

Lemma 3.9 *Let K be a convex body in $\mathbb{R}^n$, $\{u_j\}_{j\leq m}$ be a sequence of unit vectors, and $\{c_j\}_{j\leq m}$ be a sequence of positive numbers such that*

$$\mathrm{Id} = \sum_{j=1}^{m} c_j u_j \otimes u_j.$$

Then,

$$|K|^{n-1} \leq \prod_{j=1}^{m} |P_{u_j}(K)|^{c_j}. \quad \square$$

Assume that K is a polytope with facets F_j and normals u_j, $j = 1, \ldots, m$, which has isotropic area measure. Because of (3.4) we can apply Lemma 3.9 to obtain

$$(n-1)\log|K| \leq \frac{n}{\partial(K)} \sum_{j=1}^{m} \log|P_{u_j}(K)||F_j|,$$

which can be written in the form

$$|K|^{\frac{n-1}{n}} \leq \exp\left(\frac{1}{\partial(K)}\sum_{j=1}^{m}\log|P_{u_j}(K)||F_j|\right). \tag{3.12}$$

Using this fact and an approximation argument we can prove the following:

Proposition 3.10 *Let K be a convex body in $\mathbb{R}^n$ whose area measure σ_K is isotropic. Then,*

$$|K|^{\frac{n-1}{n}} \leq \exp\left(\frac{1}{\partial(K)}\int_{S^{n-1}}\log|P_\theta(K)|\sigma_K(d\theta)\right).$$

Proof: Assume that $\{K_i\}$ is a sequence of polytopes with σ_{K_i} isotropic and $K_i \to K$, By (3.12) we have

$$|K_i|^{\frac{n-1}{n}} \leq \exp\left(\frac{1}{\partial(K_i)}\int_{S^{n-1}}\log|P_\theta(K_i)|\sigma_{K_i}(d\theta)\right)$$

for every i, and taking limit as $i \to \infty$ we conclude the proof. □

Combining Cauchy's formula with the isoperimetric inequality and Ball's reverse isoperimetric inequality we have:

Proposition 3.11 *Let K be a surface isotropic symmetric convex body in $\mathbb{R}^n$. Then,*

$$\max_\theta|P_\theta(K)| \geq [\omega_{n-1}/\omega_n^{\frac{n-1}{n}}]|K|^{\frac{n-1}{n}} \geq c|K|^{\frac{n-1}{n}}$$

and

$$\min_\theta|P_\theta(K)| \leq [2\omega_{n-1}/\omega_n]|K|^{\frac{n-1}{n}} \leq c\sqrt{n}|K|^{\frac{n-1}{n}}.\square$$

The first inequality is sharp for a ball: the ball of volume 1 has all its projections of volume of the order of 1. We now show that the second inequality is also exact up to an absolute constant:

Proposition 3.12 *There exists a surface isotropic symmetric convex body such that*

$$c\sqrt{n}|K|^{\frac{n-1}{n}} \leq |P_\theta(K)| \leq \sqrt{n}|K|^{\frac{n-1}{n}}$$

for every $\theta \in S^{n-1}$.

Proof: We shall follow Ball's counterexample on Shephard's problem [1]. What we want to make sure is that the example will be surface isotropic.

Let $\{e_j : j \leq n\}$ be an orthonormal basis in $\mathbb{R}^n$. It is a known fact going back to the work of Kashin (see [7, Chapter 6]) that there exists an orthogonal transformation $T \in O(n)$ such that

$$c_1\sqrt{n}|x| \leq \|x\|_1 + \|Tx\|_1 \leq c_2\sqrt{n}|x| \tag{3.13}$$

for every $x \in \mathbb{R}^n$, where $\|x\|_1 = \sum_{j=1}^n |\langle x, e_j\rangle|$. It is clear that the sum of two isotropic measures on S^{n-1} is isotropic, therefore the measure

$$\sigma = \sigma_Q + \sigma_{T^{-1}Q}$$

is isotropic. From Minkowski's existence theorem, there exists a symmetric polytope K with $\sigma_K = \sigma$.
K has normals $\pm e_j, \pm T^* e_j$, and the area of each facet of K is equal to 1. Using (3.13) we get

$$|P_\theta(K)| = \sum_{j=1}^n (|\langle \theta, e_j\rangle| + |\langle \theta, T^* e_j\rangle|) = \|\theta\|_1 + \|T\theta\|_1 \geq c_1\sqrt{n} \tag{3.14}$$

for every $\theta \in S^{n-1}$. As in [1], consider the body

$$L = \{x \in \mathbb{R}^n : |\langle x, e_j\rangle| \leq 1 \ , \ |\langle x, T^* e_j\rangle| \leq 1 \ , \ j = 1, \ldots, n\}.$$

By a result of Vaaler [10], $|L|^{1/n} \geq \sqrt{2}$, and by Minkowski's inequality,

$$|K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}} \leq \frac{1}{n} \int_{S^{n-1}} h_L(u)\sigma_K(du) = \frac{2}{n} \sum_{j=1}^n (h_L(e_j) + h_L(T^* e_j)) \leq 4. \tag{3.15}$$

It follows that

$$|P_\theta(K)| \geq \frac{c_1}{4}\sqrt{n}|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}} \geq c\sqrt{n}|K|^{\frac{n-1}{n}}, \tag{3.16}$$

which completes the proof. □

Corollary 3.13 *There exist two surface isotropic symmetric convex bodies K_1 and K_2 in $\mathbb{R}^n$ such that $|P_\theta(K_1)| \leq |P_\theta(K_2)|$ for every $\theta \in S^{n-1}$, but $|K_1| \geq c\sqrt{n}|K_2|$, where $c > 0$ is an absolute constant.* □

4 Stability of the surface isotropic position

Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. Assume that it is close to the surface isotropic position in the sense that all the integrals

$$\int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du) \quad , \quad \theta \in S^{n-1}$$

are equivalent up to some constant $\beta > 1$. We shall show that the surface area $\partial(K)$ of K is close to the minimal surface area ∂_K. More precisely, we have the following stability result:

Theorem 4.1 *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. Assume that there exist $A > 0$ and $\beta > 1$ such that*

$$A \leq \int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du) \leq \beta A \tag{4.1}$$

for every $\theta \in S^{n-1}$. *Then,*

$$\partial_K \leq \partial(K) \leq c\beta\partial_K, \tag{4.2}$$

where $c > 0$ *is an absolute constant.*

Proof: Choosing $\theta = e_i$, $i = 1, \ldots, n$ in (4.1) and adding all inequalities we get

$$\frac{1}{\beta}\frac{\partial(K)}{n} \leq \int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du) \leq \beta\frac{\partial(K)}{n} \quad , \quad \theta \in S^{n-1}. \tag{4.3}$$

We may assume that K is a polytope with facets F_j and normals u_j, $j = 1, \ldots, m$. Consider the ellipsoid E defined by

$$\|x\|_E^2 = \sum_{j=1}^m |F_j|\langle x, u_j\rangle^2 = \int_{S^{n-1}} \langle u, x\rangle^2 \sigma_K(du). \tag{4.4}$$

Then $E = TD_n$ for some linear transformation T, and (4.3) shows that

$$\frac{1}{\beta}\frac{\partial(K)}{n}|Tx|^2 \leq |x|^2 = \|Tx\|_E^2 = \sum_{j=1}^m |F_j|\langle x, T^*(u_j)\rangle^2 \leq \beta\frac{\partial(K)}{n}|Tx|^2. \tag{4.5}$$

In particular,

$$|\det T|^{1/n} \leq \|T : \ell_2^n \to \ell_2^n\| \leq \sqrt{\beta n/\partial(K)}. \tag{4.6}$$

Consider the body $K_1 = T^{-1}K$. It has normals $u_j' = T^*(u_j)/|T^*(u_j)|$ and facets $|F_j'| = |T^*(u_j)||F_j|/|\det T|$. From (4.5) it is clear that

$$|x|^2 = \sum_{j=1}^m |F_j||T^*(u_j)|^2\langle x, u_j'\rangle^2,$$

which implies

$$\mathrm{Id} = \sum_{j=1}^m |F_j||T^*(u_j)|^2 u_j' \otimes u_j'. \tag{4.7}$$

Consider the polar projection body of K_1. We have

$$\|\theta\|_{\Pi^* K_1} = |P_\theta(K_1)| = \sum_{j=1}^m \frac{|F_j||T^*(u_j)|}{2|\det T|}|\langle \theta, u_j'\rangle|, \tag{4.8}$$

and Lemma 3.2 shows that

$$\begin{aligned} |\Pi^* K_1| &\leq \frac{2^n}{n!}\prod_{j=1}^m (2|T^*(u_j)||\det T|)^{|F_j||T^*(u_j)|^2} \\ &\leq \frac{2^n}{n!}(2\|T\||\det T|)^{\sum_j |F_j||T^*(u_j)|^2} \\ &= \frac{4^n}{n!}\|T\|^n|\det T|^n. \end{aligned}$$

Using Theorem 3.6 and (4.6), we see that

$$\frac{c}{\partial_K} \leq |K_1|^{\frac{n-1}{n}} |\Pi^* K_1|^{\frac{1}{n}} \leq \frac{4}{(n!)^{1/n}} \|T\| |\det T|^{\frac{1}{n}} \leq \frac{4e\beta}{\partial(K)},$$

which shows that

$$\partial_K \leq \partial(K) \leq \frac{4e}{c} \beta \partial_K. \quad \square$$

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