Volume of the polar of random sets and shadow systems

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Abstract

We obtain optimal inequalities for the volume of the polar of random sets, generated for instance by the convex hull of independent random vectors in Euclidean space. Extremizers are given by random vectors uniformly distributed in Euclidean balls. This provides a random extension of the Blaschke-Santaló inequality which, in turn, can be derived by the law of large numbers. The method involves generalized shadow systems, their connection to Busemann type inequalities, and how they interact with functional rearrangement inequalities.

1 Introduction

A celebrated result of Blaschke and Santaló [23] states that among symmetric convex bodies K of fixed volume in the Euclidean space $(\mathbb{R}^n, |\cdot|)$, the volume of the polar body K° is maximized by the Euclidean ball, and therefore also by ellipsoids, by SL_n-invariance (precise definitions will be recalled below, in §2); we refer to [15] for a proof based on Steiner symmetrization. In the present paper, we are interested in extending such a result to random sets. A typical example of such a random set is given by the convex hull of the columns of a random matrix, for which we can prove the following property.

Theorem 1.1. *Let $N, n \geq 1$. In the class of N-tuples $(X_1, \ldots, X_N)$ of independent random vectors in $\mathbb{R}^n$ whose laws have a density bounded by one, the expectation of the volume of the set*

$$\left(\mathrm{conv}\{\pm X_1, \ldots, \pm X_N\}\right)^\circ$$

is maximized by N independent random vectors uniformly distributed in the Euclidean ball $D_n \subset \mathbb{R}^n$ of volume one.

The density of a measure on $\mathbb{R}^n$ will always refer to the density with respect to Lebesgue measure on $\mathbb{R}^n$ (so it is implicit that the measure is absolutely continuous with respect to Lebesgue measure).

To see how the latter theorem generalizes the Blaschke-Santaló inequality, let K be a symmetric convex body and assume, without loss of generality, that $|K| = 1$, where $|\cdot|$ denotes Lebesgue measure. Let $X_1, \ldots, X_N, \ldots$ be a sequence of independent random vectors uniformly distributed in K; by this we mean that the law of X_i is λ_K, Lebesgue measure restricted to K, which has density 1_K (indeed bounded by one). It is known that $\operatorname{conv}\{\pm X_1, \ldots, \pm X_N\}$ converges almost surely to K, in the Hausdorff metric, as $N \to +\infty$. The latter also holds in the special case when $K = D_n$. Consequently, we derive from the theorem above that, in the limit, $|K^\circ| \leq |D_n^\circ|$ under the assumption that $|K| = |D_n| = 1$, which is the Blaschke-Santaló inequality; more detailed arguments as well as other applications will be given in §5.

Our work is part of the program initiated in [20] aimed at obtaining systematic random extensions (i.e. for random sets) of several inequalities in convexity. In the present paper, we treat inequalities that are *dual* to those considered in [20] (i.e. inequalities for the polar bodies, such as the Blaschke-Santaló inequality). Steiner symmetrization, and more precisely *shadow systems*, as in the work of Campi and Gronchi [9], which was our main source of inspiration, will play a central role, together with rearrangement inequalities.

In fact, we prove a general inequality, in the spirit of those established in [20]. Our main result below extends the statement of the previous theorem in several ways:

(i) the result holds in distribution, not only in expectation;

(iii) we can replace Lebesgue measure by any rotationally invariant, radially decreasing, measure (for instance we can consider the volume of the intersection with a fixed Euclidean ball);

(iii) we can perform more general (convex) operations than the convex hull.

Before stating the result, we need to introduce a bit of notation. Given N vectors $x_1, \ldots, x_N$ in some $\mathbb{R}^n$ space, we form the $n \times N$ matrix $[x_1 \cdots x_N]$ that we view as an operator from $\mathbb{R}^N$ to $\mathbb{R}^n$ or rather to $\operatorname{span}\{x_i\} \subset \mathbb{R}^n$; therefore if C is a set in $\mathbb{R}^N$, we denote

$$[x_1 \cdots x_N]C = \left\{ \sum_{i=1}^{N} c_i \, x_i : c = (c_i)_{i \leq N} \in C \right\} \subset \operatorname{span}\{x_i\} \subset \mathbb{R}^n.$$

Accordingly, if $X_1, \ldots, X_N$ are random vectors in $\mathbb{R}^n$, the random matrix $[X_1 \cdots X_N]$ is a random linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$ and for $C \subset \mathbb{R}^N$, $[X_1 \cdots X_N]C$ is a random set in $\mathbb{R}^n$.

A convex body C in $\mathbb{R}^N$ is *unconditional* if it is invariant under the coordinate reflections, i.e. $(c_1, \ldots, c_N) \in C \Rightarrow (\pm c_1, \ldots, \pm c_N) \in C$. Typical examples include the unit ball $B_p^N = \{c \in \mathbb{R}^N \ : \ \sum |c_i|^p \leq 1\}$ of the ℓ_p^N space, for $p \geq 1$.

Finally, let us denote by $\mathcal{P}_n$ the class of all Borel probability measures on $\mathbb{R}^n$ that have an L^1-density with respect to Lebesgue measure bounded by 1, i.e. with some abuse of notation,

$$\mathcal{P}_n = \Big\{\mu \ : \ d\mu(x) = f(x)\,dx \ \text{ with } f \geq 0, \int f = 1 \text{ and } \|f\|_\infty \leq 1\Big\}.$$

where $\|\cdot\|_\infty$ is the essential supremum. This set includes Lebesgue measure restricted to sets of volume one, and actually after proper scaling (dilation), any Borel probability measure that is absolutely continuous with respect to Lebesgue measure and that has a bounded density.

Our main result is as follows.

Theorem 1.2. *Let C be an unconditional convex body in $\mathbb{R}^N$ and ν be a radial measure on $\mathbb{R}^n$ of the form $d\nu(x) = \rho(|x|)\,dx$ with $\rho : [0, +\infty) \to [0 + \infty)$ decreasing. If $X_1, \ldots, X_N$ are N independent random vectors in $\mathbb{R}^n$ whose laws are in $\mathcal{P}_n$, then*

$$\mathbb{E}\big[\nu\big(([X_1 \cdots X_N]C)^\circ\big)\big] \leq \mathbb{E}\big[\nu\big(([Z_1 \cdots Z_N]C)^\circ\big)\big]. \tag{1.1}$$

where $Z_1, \ldots, Z_N$ are independent random vectors uniformly distributed in the Euclidean ball $D_n \subset \mathbb{R}^n$ of volume one.

Moreover, if $\rho^{-1/(n+1)} : [0, +\infty) \to [0, +\infty]$ is convex, then, with the same notation, we also have that

$$\forall t > 0, \quad \mathbb{P}\big[\nu\big(([X_1 \cdots X_N]C)^\circ\big) \geq t\big] \leq \mathbb{P}\big[\nu\big(([Z_1 \cdots Z_N]C)^\circ\big) \geq t\big]. \tag{1.2}$$

Throughout the text, we will use the terms increasing and decreasing in the non-strict sense.

In particular, note that both results (1.1) and (1.2) hold in the case where $\nu = |\cdot|$ is Lebesgue measure, and more generally when $\nu(\cdot) = |\cdot \cap r B_2^n|$ is the restriction of Lebesgue measure to any Euclidean ball $\{|x| \leq r\}$, $r \in (0, +\infty]$. Of course, (1.1) formally follows from (1.2), but we stated it first because we can prove it for a more general class of measures ν. We do not claim, though, that the further convexity assumption is necessary for the inequality to hold in distribution; but it is needed in our proof.

To recover Theorem 1.1 from the previous theorem, simply notice that if $C = B_1^N$, the unit ball in ℓ_1^N, then

$$[X_1 \cdots X_N]B_1^N = \text{conv}\{\pm X_1 \cdots \pm X_N\}.$$

More generally, using $C = B_q^N$, for some $1 \le q \le +\infty$, we also recover in §5 below an inequality which implies the polar L^p centroid body inequality of Lutwak-Zhang [14].

In §2 we recall some basic notation and facts from convex geometry, in particular Borell's terminology and results concerning dimensional forms of Prékopa's theorem. The content of §3 might be of independent interest. There, we first recall and extend Busemann type results, for which we explain how to derive them from the aforementioned dimensional inequalities by a simple change of variable. Then we apply these results to the measure of the polar sets along generalized shadow systems, in the spirit of the work of Campi and Gronchi [9]. In §4, we start by recalling the rearrangement inequality of Rogers-Brascamp-Lieb-Luttinger type, in the form put forward by Christ [10]. With these ingredients in hand, we procede, at the end of §4, to give the proof of our main result, Theorem 1.2. Finally, §5 presents some further applications of our result to (non-random) geometric inequalities.

2 Preliminaries

We work in Euclidean spaces $\mathbb{R}^n, \mathbb{R}^N \subset \mathbb{R}^{n+N} = \mathbb{R}^n \oplus \mathbb{R}^N$ with the canonical embeddings and we assume that $N, n \geqslant 1$. The usual inner-product is denoted $\langle \cdot, \cdot \rangle$ with associated Euclidean norm $|\cdot|$ and the standard unit vector basis is $e_1, \ldots, e_{n+N}$. We also use $|\cdot|$ for the n-dimensional Lebesgue measure and the absolute value of a scalar, the use of which will be clear from the context. The Euclidean ball of radius one is denoted B_2^n and its volume $\omega_n := |B_2^n|$. We reserve D_n for the Euclidean ball of volume one, i.e., $D_n = \omega_n^{-1/n} B_2^n$; Lebesgue measure restricted to D_n is λ_{D_n}; it belongs to $\mathcal{P}_n$. The unit sphere is $S^{n-1} \subset \mathbb{R}^n$ and is equipped with the Haar probability measure σ, which is the usual rotationally invariant measure on S^{n-1}, normalized to be a probability measure. Recall that B_p^n denotes the unit-ball of ℓ_p^n, $1 \leqslant p \leqslant \infty$.

The support function of a convex set $K \subset \mathbb{R}^n$ is given by

$$h_K(x) = \sup\{\langle y, x \rangle \ : \ y \in K\} \quad (x \in \mathbb{R}^n).$$

If K and L are convex sets in $\mathbb{R}^n$ then

$$h_K(x) + h_L(x) = h_{K+L}(x) \quad (x \in \mathbb{R}^n),$$

where $K + L$ is the Minkowski sum of K and L:

$$K + L = \{k + l \ : \ k \in K, l \in L\}.$$

If $K \subset \mathbb{R}^n$ is a convex set, the polar K° of K is defined by $K^\circ = \{y \in \mathbb{R}^n : h_K(y) \leqslant 1\}$. A *convex body* $K \subset \mathbb{R}^n$ is a compact, convex set with non-empty interior. A set $K \subset \mathbb{R}^n$ is a *star-body* with respect to the origin if it is compact, with the origin in its interior and for every $x \in K$ and $\lambda \in [0,1]$ we have $\lambda x \in K$. Then its *gauge function* is denoted by $\|\cdot\|_K$ and is defined as $\|x\|_K = \inf\{t > 0 : x \in tK\}$. We say that K is (origin) symmetric if $K = -K$. We already gave the definition of *unconditional convex bodies* which are an important subfamily of symmetric convex bodies.

For K, L being convex sets in $\mathbb{R}^n$, we let $\delta^H(K,L)$ denote the Hausdorff distance between them, i.e.,

$$\delta^H(K,L) = \inf\{\varepsilon > 0 \ : \ K \subset L + \varepsilon B_2^n, L \subset K + \varepsilon B_2^n\};$$

or equivalently, in terms of support functions,

$$\delta^H(K,L) = \sup_{\theta \in S^{n-1}} |h_K(\theta) - h_L(\theta)|.$$

Let $\mathcal{K}_\circ^n$ denote the class of all convex bodies that contain the origin in their interior. We will make use of the following basic facts (see, e.g. [18, 24]).

Lemma 2.1. *Let $K, L, K_1, K_2, \ldots \in \mathcal{K}_\circ^n$ be such that $K_N \xrightarrow{\delta_H} K$ as $N \to \infty$. Then*

(i) $K_N^\circ \xrightarrow{\delta_H} K^\circ$ *as* $N \to \infty$

(ii) $K_N \cap L \xrightarrow{\delta_H} K \cap L$ *as* $N \to \infty$

(iii) $K_N + L \xrightarrow{\delta_H} K + L$ *as* $N \to \infty$

If $A \subset \mathbb{R}^n$ is a Borel set with finite volume, the symmetric rearrangement A^* of A is the (open) Euclidean ball centered at the origin whose volume is equal to that of A. The symmetric decreasing rearrangement of 1_A is defined by $(1_A)^* := 1_{A^*}$. If $f : \mathbb{R}^n \to \mathbb{R}^+$ is an integrable function, we define its symmetric decreasing rearrangement f^* by

$$f^*(x) = \int_0^\infty 1^*_{\{f>t\}}(x)dt = \int_0^\infty 1_{\{f>t\}^*}(x)dt.$$

The latter should be compared with the "layer-cake representation" of f:

$$f(x) = \int_0^\infty 1_{\{f>t\}}(x)dt; \tag{2.1}$$

see [13, Theorem 1.13]. The function f^* is radially-symmetric, decreasing and equimeasurable with f, i.e., $\{f > \alpha\}$ and $\{f^* > \alpha\}$ have the same volume for each

$\alpha > 0$. By equimeasurability one has $\|f\|_p = \|f^*\|_p$ for each $1 \leqslant p \leqslant \infty$, where $\|\cdot\|_p$ denotes the $L_p(\mathbb{R}^n)$-norm. If $\mu \in \mathcal{P}_n$ has density f_μ, we let μ^* denote the measure in $\mathcal{P}_n$ with density f_μ^*. For completeness, recall that for a nonnegative function f in $\mathbb{R}^n$, the rearrangement f^* can be reached by a sequence of *Steiner symmetrizations* $f^*(\cdot|\theta)$, which correspond to symmetrization in dimension one in the direction $\theta \in S^{n-1}$; namely $f^*(\cdot|\theta)$ is obtained by rearranging f (in dimension 1) along every line parallel to θ. The function $f^*(\cdot|\theta)$ is symmetric with respect to $\theta^\perp$ (by this we mean invariant under the hyperplane reflection $\sigma_\theta(x) := x - 2\langle x, \theta\rangle\theta$). We refer the reader to the book [13] for further background material on rearrangements of functions.

Let us now recall the results and terminology of Borell [4, 5].

Definition 2.2 (Borell's terminology)**.** *Let $s \in [-\infty, 1]$. A Borel measure μ on $\mathbb{R}^n$ is called* s-concave *if*

$$\mu\left((1-\lambda)A + \lambda B\right) \geq \left((1-\lambda)\mu(A)^s + \lambda\mu(B)^s\right)^{1/s}$$

for all compact sets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$. For $s = 0$, one says that μ is log-*concave and the inequality is read as $\mu\left((1-\lambda)A + \lambda B\right) \geq \mu(A)^{1-\lambda}\mu(B)^\lambda$. For $s = -\infty$, the measure is said to be* convex *and the inequality is replaced by*

$$\mu\left((1-\lambda)A + \lambda B\right) \geq \min\left(\mu(A), \mu(B)\right).$$

Notice that the class of s-concave measures on $\mathbb{R}^n$ is decreasing in s, so that convex measures form the largest one.

We have an analogous notion of γ-concavity for functions. Namely, by definition, a nonnegative, non-identically zero, function ψ is *γ-concave* if: (i) for $\gamma > 0$, ψ^γ is concave on $\{\psi > 0\}$; (ii) for $\gamma = 0$, $\log\psi$ is concave on $\{\psi > 0\}$; (iii) for $\gamma < 0$, ψ^γ is convex on $\{\psi > 0\}$.

In [4, 5], Borell established the following complete characterization of convex measures. An s-concave measure μ is always supported on some convex subset of an affine subspace E where it has a density. Moreover, if μ is a measure on $\mathbb{R}^n$ absolutely continuous with respect to Lebesgue measure with density ψ, then it is s-concave if and only if its density ψ is γ-concave where $\gamma = s/(1-ns)$. In particular, a measure μ with density ψ on $\mathbb{R}^n$ is a convex measure if and only if ψ is $-1/n$-concave. A crucial tool in our arguments will be the dimensional form of Prékopa's theorem obtained in [5, 7] as a corollary of the functional versions of the Brunn-Minkowski inequality, known as the *Borell-Brascamp-Lieb inequalities.* It can also be seen as a direct consequence of the aforementioned characterization of Borell and the fact that the marginals of an s-concave measure are always s-concave. Thus the following theorem could be called by many names such as "Borell-Brascamp-Lieb restricted to convex functions" or "the dimensional Prékopa's theorem" or "the functional Brunn's principle". In this paper, we shall use the last two names.

Theorem 2.3. (Functional Brunn's principle) *Let $\varphi : \mathbb{R}^{n+1} \to (0, \infty]$ be a positive convex function and let $\alpha > 0$. Then the function Φ defined on $\mathbb{R}$ by*

$$\Phi(t) = \left(\int_{\mathbb{R}^n} \varphi(t, x)^{-n-\alpha} \, dx \right)^{-\frac{1}{\alpha}}$$

is convex.

We shall also sometimes combine it with the following easy and well-known lemma.

Lemma 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a convex function. Then the function $\varphi : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}_+$ defined for $(z, s) \in \mathbb{R}^n \times (0, +\infty)$ by $\varphi(z, s) = sf(z/s)$ is convex.*

Proof. Notice that φ is positively homogeneous in the sense that $\varphi(\lambda z, \lambda s) = \lambda \varphi(z, s)$ for every $z \in \mathbb{R}^n$ and $s, \lambda > 0$. For every $s_1, s_2 > 0$, $\lambda_1, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 = 1$ and $z_1, z_2 \in \mathbb{R}^n$, with $f(z_1/s_1) > 0$ and $f(z_2/s_2) > 0$ one has

$$\begin{aligned} \varphi(\lambda_1 z_1 + \lambda_2 z_2, \lambda_1 s_1 + \lambda_2 s_2) &= (\lambda_1 s_1 + \lambda_2 s_2) f \left(\frac{\lambda_1 s_1 \frac{z_1}{s_1} + \lambda_2 s_2 \frac{z_2}{s_2}}{\lambda_1 s_1 + \lambda_2 s_2} \right) \\ &\leq \lambda_1 s_1 f \left(\frac{z_1}{s_1} \right) + \lambda_2 s_2 f \left(\frac{z_2}{s_2} \right) \\ &= \lambda_1 \varphi(z_1, s_1) + \lambda_2 \varphi(z_2, s_2). \end{aligned}$$

□

3 Busemann's theorem and shadow systems for convex measures

In this section, we first recall a generalization of Busemann's inequality, that we derive from the functional Brunn's Principle (Theorem 2.3) by an elementary argument; we then use it to deduce an extension of Busemann's theorem to convex measures. In the second part of this section, we combine these inequalities to extend a theorem of Campi and Gronchi [9] to generalized shadow systems and convex measures.

3.1 Busemann's theorem for convex measures

The following theorem is Bobkov's generalization [3] of a theorem due to Ball [2] in the log-concave case. The short proof that we give below shows that it follows from the functional Brunn's principle by a change of variable (which simplifies the argument given by Bobkov [3]). This theorem enables one to attach to a function with some concavity properties a family of convex bodies (the unit balls of the gauges given by the theorem), sometimes called Ball's bodies. It is a key technique due to Ball to extend results from convex sets to log-concave functions.

Theorem 3.1 ([2, 3])**.** *Let $p > 0$ and $f : \mathbb{R}^n \to \mathbb{R}_+$ be γ-concave for some $\gamma \geq -1/(p+1)$. Then the function $F : \mathbb{R}^n \to \mathbb{R}_+$ defined by*

$$F(x) = \left(\int_0^{+\infty} f(rx)\, r^{p-1} dr \right)^{-\frac{1}{p}}$$

is a gauge on $\mathbb{R}^n$.

Proof. Let us denote by φ the convex function such that $\varphi = f^{-1/(p+1)}$. Using the change of variable $r = 1/s$, we get

$$\int_0^{+\infty} f(rx) r^{p-1} dr = \int_0^{+\infty} \left(s\varphi(x/s) \right)^{-(p+1)} ds$$

From Lemma 2.4, we know that $(x, s) \to s\varphi(x/s)$ is convex on $(0, +\infty) \times \mathbb{R}^n$. From the functional Brunn's principle (Theorem 2.3), we conclude that F is convex. $\square$

The previous proof clarifies the relation between Brunn-Minkowski type results and Busemann's theorem, which follows from Theorem 3.1 for $p = 1$, as we shall see below. Indeed, even if we are interested in the case of a log-concave function f (which means that f is 0-concave, and therefore also $-1/2$-concave), the shortest proof uses the dimensional Prékopa theorem for $-1/2$-concave densities (rather than the usual Prékopa theorem for log-concave densities).

Actually, combining the previous theorem with one more instance of the functional Brunn's principle, one may also extend Milman-Pajor's generalization of Busemann's theorem [17] to densities with less concavity.

Proposition 3.2. *Let E be a k-dimensional subspace of $\mathbb{R}^n$ and $p > 0$. If $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ is a γ-concave function for some $\gamma \geq -1/(k+p+1)$, then the function $\Phi : E^\perp \to \mathbb{R}_+$ defined for $v \in E^\perp \setminus \{0\}$ by*

$$\Phi(v) = |v|^{\frac{2p-1}{p}} \left(\int_{E \oplus \mathbb{R}_+ v} \langle x, v \rangle^{p-1} \varphi(x)\, dx \right)^{-1/p}$$

is a gauge on $E^\perp$.

This result formally contains Theorem 3.1 (the case $E = \{0\}$, $k = 0$), but in the application below we will rather use the cases $k = n-2$ or $k = (n+1)-2 = n-1$ and $p = 1$.

Proof. The homogeneity is clear so it suffices to prove the convexity. Introduce the function $f : E^{\perp} \to \mathbb{R}_+$ defined by

$$f(x) = \int_E \varphi(x+y)dy.$$

From the functional Brunn's principle (Theorem 2.3), it follows that f is $-1/(p+1)$-concave. By Fubini and the normal parametrization of $\mathbb{R}_+ v$ by $s\frac{v}{|v|}$ we have that, for $v \neq 0$,

$$\begin{aligned}
\Phi(v) &= |v|^{\frac{2p-1}{p}} \left(\int_{\mathbb{R}_+ v} \langle x, v\rangle^{p-1} f(x)\, dx \right)^{-1/p} \\
&= |v|^{\frac{2p-1}{p}} \left(\int_0^{+\infty} |v|^{p-1} s^{p-1} f\big(s\frac{v}{|v|}\big)\, ds \right)^{-1/p} \\
&= \left(\int_0^{+\infty} r^{p-1} f(rv)\, dr \right)^{-1/p}.
\end{aligned}$$

So the result follows from Theorem 3.1 with $E^{\perp}$ in place of $\mathbb{R}^n$. $\square$

We now apply the previous results to extend Busemann's theorem [8] to convex measures. The case of log-concave measures is due to Kim, Yaskin and Zvavitch [12] who proved it somewhat differently by applying the usual Busemann's theorem to Ball's body associated to the measure. The same method could also be used in our more general setting.

For any measure ν on $\mathbb{R}^n$ with a density ψ, $d\nu(x) = \psi(x)\, dx$, and for every hyperplane H we define the ν-measure of H to be

$$\nu^+(H) = \int_H \psi(x)dx,$$

where dx denotes Lebesgue measure on H.

Theorem 3.3 (Busemann's theorem for convex measures)**.** *Let ν be a convex measure with even density ψ on $\mathbb{R}^n$. Then the function Φ defined on $\mathbb{R}^n$ by $\Phi(0) = 0$ and for $z \neq 0$*

$$\Phi(z) = \frac{|z|}{\nu^+(z^{\perp})} = \frac{|z|}{\int_{z^{\perp}} \psi(x)dx}$$

is a norm.

Proof. The homogeneity and symmetry are clear so it suffices to prove the convexity. This is equivalent to proving that the restriction of Φ to any linear 2-dimensional subspace is convex.

So let F be a 2-dimensional subspace of $\mathbb{R}^n$ and set $E = F^{\perp}$. Introduce the rotation R of angle $\pi/2$ in the plane F. Then for all $z \in F$, $z \neq 0$,

$$\int_{(Rz)^{\perp}} \psi = \int_{E \oplus \mathbb{R}z} \psi = 2 \int_{E \oplus \mathbb{R}_+ z} \psi.$$

We now apply Proposition 3.2 with $k = n-2$, $p = 1$, and $\varphi = \psi$, which is $-1/n = -1/(k+p+1)$-concave by assumption,. It gives exactly that $z \to \Phi(Rz)$ is convex. Since R is linear, the convexity of Φ follows. $\square$

Remark 3.4. Since the restriction of a convex measure to a convex set K with non-empty interior remains a convex measure, the theorem also implies that the function

$$z \to \frac{|z|}{\nu^+(K \cap z^{\perp})} = \frac{|z|}{\int_{K \cap z^{\perp}} \psi(z) dz}$$

is a norm.

3.2 Campi-Gronchi type results for convex measures

We now generalize a theorem of Campi and Gronchi [9] on shadow systems. The inspiring argument of Campi and Gronchi relies on the formula $|K| = \omega_n \int_{S^{n-1}} h_K^{-n} \, d\sigma$ (followed by a stereographic projection) and on the dimensional Prékopa inequality. Our more general situation requires a slightly different look at the problem and it turns out that the Busemann theorem for convex measures (Theorem 3.3) is a good tool to work with.

Shadow systems were defined by Shephard [25] in the following way. Let C be a closed convex set in $\mathbb{R}^{n+1}$. Let $(e_1, \cdots, e_{n+1})$ be an orthonormal basis of $\mathbb{R}^{n+1}$, we write $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}e_{n+1}$, so that $\mathbb{R}^n = e_{n+1}^{\perp}$. Let $\theta \in \mathbb{R}^n$, with $|\theta| = 1$. For every $t \in \mathbb{R}$ let P_t be the projection onto $\mathbb{R}^n$ parallel to $e_{n+1} - t\theta$: for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$P_t(x + se_{n+1}) = x + ts\theta.$$

We denote $K_t = P_t C \subset \mathbb{R}^n$. Then the family (K_t) is a shadow system of convex sets. The next theorem extends the Campi-Gronchi result [9], originally proved for Lebesgue measure, to the setting of convex measures.

Theorem 3.5. *Let ν be a measure on $\mathbb{R}^n$ with a density ψ which is even and γ-concave on $\mathbb{R}^n$ for some $\gamma \geq -1/(n+1)$. Let (K_t) be a shadow system of centrally symmetric convex sets. Then the function $t \to \nu(K_t^{\circ})^{-1}$ is convex.*

Proof. Write

$$K_t^\circ \;=\; \{x \in \mathbb{R}^n : \langle x, P_t y\rangle \le 1,\ \forall y \in C\} = \{x \in \mathbb{R}^n : P_t^* x \in C^\circ\}$$

where P_t^* is the adjoint of P_t. Observe that P_t^* is the projection on $(e_{n+1} - t\theta)^\perp$ parallel to e_{n+1} and that $PP_t^* = P$, where P denotes the orthogonal projection on $\mathbb{R}^n = e_{n+1}^\perp$. It follows that

$$K_t^\circ = P(C^\circ \cap (e_{n+1} - t\theta)^\perp),$$

We now perform the change of variables $x = P(y)$ which is a diffeomorphism from $(e_{n+1} - t\theta)^\perp$ onto $\mathbb{R}^n$ with Jacobian equal to $1/\sqrt{1+t^2}$. We get

$$\nu(K_t^\circ) = \int_{P(C^\circ\cap(e_{n+1}-t\theta)^\perp)} \psi(x)dx = \int_{C^\circ\cap(e_{n+1}-t\theta)^\perp} \psi(P(y))\frac{dy}{\sqrt{1+t^2}}.$$

Since K_t is symmetric, so is $C^\circ \cap (e_{n+1} - t\theta)^\perp$. Thus, the function φ defined on $\mathbb{R}^{n+1}$ by

$$\varphi(y) = 1_{C^\circ(y)}\,\psi(P(y))$$

is $-1/(n+1)$-concave on $\mathbb{R}^{n+1}$ and its restriction to $(e_{n+1} - t\theta)^\perp$ is even. It follows that the measure ν with density φ on $\mathbb{R}^{n+1}$ is a convex measure. Since

$$\nu(K_t^\circ) = \frac{\int_{(e_{n+1}-t\theta)^\perp} \varphi(y)dy}{\sqrt{1+t^2}} = \frac{\nu^+((e_{n+1} - t\theta)^\perp)}{|e_{n+1} - t\theta|},$$

we conclude from Busemann's theorem for measures (Theorem 3.3 with $n + 1$ in place of n) that the function $t \to \nu(K_t^\circ)^{-1}$ is convex. □

Remark 3.6. From Theorem 3.5, one easily deduces a new and simple proof of the following result established by Meyer-Reisner [16] in the log-concave case and generalized by Bobkov [3]. For $a \in \mathbb{R}^n$, denote $B(a) = \{x \in \mathbb{R}^n : |\langle x, a\rangle| \le 1\}$. Let ν be a measure on $\mathbb{R}^n$ with a density ψ which is even and γ-concave on $\mathbb{R}^n$ for some $\gamma \ge -1/(n + 1)$. Then the function $W(a) = \nu(B(a))^{-1}$ is convex on $\mathbb{R}^n$.

Indeed, we can use the simplest shadow system: take $C = [-(a+e_{n+1}), a+e_{n+1}]$ and $\theta \in S^{n-1}$. One gets $K_t = [-(a + t\theta), a + t\theta]$ and $K_t^\circ = B(a + t\theta)$. Thus Theorem 3.5 implies that $t \to W(a + t\theta)$ is convex for all $a \in \mathbb{R}^n$ and $\theta \in S^{n-1}$, which implies that W is convex on $\mathbb{R}^n$.

Arguments similar to those in the proof of Theorem 3.5 also apply to the following generalization of shadow systems.

Proposition 3.7. *Let n, N be positive integers and $\mathcal{C}$ be a centrally symmetric closed convex set in $\mathbb{R}^n \times \mathbb{R}^N$. Let $\theta \in S^{n-1}$. For $t \in \mathbb{R}^N$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^N$,* *we define $P_t(x,y) = x + \langle y, t\rangle\theta$ and $K_t = P_t(\mathcal{C})$. Let ν be a measure on $\mathbb{R}^n$ with a density ψ with respect to Lebesgue measure that is even and $-1/(n+1)$-concave on $\mathbb{R}^n$. Then*

i) $t \to \nu(K_t^\circ)^{-1}$ is convex on $\mathbb{R}^N$.

ii) if $\mathcal{C}$ and ψ are symmetric with respect to $\theta^\perp$ then $t \to \nu((K_t)^\circ)^{-1}$ is even and convex on $\mathbb{R}^N$.

Proof. (i) The proof follows that of Theorem 3.5. Again, the result relies on a proper application of Busemann's theorem for measures (as before in dimension $n+1$ for n-dimensional sections); actually, it will be more handy, but equivalent, to go back to the formulation of Proposition 3.2 rather than to quote Theorem 3.3.

We work on $\mathbb{R}^n \oplus \mathbb{R}^N$. For $t = (t_1, \dots, t_N) \in \mathbb{R}^N$, the linear map P_t in the proposition is the projection onto $\mathbb{R}^n$ parallel to the N-dimensional subspace $\mathrm{span}(e_{n+1} - t_1\theta, \dots, e_{n+N} - t_N\theta)$. Introducing the $(n-1)$ dimensional subspace $E = \theta^\perp \cap \mathbb{R}^n$, we observe that P_t^* is the projection onto the n-dimensional space $E \oplus \mathbb{R}(\theta + \sum_{i=1}^N t_i e_{n+i})$ parallel to $\mathbb{R}^N$. Let us denote by P the orthogonal projection onto $\mathbb{R}^n$. Then one has that

$$K_t = P(C^\circ \cap (E \oplus \mathbb{R}(\theta + \sum_{i=1}^N t_i e_{n+i}))).$$

The projection P induces a diffeomorphism from $E \oplus \mathbb{R}(\theta + \sum_{i=1}^N t_i e_{n+i})$ to $\mathbb{R}^n$ with Jacobian $1/\sqrt{1 + t_1^2 + \cdots + t_N^2}$. Therefore we get

$$\nu(K_t^\circ) = \int_{C^\circ \cap (E \oplus \mathbb{R}(\theta + \sum_{i=1}^N t_i e_{n+i}))} \frac{\psi(P(y))}{\sqrt{1 + t_1^2 + \cdots + t_N^2}} dy.$$

The function $\varphi(y) = 1_{C^\circ}(y)\psi(P(y))$ is $-1/(n+1)$-concave on $\mathbb{R}^n \simeq E \oplus \mathbb{R}(\theta + \sum_{i=1}^N t_i e_{n+i})$. Using Proposition 3.2 for φ, $p = 1$ and $k = n-1$ we can conclude, after composition with the linear map $t \to v = \theta + \sum_{i=1}^N t_i e_{n+i}$, that the function $t \to \nu((K_t)^\circ)^{-1}$ is convex.

(ii) Let us denote by σ_θ the orthogonal symmetry with respect to $\theta^\perp$ in $\mathbb{R}^n \times \mathbb{R}^N$. Since $P_{-t} = \sigma_\theta \circ P_t \circ \sigma_\theta$, we deduce that

$$K_{-t} = P_{-t}\mathcal{C} = \sigma_\theta \circ P_t \circ \sigma_\theta \mathcal{C} = \sigma_\theta \circ P_t \mathcal{C} = \sigma_\theta K_t.$$

Therefore, using $\psi \circ \sigma_\theta = \psi$, we get

$$\nu((K_{-t})^\circ) = \nu((\sigma_\theta K_t)^\circ) = \nu(\sigma_\theta((K_t)^\circ)) = \nu(K_t^\circ).$$

We conclude that $t \to \nu((K_t)^\circ)^{-1}$ is even. □

Let us mention that such generalized shadow systems and even more general notions were considered by Shephard in his seminal article [25].

Lastly, we state a key corollary that will be used in the proof of Theorem 1.2.

Corollary 3.8. *Let $r \geq 0$, C be an origin-symmetric convex set in $\mathbb{R}^N$, let $\theta \in S^{n-1}$ and $y_1, \ldots, y_N \in \theta^\perp$. Let ν be a measure on $\mathbb{R}^n$ with a density ψ which is $-1/(n+1)$-concave on $\mathbb{R}^n$, even and symmetric with respect to $\theta^\perp$. Then, the map*

$$(t_1, \ldots, t_N) \to \nu(([y_1 + t_1\theta \cdots y_N + t_N\theta]C + rB_2^n)^\circ)^{-1}$$

is even and convex on $\mathbb{R}^N$.

Proof. Let $\mathcal{C} = [y_1 + e_{n+1} \cdots y_N + e_{n+N}]C + rB_2^n$. Then $\mathcal{C}$ is an origin-symmetric convex set in $\mathbb{R}^n \times \mathbb{R}^N$ which is symmetric with respect to $\theta^\perp$ in $\mathbb{R}^{n+N}$ since $[y_1 + e_{n+1} \cdots y_N + e_{n+N}]C \subset \theta^\perp$. Let $P_t : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ be defined as in Proposition 3.7 and let $K_t = P_t\mathcal{C}$. Then one has

$$\begin{aligned} K_t &= P_t([y_1 + e_{n+1} \cdots y_N + e_{n+N}]C + rB_2^n) \\ &= [P_t(y_1 + e_{n+1}) \cdots P_t(y_N + e_{n+N})]C + rP_tB_2^n \\ &= [y_1 + t_1\theta \cdots y_N + t_N\theta]C + rB_2^n. \end{aligned}$$

By $ii)$ of the preceding proposition we can conclude. $\square$

4 Proof of Theorem 1.2

We start by recalling the rearrangement inequalities that are at the heart of the argument. Recall from Section §2 the notation g^* for the radially decreasing rearrangement of a nonnegative function g. The Brascamp-Lieb-Luttinger inequality [6], which was actually anticipated by Rogers [22] as pointed out in [26], states that given k (integrable) nonnegative functions $g_1, \ldots, g_k$ on $\mathbb{R}$ and Nk constants $\{c_{i,j}\}_{i \leq k, j \leq N}$ we have that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{k} g_i\big(c_{i1}s_1 + \ldots + c_{iN}s_N\big)\, ds_1 \ldots ds_N \leq \int_{\mathbb{R}^N} \prod_{i=1}^{k} g_i^*\big(c_{i1}s_1 + \ldots + c_{iN}s_N\big)\, ds_1 \ldots ds_N. \tag{4.1}$$

Christ [10] derived a useful consequence of the Rogers-Brascamp-Lieb-Luttinger inequality. In particular, it was shown in [20] that Christ's formulation is very well adapted to geometric inequalities in convexity, as it provides a handy interface between generalized Steiner symmetrization (as in (4.3) below) and the Rogers-Brascamp-Lieb-Luttinger inequality (see also [1, page 15] and [11, Lemma 3.3]). The result is as follows.

Theorem 4.1 ([10, 20]). *Let $F : (\mathbb{R}^n)^N = \otimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}_+$. We have that*

$$\int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N)\, f_1(x_1) \cdots f_N(x_N)\, dx_1 \ldots dx_N \leq \int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N)\, f_1^*(x_1) \cdots f_N^*(x_N)\, dx_1 \ldots dx_N \quad (4.2)$$

holds for any integrable $f_1, \ldots, f_N : \mathbb{R}^n \to \mathbb{R}_+$ provided that F satisfies the following condition: for every $z \in S^{n-1} \subset \mathbb{R}^n$ and for every $Y = (y_1, \ldots, y_N) \subset (z^\perp)^N \subset (\mathbb{R}^n)^N$, the function $F_{z,Y} : \mathbb{R}^N \to \mathbb{R}_+$ defined by

$$F_{z,Y}(t) := F(y_1 + t_1 z, \ldots, y_N + t_N z). \quad (4.3)$$

is even and quasi-concave.

Briefly, the argument from (4.1) towards this result, goes as follows. First, by putting extra functions of the form $1_{[-r_j, r_j]}$ and using the fact that a symmetric convex set is the intersection of symmetric strips $\{|x \cdot a_j| \leq r_j\}$, we see that the inequality (4.1) remains true if we integrate on a symmetric convex subset of $\mathbb{R}^N$. Then, by the decomposition (2.1) of a function into its level sets, we see that (4.1) remains true if we integrate against an even quasi-concave function G on $\mathbb{R}^N$. In particular, we have for N nonnegative functions g_i on $\mathbb{R}$, that

$$\int_{\mathbb{R}^N} G(s) \prod_{i=1}^N g_i\big(s_i\big)\, ds_1 \ldots ds_N \leq \int_{\mathbb{R}^N} G(s) \prod_{i=1}^N g_i^*\big(s_i\big)\, ds_1 \ldots ds_N.$$

Then, to move from $n = 1$ to arbitrary $n \geqslant 1$, i.e. for functions f_i on $\mathbb{R}^n$ and integration against F on $\mathbb{R}^{nN}$ as in the theorem, we approximate the rearranged function f_i^* by a suitable sequence of Steiner symmetrizations $f_i^*(\cdot|\theta)$, $\theta \in S^{n-1}$. We can use, by Fubini, on N affine lines in $\mathbb{R}^n$ parallel to θ, $y_i + \mathbb{R}\theta$, the previous rearrangement inequality with the g_i's being the restrictions of the f_i's, the condition (4.3) guaranteeing exactly that the restriction G of F is indeed even and quasi-concave on $\mathbb{R}^N$. We refer to [20] for further details.

With this rearrangement inequality in hand, we can put together all the pieces needed for the proof of Theorem 1.2. Let us recall the class of measures ν on $\mathbb{R}^n$ we can work with, namely the spherically-invariant measures with

$$d\nu(x) = \rho(|x|)\, dx \quad \text{with } \rho : [0, +\infty) \to [0, +\infty) \text{ decreasing}, \quad (4.4)$$

together with the sub-class of those of the form

$$d\nu(x) = k^{-(n+1)}(|x|)\, dx \quad \text{with } k : [0, +\infty) \to [0, +\infty] \text{ convex increasing}. \quad (4.5)$$

We will prove the following more general statement (Theorem 1.2 corresponds to (ii) below, with $r = 0$) .

Theorem 4.2. *Let $X_1, \ldots, X_N$ be N independent random vectors in $\mathbb{R}^n$ whose laws are in $\mathcal{P}_n$ and let $r \geq 0$.*

(i) If C is an origin-symmetric convex body in $\mathbb{R}^N$ and ν a measure on $\mathbb{R}^n$ of the form (4.4)*, then*

$$\mathbb{E}\big[\nu\big(([X_1 \cdots X_N]C + rB_2^n)^\circ\big)\big] \leq \mathbb{E}\big[\nu\big(([X_1^* \cdots X_N^*]C + rB_2^n)^\circ\big)\big] \tag{4.6}$$

where $X_1^*, \ldots, X_N^*$ are independent random vectors in $\mathbb{R}^n$ whose densities are the symmetric decreasing rearrangement of the densities of $X_1, \ldots, X_N$. Moreover if ν is of the form (4.5) *we also have that for every $t \geq 0$,*

$$\mathbb{P}\big[\nu\big(([X_1 \cdots X_N]C + rB_2^n)^\circ\big) \geq t\big] \leq \mathbb{P}\big[\nu\big(([X_1^* \cdots X_N^*]C + rB_2^n)^\circ\big) \geq t\big]. \tag{4.7}$$

(ii) If C is an unconditional convex body in $\mathbb{R}^N$ and ν a measure on $\mathbb{R}^n$ of the form (4.4)*, then*

$$\mathbb{E}\big[\nu\big(([X_1 \cdots X_N]C + rB_2^n)^\circ\big)\big] \leq \mathbb{E}\big[\nu\big(([Z_1 \cdots Z_N]C + rB_2^n)^\circ\big)\big] \tag{4.8}$$

where $Z_1, \ldots, Z_N$ are independent random vectors distributed according to λ_{D_n}. Moreover if ν is of the form (4.5)*, we also have that for every $t \geq 0$,*

$$\mathbb{P}\big[\nu\big(([X_1 \cdots X_N]C + rB_2^n)^\circ\big) \geq t\big] \leq \mathbb{P}\big[\nu\big(([Z_1 \cdots Z_N]C + rB_2^n)^\circ\big) \geq t\big]. \tag{4.9}$$

Proof. (i) Let G and F be defined on $(\mathbb{R}^n)^N$ by

$$G(x_1, \ldots, x_N) = \nu(([x_1 \cdots x_N]C + rB_2^n)^\circ) \quad \text{and} \quad F = 1_{\{G > \alpha\}} \tag{4.10}$$

Let $\theta \in S^{n-1}$ and $Y = (y_1, \ldots, y_N) \subset (\theta^\perp)^N$ and let $F_{\theta,Y}$ and $G_{\theta,Y}$ be the restrictions of F and G as in (4.3) with $z = \theta$. Note that $F_{\theta,Y} = 1_{\{G_{\theta,Y} > \alpha\}}$.

Assume first that ν is of the form (4.5). The rotational invariance and the convexity assumption on the density ensure that the assumptions of Corollary 3.8 are satisfied. Thus $G_{\theta,Y}^{-1}$ is even and convex on $\mathbb{R}^N$. Hence $G_{\theta,Y}$ and therefore $F_{\theta,Y}$ are quasi-concave and even. Thus we can apply Theorem 4.1 to the function F and obtain (4.7).

Next, if ν satisfies the weaker assumption (4.4), we start by applying the previous result in the case of Lebesgue measure restricted to an Euclidean ball of radius $R > 0$ (the density $1_{RB_2^n}$ is $+\infty$-concave and therefore $-1/(n+1)$)-concave). To condense the notation, we will write $[x_i]$ rather than $[x_1 \cdots x_N]$. We have

$$\forall t > 0, \quad \mathbb{P}\left(|([X_i]C + rB_2^n)^\circ \cap RB_2^n| > t\right) \leqslant \mathbb{P}\left(|([X_i^*]C + rB_2^n)^\circ \cap RB_2^n| > t\right). \tag{4.11}$$

With the notation (4.4), note that for $t \in (0, \rho(0))$, the set $\{\rho \geq t\}$ is an Euclidean ball, open or closed, but the difference is of Lebesgue measure zero, so we will take later closed balls; we denote by $R(t)$ the corresponding radius. By Fubini, for any Borel set $A \subset \mathbb{R}^n$, we can write

$$\nu(A) = \int_0^{+\infty} |A \cap \{\rho \geq t\}| \, dt = \int_0^{\rho(0)} |A \cap R(t)B_2^n| \, dt,$$

which gives

$$\begin{aligned}\mathbb{E}\nu(([X_i]C + rB_2^n)^\circ) &= \mathbb{E} \int_0^{+\infty} |([X_i]C + rB_2^n)^\circ \cap R(t)B_2^n| dt \\ &= \int_0^{+\infty} \int_0^{+\infty} \mathbb{P}\left(|([X_i]C + rB_2^n)^\circ \cap R(t)B_2^n| > \alpha\right) d\alpha dt \quad (4.12)\end{aligned}$$

Thus (4.6) follows from (4.11).

(ii) After this step, we have arrived to radially decreasing probability distributions. It remains to go to the uniform distributions on D_n, namely to the inequalities (4.8) and (4.9) in the case where C is an unconditional convex body. Note that in this case, the functions G and F defined above are coordinate-wise decreasing in the sense that

$$\begin{aligned}&\forall x_1, \ldots, x_N \in \mathbb{R}^n, \quad (0 \leqslant s_i \leqslant t_i, \ \forall i \leqslant N) \\ &\qquad\qquad \Longrightarrow F(s_1x_1, \ldots, s_Nx_N) \geqslant F(t_1x_1, \ldots, t_Nx_N). \quad (4.13)\end{aligned}$$

This follows from the fact that, for such s_i's and t_i's, the unconditionality of C implies that, for every $x_1, \ldots, x_N \in \mathbb{R}^n$,

$$[s_1x_1 \cdots s_Nx_N]C \subset [t_1x_1 \cdots t_Nx_N]C.$$

Then we can apply the following fact from [21, Prop. 3.5], for which we recall a proof below for completeness.

Lemma 4.3 ([21]). *Let $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ be a function that satisfies the condition (4.13). If $g_1, \ldots, g_N : \mathbb{R}^+ \to [0, 1]$ are nonnegative, bounded by 1, integrable functions with $\int_{\mathbb{R}^n} g_i(|x|) \, dx = 1$ for all $i = 1, \ldots N$, then*

$$\begin{aligned}&\int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N) \prod_{i=1}^N g_i(|x_i|) \, dx_1 \ldots dx_N \\ &\qquad\qquad \leq \int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N) \prod_{i=1}^N 1_{[0,r_n]}(|x_i|) \, dx_1 \ldots dx_N\end{aligned}$$

where r_n is the radius of D_n.

We can therefore apply this lemma in the case where g_i is the law of X_i^* which satisfies the assumptions (the fact that the density of X_i is bounded by 1 implies indeed that its radial rearrangement is also bounded by 1) and to our function F (4.10) which now satisfies (4.13). This yields

$$\mathbb{P}\big[\nu\big(([X_i^*]C + rB_2^n)^\circ\big) \geq t\big] \leq \mathbb{P}\big[\nu\big(([Z_i]C + rB_2^n)^\circ\big) \geq t\big].$$

Combined with (4.7), we arrive at (4.9). Note that in the proof of (ii) we have not exploited the fact that the g_i's are radially decreasing, only the fact that they are radial.

To get (4.8) for the larger class of measure ν, we can either use (4.6) and the fact above applied to the function G, or else deduce it from (4.9) with the same trick (4.12) as above.

$\square$

Proof of Lemma 4.3. Using Fubini, we see that it is enough to treat each coordinate one after the other, and so the fact boils down to the following $N = 1$ dimensional statement:

$$\int_{\mathbb{R}^n} F(x) g(|x|)\, dx \leq \int_{\mathbb{R}^n} F(x) 1_{[0,r_n]}(|x|)\, dx$$

when g is a nonnegative function bounded by 1 with $\int_{\mathbb{R}^n} g(|x|)\, dx = 1$, and F is an even function on $\mathbb{R}^n$ satisfying $F(sx) \geq F(x)$ for all $x \in \mathbb{R}^n$ and $s \in [0,1]$. This property of F implies that the function $r \to F(rx_0)$ is decreasing on $\mathbb{R}^+$ for any fixed $x_0 \in \mathbb{R}^n$. Therefore, by integration in polar coordinates, we see that it suffices to prove that

$$\int_0^{+\infty} f(r) g(r)\, r^{n-1}\, dr \leq \int_0^{r_n} f(r)\, r^{n-1}\, dr$$

when f is a decreasing function and g has values in $[0,1]$ with $\int_0^{+\infty} g(r)\, r^{n-1}\, dr = \int_0^{r_n} r^{n-1}\, dr$. This is now standard. Denote $\alpha(r) := (1_{[0,r_n]}(r) - g(r)) r^{n-1}$, and observe that

$$\int_0^{+\infty} f(r)\, \alpha(r)\, dr = \int_0^{+\infty} (f(r) - f(r_n))\, \alpha(r)\, dr \geq 0$$

since the integrand in the second integral is point-wise nonnegative. $\square$

5 Applications

Here we present some applications of our random theorems to deterministic geometric inequalities using the law of large numbers. In particular, we give a more rigorous argument than that sketched in the introduction on how to recover Blaschke-Santaló type inequalities.

The following result shows that we can pass to the limit in our main statement when there is almost-sure convergence in the Hausdorff metric.

Theorem 5.1. *Let (X_i) and (Z_i) be sequences of independent random vectors in $\mathbb{R}^n$ with each X_i distributed according to the same fixed $\mu \in \mathcal{P}_n$ and each Z_i according to λ_{D_n}. Assume that $C_N, C_{N+1}, \ldots$ are unconditional convex bodies with $C_N \subset \mathbb{R}^N$, $N = n, n+1, \ldots$, such that*

$$[X_1 \cdots X_N]C_N \text{ converges } \otimes_{i=1}^{\infty} \mu\text{-a.s. in } \delta^H \tag{5.1}$$

and

$$[Z_1 \cdots Z_N]C_N \text{ converges } \otimes_{i=1}^{\infty} \lambda_{D_n}\text{-a.s. in } \delta^H. \tag{5.2}$$

Then, if ν is a measure on $\mathbb{R}^n$ with a spherically-symmetric, decreasing density, we have

$$\mathbb{E}\nu\left(\left(\lim_{N\to\infty}[X_1 \cdots X_N]C_N\right)^{\circ}\right) \leqslant \mathbb{E}\nu\left(\left(\lim_{N\to\infty}[Z_1 \cdots Z_N]C_N\right)^{\circ}\right). \tag{5.3}$$

To prove the theorem, we will need the following lemma.

Lemma 5.2. *Let ν be a measure on $\mathbb{R}^n$ with a spherically-symmetric, decreasing density. Then ν is continuous on $\mathcal{K}_{\circ}^n$ with respect to δ^H.*

Proof. We can restrict ourselves to continuity for sets included in some compact set. Then by uniform approximation, we may assume that the density $f_\nu = \frac{d\nu}{dx}$ of ν is of the form:

$$f_\nu(x) = \sum_{j=1}^{M} a_j 1_{r_j B_2^n}(x) \quad (x \in \mathbb{R}^n)$$

where $a_j > 0$, $j = 1, \ldots, M$ and $r_1 > r_2 > \ldots > r_M > 0$. Suppose now that $K, K_1, K_2, \ldots \in \mathcal{K}_{\circ}^n$ and $\delta^H(K_N, K) \to 0$ as $N \to \infty$. Then, as $N \to \infty$,

$$\nu(K_N) = \sum_{j=1}^{M} a_j |K_N \cap (r_j B_2^n)| \to \sum_{j=1}^{M} a_j |K \cap (r_j B_2^n)| = \nu(K) \tag{5.4}$$

by Lemma 2.1. $\square$

Proof of Theorem 5.1. Let $\varepsilon > 0$. Note that

$$[X_1 \cdots X_N]C_N + \varepsilon B_2^n \supseteq \varepsilon B_2^n,$$

hence

$$\nu\left(([X_1 \cdots X_N]C_N + \varepsilon B_2^n)^\circ\right) \leqslant \nu(\varepsilon^{-1} B_2^n)$$

for each $N \geqslant n$; the same holds for $Z_1, \ldots, Z_N$. By dominated convergence, Lemmas 2.1, 5.2 and Theorem 4.2, we have

$$\begin{aligned}
&\mathbb{E}\nu\left((\lim_{N\to\infty}[X_1 \cdots X_N]C_N + \varepsilon B_2^n)^\circ\right) \\
&\quad = \mathbb{E}\lim_{N\to\infty} \nu\left(([X_1 \cdots X_N]C_N + \varepsilon B_2^n)^\circ\right) \\
&\quad = \lim_{N\to\infty} \mathbb{E}\nu\left(([X_1 \cdots X_N]C_N + \varepsilon B_2^n)^\circ\right) \\
&\quad \leqslant \lim_{N\to\infty} \mathbb{E}\nu\left(([Z_1 \cdots Z_N]C_N + \varepsilon B_2^n)^\circ\right) \\
&\quad = \mathbb{E}\lim_{N\to\infty} \nu\left(([Z_1 \cdots Z_N]C_N + \varepsilon B_2^n)^\circ\right) \\
&\quad = \mathbb{E}\nu\left((\lim_{N\to\infty}[Z_1 \cdots Z_N]C_N + \varepsilon B_2^n)^\circ\right).
\end{aligned}$$

If $\mathbb{E}\nu\left((\lim_{N\to\infty}[Z_1 \cdots Z_N]C_N)^\circ\right) = \infty$, (5.3) is trivial. Otherwise, since

$$\lim_{N\to\infty}[Z_1 \cdots Z_N]C_N + \varepsilon B_2^n \supseteq \lim_{N\to\infty}[Z_1 \cdots Z_N]C_N,$$

we have

$$\nu\left((\lim_{N\to\infty}[Z_1 \cdots Z_N]C_N + \varepsilon B_2^n)^\circ\right) \leqslant \nu\left((\lim_{N\to\infty}[Z_1 \cdots Z_N]C_N)^\circ\right)$$

for each $\varepsilon > 0$. Thus we can appeal to dominated convergence once more and let $\varepsilon \to 0$ to conclude the proof. $\square$

Recall that given a measure $\mu \in \mathcal{P}_n$ and $p \geqslant 1$, the L_p-centroid body $Z_p(\mu)$ of μ is the convex body with support function

$$h_{Z_p(\mu)}(y) = \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^p d\mu(x)\right)^{1/p} \qquad (y \in \mathbb{R}^n).$$

Such bodies were originally defined for compact star-shaped sets rather than measures, under an alternate normalization, in [14].

If the X_i's are sampled according to μ, then

$$Z_p(\mu) = \lim_{N\to\infty} N^{-1/p}[X_1 \cdots X_N]B_q^N, \tag{5.5}$$

where $1/p + 1/q = 1$, and convergence occurs a.s. in δ^H; the latter follows from the law of large numbers (see [20]). In particular, if K is an origin-symmetric convex body, then

$$K = \lim_{N\to\infty} [X_1 \cdots X_N] B_1^N = \lim_{N\to\infty} \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}, \tag{5.6}$$

where $X_1, X_2, \dots$ are independent random vectors sampled in K and convergence occurs a.s. in δ^H.

Corollary 5.3. *Let ν be a measure on $\mathbb{R}^n$ with a spherically-symmetric, decreasing density. Let $\mu \in \mathcal{P}_n$, $p \geqslant 1$, and $Z_p(\mu)$ be the L_p-centroid body of μ. Then,*

$$\nu(Z_p^\circ(\mu)) \leqslant \nu(Z_p^\circ(\lambda_{D_n})).$$

When ν is Lebesgue measure on $\mathbb{R}^n$ and μ is the uniform measure on a compact star-shaped set, the latter result is due to Lutwak-Zhang [14]; a straightforward generalization from star-shaped sets to measures μ appears in [19].

Proof. By (5.5), Theorem 5.1 applies. □

When ν is not Lebesgue measure, the result is very sensitive to scaling, since we lose affine invariance. When we drop the volume normalization, we can still prove the following result.

Corollary 5.4. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and suppose that $|K| = |t_K B_2^n|$. Then for any Lebesgue absolutely continuous measure ν with a spherically-symmetric, decreasing density, we have*

$$\nu(K^\circ) \leqslant \nu((t_K B_2^n)^\circ). \tag{5.7}$$

Proof. Let $\overline{K} = K/|K|^{1/n}$ be the volume one homothetic copy of K. If $X_1, X_2, \dots$ are independent random vectors sampled in $\overline{K}$, then $\overline{K} = \lim_{N\to\infty}[X_1 \cdots X_N]B_1^N$ and hence

$$K^\circ = \lim_{N\to\infty} \left([X_1 \cdots X_N] B_1^N / |K|^{1/n}\right)^\circ,$$

where the convergence is a.s. in δ^H. Similarly, if $Z_1, Z_2, \dots$ are independent random vectors sampled in D_n, then we have a.s. convergence in δ^H:

$$D_n^\circ = \lim_{N\to\infty} ([Z_1 \cdots Z_N] B_1^N)^\circ.$$

Thus

$$\begin{aligned}
\nu(K^\circ) &= \mathbb{E}\nu\left(\lim_{N\to\infty}([X_1\cdots X_N]B_1^N/|K|^{1/n})^\circ\right)\\
&\leqslant \mathbb{E}\nu\left(\lim_{N\to\infty}([X_1\cdots X_N]B_1^N/|K|^{1/n})^\circ\right)\\
&= \nu(|K|^{-1/n}D_n^\circ)\\
&= \nu((t_K B_2^n)^\circ).
\end{aligned}$$

□

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