On the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies

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Abstract

It is known that every isotropic convex body K in $\mathbb{R}^n$ has a "subgaussian" direction with constant $r = O(\sqrt{\log n})$. This follows from the upper bound $|\Psi_2(K)|^{1/n} \leqslant \frac{c\sqrt{\log n}}{\sqrt{n}} L_K$ for the volume of the body $\Psi_2(K)$ with support function $h_{\Psi_2(K)}(\theta) := \sup_{2\leqslant q\leqslant n} \frac{\|\langle\cdot,\theta\rangle\|_q}{\sqrt{q}}$. The approach in all the related works does not provide estimates on the measure of directions satisfying a ψ_2-estimate with a given constant r. We introduce the function $\psi_K(t) := \sigma\left(\{\theta\in S^{n-1} : h_{\Psi_2(K)}(\theta) \leqslant ct\sqrt{\log n} L_K\}\right)$ and we discuss lower bounds for $\psi_K(t)$, $t \geqslant 1$. Information on the distribution of the ψ_2-norm of linear functionals is closely related to the problem of bounding from above the mean width of isotropic convex bodies.

1 Introduction

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered (i.e. it has its center of mass at the origin), and there exists a constant $L_K > 0$ such that

$$\int_K \langle x,\theta\rangle^2 dx = L_K^2 \tag{1.1}$$

for every $\theta \in S^{n-1}$. It is known (see [19]) that for every convex body K in $\mathbb{R}^n$ there exists an invertible affine transformation T such that $T(K)$ is isotropic. Moreover, this isotropic position of K is uniquely determined up to orthogonal transformations; therefore, if we define $L_K = L_{\tilde{K}}$ where $\tilde{K}$ is an isotropic affine image of K, then L_K is well defined for the affine class of K.

A central question in asymptotic convex geometry asks if there exists an absolute constant $C > 0$ such that $L_K \leqslant C$ for every convex body K. Bourgain [4] proved that $L_K \leqslant c\sqrt[4]{n}\log n$ for every symmetric convex body K in $\mathbb{R}^n$. The best known general estimate is currently $L_K \leqslant c\sqrt[4]{n}$; this was proved by Klartag in [10] – see also [12].

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. We say that $\theta \in S^{n-1}$ is a subgaussian direction for K with constant $r > 0$ if $\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant r\|\langle \cdot, \theta\rangle\|_2$, where

$$\|f\|_{\psi_\alpha} = \inf\left\{t > 0 : \int_K \exp\left((|f(x)|/t)^\alpha\right) dx \leqslant 2\right\}, \qquad \alpha \in [1, 2]. \tag{1.2}$$

V. Milman asked if every centered convex body K has at least one "subgaussian" direction (with constant $r = O(1)$). By the formulation of the problem, it is clear that one can work within the class of isotropic convex bodies. Affirmative answers have been given in some special cases. Bobkov and Nazarov (see [2] and [3]) proved that if K is an isotropic 1–unconditional convex body, then $\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant c\sqrt{n}\|\theta\|_\infty$ for every $\theta \in S^{n-1}$; a direct consequence is that the diagonal direction is a subgaussian direction with constant $O(1)$. In [23] it is proved that every zonoid has a subgaussian direction with a uniformly bounded constant. Another partial result was obtained in [24]: if K is isotropic and $K \subseteq (\gamma\sqrt{n}L_K)B_2^n$ for some $\gamma > 0$, then

$$\sigma\big(\theta \in S^{n-1} : \|\langle \cdot, \theta\rangle\|_{\psi_2} \geqslant c_1\gamma t L_K\big) \leqslant \exp(-c_2\sqrt{n}t^2/\gamma) \tag{1.3}$$

for every $t \geqslant 1$, where σ is the rotationally invariant probability measure on S^{n-1} and $c_1, c_2 > 0$ are absolute constants.

The first general answer to the question was given by Klartag who proved in [11] that every isotropic convex body K in $\mathbb{R}^n$ has a "subgaussian" direction with a constant which is logarithmic in the dimension. An alternative proof with a slightly better estimate was given in [6]. The best known estimate, which appears in [7], follows from an upper bound for the volume of the body $\Psi_2(K)$ with support function

$$h_{\Psi_2(K)}(\theta) := \sup_{2\leqslant q\leqslant n} \frac{\|\langle \cdot, \theta\rangle\|_q}{\sqrt{q}}. \tag{1.4}$$

It is known that $\|\langle \cdot, \theta\rangle\|_{\psi_2} \simeq \sup_{2\leqslant q\leqslant n} \frac{\|\langle \cdot, \theta\rangle\|_q}{\sqrt{q}}$, and hence, $h_{\Psi_2(K)}(\theta) \simeq \|\langle \cdot, \theta\rangle\|_{\psi_2}$. The main result in [7] states that

$$\frac{c_1}{\sqrt{n}}L_K \leqslant |\Psi_2(K)|^{1/n} \leqslant \frac{c_2\sqrt{\log n}}{\sqrt{n}}L_K, \tag{1.5}$$

where $c_1, c_2 > 0$ are absolute constants. A direct consequence of the right hand side inequality in (1.5) is the existence of subgaussian directions for K with constant $r = O(\sqrt{\log n})$. With a small amount of extra work, one can also show that if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then there exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant ct\|\langle \cdot, \theta\rangle\|_2\}| \leqslant e^{-\frac{t^2}{\log(t+1)}} \tag{1.6}$$

for all $t \geqslant 1$, where $c > 0$ is an absolute constant.

The approach in [11], [6] and [7] does not provide estimates on the measure of directions for which an isotropic convex body satisfies a ψ_2-estimate with a given constant r. Klartag obtains some information on this question, but for a different position of K. More precisely, in [11] he proves that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then, there exists $T \in SL(n)$ such that the body $K_1 = T(K)$ has the following property: there exists $A \subseteq S^{n-1}$ with measure $\sigma(A) \geqslant \frac{4}{5}$ such that, for every $\theta \in A$ and every $t \geqslant 1$,

$$|\{x \in K_1 : |\langle x, \theta\rangle| \geqslant ct\|\langle\cdot,\theta\rangle\|_2\}| \leqslant e^{-\frac{ct^2}{\log^2 n \log^5(t+1)}} \tag{1.7}$$

In this result, K_1 is the ℓ-position of K (this is the position of the body which essentially minimizes its mean width; see [27]). The first aim of this note is to pose the problem of the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies and to provide some first measure estimates. To this end, we introduce the function

$$\psi_K(t) := \sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leqslant ct\sqrt{\log n} L_K\}\right). \tag{1.8}$$

The problem is to give lower bounds for $\psi_K(t)$, $t \geqslant 1$. We present a general estimate in Section 4:

Theorem 1.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geqslant 1$ we have*

$$\psi_K(t) \geqslant \exp(-cn/t^2), \tag{1.9}$$

where $c > 0$ is an absolute constant.

For the proof of Theorem 1.1 we first obtain, for every $1 \leqslant k \leqslant n$, some information on the ψ_2-behavior of directions in an arbitrary k-dimensional subspace of $\mathbb{R}^n$:

Theorem 1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$.*
(i) *For every $\log^2 n \leqslant k \leqslant n/\log n$ and every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that*

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \leqslant C\sqrt{n/k}\, L_K, \tag{1.10}$$

(ii) *For every $1 \leqslant k \leqslant \log^2 n$ and every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that*

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \leqslant C\sqrt{n/k}\sqrt{\log 2k}\, L_K, \tag{1.11}$$

(iii) *For every $n/\log n \leqslant k \leqslant n$ and every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that*

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \leqslant C\sqrt{\log n}\, L_K, \tag{1.12}$$

where $C > 0$ is an absolute constant.

It is known (for example, see [14]) that every isotropic convex body K is contained in $[(n+1)L_K]B_2^n$. This implies that the ψ_2-norm is Lipschitz with constant

$O(\sqrt{n}L_K)$. Then, Theorem 1.2 is combined with a simple argument which is based on the fact that the ψ_2-norm is stable on a spherical cap of the appropriate radius.

Note that $\psi_K(t) = 1$ if $t \geqslant c\sqrt{n/\log n}$. Therefore, the bound of Theorem 1.1 is of some interest only when $1 \leqslant t \leqslant c\sqrt{n/\log n}$. Actually, if $t \geq c\sqrt[4]{n}$ then we have much better information. In Section 5 we give some estimates on the mean width of the L_q–centroid bodies of K and of $\Psi_2(K)$; as a consequence, we get:

Proposition 1.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geqslant c_1\sqrt[4]{n}/\sqrt{\log n}$ one has*

$$\psi_K(t) \geqslant 1 - e^{-c_2 t^2 \log n}, \tag{1.13}$$

where $c_1, c_2 > 0$ are absolute constants.

Deeper understanding of the function $\psi_K(t)$ would have important applications. The strength of the available information can be measured on the problem of bounding from above the mean width of isotropic convex bodies. From the inclusion $K \subseteq [(n+1)L_K]B_2^n$, one has the obvious bound $w(K) \leqslant cnL_K$. However, a better estimate is always possible: for every isotropic convex body K in $\mathbb{R}^n$ one has

$$w(K) \leqslant cn^{3/4}L_K, \tag{1.14}$$

where $c > 0$ is an absolute constant. There are several approaches that lead to the estimate (1.14). The first one appeared in the PhD Thesis of M. Hartzoulaki [9] and was based on a result from [5] regarding the mean width of a convex body under assumptions on the regularity of its covering numbers. The second one is more recent and is due to P. Pivovarov [28]; it relates the question to the geometry of random polytopes with vertices independently and uniformly distributed in K and makes use of the concentration inequality of [25]. A third – very direct – proof of this bound can be based on the "theory of L_q–centroid bodies" which was developed by the second named author (see Section 5). In Section 6 we propose one more approach, which can exploit our knowledge on $\psi_K(t)$.

2 Background material

§2.1. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. Let $k \leqslant n$ and $F \in G_{n,k}$. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

§2.2. A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if it has center of mass at the origin, i.e. $\int_C \langle x, \theta\rangle \, dx = 0$ for every $\theta \in S^{n-1}$. The support function of a convex body C is defined by

$$h_C(y) = \max\{\langle x, y\rangle : x \in C\}, \tag{2.1}$$

and the mean width of C is

$$w(C) = \int_{S^{n-1}} h_C(\theta)\sigma(d\theta). \tag{2.2}$$

For each $-\infty < p < \infty$, $p \neq 0$, we define the p-mean width of C by

$$w_p(C) = \left(\int_{S^{n-1}} h_C^p(\theta)\sigma(d\theta)\right)^{1/p}. \tag{2.3}$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}. \tag{2.4}$$

A centered convex body C is called *almost isotropic* if C has volume one and $C \simeq T(C)$ where $T(C)$ is an isotropic linear transformation of C. Finally, we write $\overline{C}$ for the homothetic image of volume 1 of a convex body $C \subseteq \mathbb{R}^n$, i.e. $\overline{C} := \frac{C}{|C|^{1/n}}$.

§2.3. Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ and $y \in \mathbb{R}^n$ we define

$$h_{Z_q(K)}(y) := \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}. \tag{2.5}$$

We define the L_q-centroid body $Z_q(K)$ of K to be the centrally symmetric convex set with support function $h_{Z_q(K)}$. Note that K is isotropic if and only if $Z_2(K) = L_K B_2^n$. It is clear that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \text{conv}\{K, -K\}$. If $T \in SL(n)$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, as a consequence of Borell's lemma (see [20, Appendix III]), one can check that

$$Z_q(K) \subseteq cq\, Z_2(K) \tag{2.6}$$

for every $q \geqslant 2$ and, more generally,

$$Z_q(K) \subseteq c\frac{q}{p} Z_p(K) \tag{2.7}$$

for all $1 \leqslant p < q$, where $c \geqslant 1$ is an absolute constant. Also, if K is centered, then

$$Z_q(K) \supseteq c_1\, K \tag{2.8}$$

for all $q \geqslant n$, where $c_1 > 0$ is an absolute constant.

§2.4. Let C be a symmetric convex body in $\mathbb{R}^n$. We write $\|\cdot\|_C$ for the norm induced on $\mathbb{R}^n$ by C. We also define $k_*(C)$ as the largest positive integer $k \leq n$ for which the measure of $F \in G_{n,k}$ for which $\frac{1}{2}w(C)B_F \subseteq P_F(C) \subseteq 2w(C)B_F$ is greater than $\frac{n}{n+k}$. The parameter $k_*(C)$ is determined, up to an absolute constant, by the mean width and the radius of C: There exist $c_1, c_2 > 0$ such that

$$c_1 n\frac{w(C)^2}{R(C)^2} \leq k_*(C) \leq c_2 n\frac{w(C)^2}{R(C)^2} \tag{2.9}$$

for every symmetric convex body C in $\mathbb{R}^n$. The lower bound follows from Milman's proof of Dvoretzky's theorem (see [18]) and the upper bound was proved in [21].

The q-mean width $w_q(C)$ is equivalent to $w(C)$ as long as $q \leq k_*(C)$. As Litvak, Milman and Schechtman prove in [16], there exist $c_1, c_2, c_3 > 0$ such that for every symmetric convex body C in $\mathbb{R}^n$ we have:

1. If $1 \leq q \leq k_*(C)$ then $w(C) \leq w_q(C) \leq c_1 w(C)$.
2. If $k_*(C) \leq q \leq n$ then $c_2\sqrt{q/n}\, R(C) \leq w_q(C) \leq c_3\sqrt{q/n}\, R(C)$.

§2.5. For every $q > -n$, $q \neq 0$, we define the quantities $I_q(K)$ by

$$I_q(K) := \left(\int_K \|x\|_2^q \, dx\right)^{1/q}. \tag{2.10}$$

In [26] and [25] it is proved that for every $1 \leq q \leq n/2$,

$$I_{-q}(K) \simeq \sqrt{n/q}\, w_{-q}(Z_q(K)) \tag{2.11}$$

and

$$I_q(K) \simeq \sqrt{n/q}\, w_q(Z_q(K)). \tag{2.12}$$

We define

$$q_*(K) := \max\{k \leq n : k_*(Z_k(K)) \geq k\}. \tag{2.13}$$

Then, the main result of [26] states that, for every centered convex body K of volume 1 in $\mathbb{R}^n$, one has

$$I_{-q}(K) \simeq I_q(K) \tag{2.14}$$

for every $1 \leq q \leq q_*(K)$. In particular, for all $q \leq q_*(K)$ one has $I_q(K) \leq CI_2(K)$, where $C > 0$ is an absolute constant.

If K is isotropic, one can check that $q_*(K) \geq c\sqrt{n}$, where $c > 0$ is an absolute constant (for a proof, see [25]). Therefore,

$$I_q(K) \leq C\sqrt{n}L_K \quad \text{for every } q \leq \sqrt{n}. \tag{2.15}$$

In particular, from (2.12) and (2.15) we see that, for all $q \leqslant \sqrt{n}$,

$$w(Z_q(K)) \simeq w_q(Z_q(K)) \simeq \sqrt{q}L_K. \tag{2.16}$$

§2.6. Let C be a symmetric convex body in $\mathbb{R}^n$. For every $\delta \geq 1$ we define

$$d_*(C, \delta) = \max\{q \geq 1 : w(C) \leq \delta w_{-q}(C)\}. \tag{2.17}$$

It was proved in [13] and [15] that

$$k_*(C) \leq c d_*(C, 2) \tag{2.18}$$

§2.7. For every k-dimensional subspace F of $\mathbb{R}^n$ we denote by E the orthogonal subspace of F. For every $\phi \in F \setminus \{0\}$ we define $E^+(\phi) = \{x \in \text{span}\{E, \phi\} : \langle x, \phi\rangle \geq 0\}$. K. Ball (see [1] and [19]) proved that, if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then, for every $q \geqslant 0$, the function

$$\phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left(\int_{K \cap E^+(\phi)} \langle x, \phi\rangle^q dx \right)^{-\frac{1}{q+1}} \tag{2.19}$$

is the gauge function of a convex body $B_q(K, F)$ on F. A basic identity from [25] states that for every $F \in G_{n,k}$ and every $q \geqslant 1$ we have that

$$P_F(Z_q(K)) = \left(\frac{k+q}{2}\right)^{1/q} |B_{k+q-1}(K, F)|^{1/k+1/q} Z_q(\overline{B}_{k+q-1}(K, F)). \tag{2.20}$$

It is a simple consequence of Fubini's theorem that if K is isotropic then $\overline{B}_{k+1}(K, F)$ is almost isotropic. Moreover, using (2.20) one can check that

$$c_1 \frac{k}{k+q} \frac{Z_q(\overline{B}_{k+1}(K, F))}{L_{\overline{B}_{k+1}(K,F)}} \subseteq \frac{P_F(Z_q(K))}{L_K} \subseteq c_2 \frac{k+q}{k} \frac{Z_q(\overline{B}_{k+1}(K, F))}{L_{\overline{B}_{k+1}(K,F)}} \tag{2.21}$$

for all $1 \leqslant k, q \leqslant n$. In particular, for all $q \leqslant k$ we have

$$\frac{Z_q(\overline{B}_{k+1}(K, F))}{L_{\overline{B}_{k+1}(K,F)}} \simeq \frac{P_F(Z_q(K))}{L_K}. \tag{2.22}$$

§2.8. Recall that if A and B are convex bodies in $\mathbb{R}^n$, then the covering number $N(A, B)$ of A by B is the smallest number of translates of B whose union covers A. A simple and useful observation is that, if A and B are both symmetric and if $S_t(A, B)$ is the maximal number of points $z_i \in A$ which satisfy $\|z_i - z_j\|_B \geqslant t$ for all $i \neq j$, then

$$N(A, tB) \leqslant S_t(A, B) \leqslant N(A, (t/2)B). \tag{2.23}$$

3 Covering numbers of projections of L_q-centroid bodies

Let K be an isotropic convex body in $\mathbb{R}^n$. We first give an alternative proof of some estimates on the covering numbers $N(Z_q(K), t\sqrt{q}L_K B_2^n)$ that were recently obtained in [7]; they improve upon previous estimates from [6].

Proposition 3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$, let $1 \leqslant q \leqslant n$ and $t \geqslant 1$. Then,*

$$\log N\left(Z_q(K), c_1 t\sqrt{q}L_K B_2^n\right) \leqslant c_2\frac{n}{t^2} + c_3\frac{\sqrt{qn}}{t}, \tag{3.1}$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

Note that the upper bound in (3.1) is of the order n/t^2 if $t \leqslant \sqrt{n/q}$ and of the order $\sqrt{qn}/t$ if $t \geqslant \sqrt{n/q}$. Our starting point is a "small ball probability" type estimate which appears in [22, Fact 3.2(c)]:

Lemma 3.2. *Let $\theta \in S^{n-1}$, $1 \leqslant k \leqslant n-1$ and $r \geqslant \sqrt{e}$. Then,*

$$\mu_{n,k}\left(\left\{F \in G_{n,k} : \|P_F(\theta)\|_2 \leqslant \frac{1}{r}\sqrt{\frac{k}{n}}\right\}\right) \leqslant \left(\frac{\sqrt{e}}{r}\right)^k. \tag{3.2}$$

Under the restriction $\log N(C, tB_2^n) \leqslant k$, Lemma 3.2 allows us to compare the covering numbers $N(C, tB_2^n)$ of a convex body C with the covering numbers of its random k-dimensional projections.

Lemma 3.3. *Let C be a convex body in $\mathbb{R}^n$, let $r \geqslant \sqrt{e}$, $s > 0$ and $1 \leqslant k \leqslant n-1$. If $N_s := N(C, sB_2^n)$, then there exists $\mathcal{F} \subseteq G_{n,k}$ such that $\mu_{n,k}(\mathcal{F}) \geqslant 1 - N_s^2 e^{k/2} r^{-k}$ and*

$$N\left(P_F(C), \frac{s}{2r}\sqrt{\frac{k}{n}}B_F\right) \geqslant N_s \tag{3.3}$$

for all $F \in \mathcal{F}$.

Proof. Let $N_s = N(C, sB_2^n)$. From (2.23) we see that there exist $z_1, \ldots, z_{N_s} \in C$ such that $\|z_i - z_j\|_2 \geqslant s$ for all $1 \leqslant i, j \leqslant N_s$, $i \neq j$. Consider the set $\{w_m : 1 \leqslant m \leqslant \frac{N_s(N_s-1)}{2}\}$ of all differences $z_i - z_j$ $(i \neq j)$. Note that $\|w_m\|_2 \geqslant s$ for all m. Lemma 3.2 shows that

$$\mu_{n,k}\left(\left\{F \in G_{n,k} : \|P_F(w_m)\|_2 \leqslant \frac{1}{r}\sqrt{\frac{k}{n}}\|w_m\|_2\right\}\right) \leqslant \left(\frac{\sqrt{e}}{r}\right)^k, \tag{3.4}$$

and hence,

$$\mu_{n,k}\left(\left\{F : \|P_F(w_m)\|_2 \geqslant \frac{1}{r}\sqrt{\frac{k}{n}}\|w_m\|_2 \text{ for all } m\right\}\right) \geqslant 1 - N_s^2 e^{k/2} r^{-k}. \tag{3.5}$$

Let $\mathcal{F}$ be the subset of $G_{n,k}$ described in (3.5). Then, for every $F \in \mathcal{F}$ and all $i \neq j$,

$$\|P_F(z_i) - P_F(z_j)\|_2 \geqslant \frac{1}{r}\sqrt{\frac{k}{n}}\|z_i - z_j\|_2 \geqslant \frac{s}{r}\sqrt{\frac{k}{n}}. \tag{3.6}$$

Since $P_F(z_i) \in P_F(C)$, the right hand side inequality of (2.23) implies that

$$N\left(P_F(C), \frac{s}{2r}\sqrt{\frac{k}{n}}B_F\right) \geqslant N_s, \tag{3.7}$$

as claimed. □

Finally, we will use the following regularity estimate for the covering numbers of L_q-centroid bodies (see [6, Proposition 3.1] for a proof of the first inequality and [9] for a proof of the second one): For all $t > 0$ and $1 \leqslant q \leqslant n$,

$$\log N\left(Z_q(K), ct\sqrt{q}L_K B_2^n\right) \leqslant \frac{\sqrt{qn}}{\sqrt{t}} + \frac{n}{t} \text{ and } \log N\left(K - K, t\sqrt{n}L_K B_2^n\right) \leqslant \frac{n}{t}, \tag{3.8}$$

where $c > 0$ is an absolute constant. Note that the upper bound in (3.8) is of the order n/t if $t \leqslant n/q$ and of the order $\sqrt{qn}/\sqrt{t}$ if $t \geqslant n/q$.

Proof of Proposition 3.1. We set $s = ct\sqrt{q}L_K$ and $N_s := N(Z_q(K), sB_2^n)$. Because of (3.8) we may assume that $3 \leqslant N_s \leqslant e^{cn}$, and then, we choose $1 \leqslant k \leqslant n$ so that $\log N_s \leqslant k \leqslant 2\log N_s$. We distinguish two cases:
(a) Assume that $1 \leqslant t \leqslant \sqrt{n/q}$. Applying Lemma 3.3 with $r = e^3$ we have that, with probability greater than $1 - N_s^2 e^{-5k/2} \geqslant 1 - e^{-k/2}$, a random subspace $F \in G_{n,k}$ satisfies

$$\frac{k}{2} \leqslant \log N_s \leqslant \log N\left(P_F(Z_q(K)), c_1 s\sqrt{\frac{k}{n}}B_F\right), \tag{3.9}$$

where $c_1 > 0$ is an absolute constant.

If $\log N_s \leqslant q$ then we trivially get $\log N_s \leqslant n/t^2$ because $q \leqslant n/t^2$. So, we may assume that $\log N_s \geqslant q$; in particular, $q \leqslant k$. Then, using (2.21) we get

$$\frac{k}{2} \leqslant \log N\left(Z_q(\overline{B}_{k+1}(K,F)), c\frac{L_{\overline{B}_{k+1}(K,F)}}{L_K}\sqrt{\frac{k}{n}}sB_F\right). \tag{3.10}$$

Observe that $\frac{s\sqrt{k/n}}{\sqrt{q}L_K} = ct\sqrt{k/n} \leqslant ct \leqslant cn/q$. Therefore, applying the estimate (3.8) for the k-dimensional isotropic convex body $\overline{B}_{k+1}(K,F)$, we get

$$\frac{k}{2} \leqslant c_2\frac{k}{t\sqrt{k/n}} = c_2\frac{\sqrt{kn}}{t}, \tag{3.11}$$

which shows that

$$\log N(Z_q(K), t\sqrt{q}L_K B_2^n) = \log N_s \leqslant k \leqslant c_3\frac{n}{t^2}, \tag{3.12}$$

where $c_3 = 4c_2^2$.

(b) Assume that $t \geqslant \sqrt{n/q}$. We set $p := \frac{\sqrt{qn}}{t} \leq q$. Then, using (2.8), we have that

$$\begin{aligned} N\left(Z_q(K), t\sqrt{q}L_K B_2^n\right) &\leq N\left(\frac{q}{p}Z_p(K), c_4 t\sqrt{q}L_K B_2^n\right) \\ &\leq N\left(Z_p(K), c_4 t\sqrt{\frac{p}{q}}\sqrt{p}L_K B_2^n\right) \\ &= N\left(Z_p(K), c_4\sqrt{\frac{n}{p}}\sqrt{p}L_K B_2^n\right). \end{aligned}$$

Applying the result of case (a) for $Z_p(K)$ with $t = \sqrt{n/p}$, we see that

$$\begin{aligned} N\left(Z_q(K), t\sqrt{q}L_K B_2^n\right) &\leq N\left(Z_p(K), c_4\sqrt{\frac{n}{p}}\sqrt{p}L_K B_2^n\right) \\ &\leq e^{c_5 p} = \exp\left(c_5\frac{\sqrt{qn}}{t}\right), \end{aligned}$$

and the proof is complete. □

Using Proposition 3.1 we can obtain analogous upper bounds for the covering numbers of $P_F(Z_q(K))$, where $F \in G_{n,k}$.

Proposition 3.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant q < k \leqslant n$, for every $F \in G_{n,k}$ and every $t \geqslant 1$, we have*

$$\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) \leqslant \frac{c_1 k}{t^2} + \frac{c_2\sqrt{qk}}{t}, \tag{3.13}$$

where $c_1, c_2 > 0$ are absolute constants. Also, for every $k \leqslant q \leqslant n$, $F \in G_{n,k}$ and $t \geqslant 1$,

$$\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) \leqslant \frac{c_3\sqrt{qk}}{t}, \tag{3.14}$$

where $c_3 > 0$ is an absolute constant.

Proof. (i) Let $1 \leqslant q \leqslant k$, $F \in G_{n,k}$ and $t \geqslant 1$. From (2.22) we see that

$$\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) \leqslant \log N\left(Z_q(\overline{B}_{k+1}(K,F)), ct\sqrt{q}L_{\overline{B}_{k+1}(K,F)} B_F\right), \tag{3.15}$$

where $c > 0$ is an absolute constant. Since $\overline{B}_{k+1}(K,F)$ is almost isotropic, we may apply Proposition 3.1 for $\overline{B}_{k+1}(K,F)$ in F: we have

$$\log N\left(Z_q(\overline{B}_{k+1}(K,F)), ct\sqrt{q}L_{\overline{B}_{k+1}(K,F)} B_F\right) \leqslant \frac{c_1 k}{t^2} + \frac{c_2\sqrt{qk}}{t}, \tag{3.16}$$

and hence,

$$\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) \leqslant \frac{c_1 k}{t^2} + \frac{c_2\sqrt{qk}}{t}. \tag{3.17}$$

(ii) Assume that $k \leqslant q \leqslant n$ and $F \in G_{n,k}$. Then, using (2.21) and the fact that $Z_q(C) \subseteq \mathrm{conv}\{C, -C\}$, for every $t \geqslant 1$ we write

$$\begin{aligned}\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) &\leqslant \log N\left(\frac{cq}{k}D_{k+1}(K,F), t\sqrt{q}L_{\overline{B}_{k+1}(K,F)}B_F\right)\\ &\leqslant \log N\left(D_{k+1}(K,F), t\sqrt{\frac{k}{q}}\sqrt{k}L_{\overline{B}_{k+1}(K,F)}B_F\right)\\ &\leqslant c_3\frac{\sqrt{qk}}{t},\end{aligned}$$

where $D_{k+1}(K,F) = \overline{B}_{k+1}(K,F) - \overline{B}_{k+1}(K,F)$, using in the end the second estimate of (3.8) for the isotropic convex body $\overline{B}_{k+1}(K,F)$. This completes the proof. □

Using these bounds we can prove the existence of directions with relatively small ψ_2-norm on any subspace of $\mathbb{R}^n$. The dependence is better as the dimension increases.

Theorem 3.5. *Let K be an isotropic convex body in $\mathbb{R}^n$.*
(i) *For every $\log^2 n \leqslant k \leqslant n/\log n$ and every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that*

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant C\sqrt{n/k}\, L_K, \tag{3.18}$$

(ii) *For every $n/\log n \leqslant k \leqslant n$ and every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that*

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant C\sqrt{\log n}\, L_K, \tag{3.19}$$

where $C > 0$ is an absolute constant.

Proof. For every integer $q \geq 1$ we define the *normalized L_q–centroid body K_q* of K by

$$K_q = \frac{1}{\sqrt{q}L_K}Z_q(K), \tag{3.20}$$

and we consider the convex body

$$T = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n\rfloor} K_{2^i}\right). \tag{3.21}$$

Then, for every $F \in G_{n,k}$ we have

$$P_F(T) = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n\rfloor} P_F(K_{2^i})\right). \tag{3.22}$$

We will use the following standard fact (see [6] for a proof): If $A_1, \dots, A_s$ are subsets of RB_2^k, then for every $t > 0$ we have

$$N(\mathrm{conv}(A_1 \cup \cdots \cup A_s), 2tB_2^k) \leqslant \left(\frac{cR}{t}\right)^s \prod_{i=1}^{s} N(A_i, tB_2^k). \tag{3.23}$$

We apply this to the sets $A_i = P_F(K_{2^i})$. Observe that $K_{2^i} \subseteq c_1 2^{i/2} B_2^n$, and hence, $N(A_i, tB_F) = 1$ if $c_1 2^{i/2} \leqslant t$. Also, $A_i \subseteq c_2\sqrt{n} B_F$ for all i.

Using Proposition 3.4, for every $t \geqslant 1$ we can write

$$\begin{aligned} N(P_F(T), 2tB_F) &\leqslant (c_2\sqrt{n})^{\lfloor \log_2 n\rfloor}\left[\prod_{i=1}^{\lfloor \log_2 n\rfloor} N(P_F(K_{2^i}), tB_F)\right] \\ &\leqslant e^{c_3 \log^2 n} \exp\left(C\sum_{i=1}^{\lfloor \log_2 n\rfloor} \frac{2^{i/2}\sqrt{k}}{t} + C\sum_{t^2 \leqslant 2^i \leqslant k} \frac{k}{t^2}\right) \\ &\leqslant e^{c_3 \log^2 n} \exp\left(C\frac{\sqrt{nk}}{t} + C\frac{k}{t^2}\log(k/t^2)\right), \end{aligned}$$

where the second term appears only if $k \geqslant ct^2$.

Now, we distinguish two cases:

(i) If $\log^2 n \leqslant k \leqslant n/\log n$ we choose $t_0 = \sqrt{n/k}$. Observe that $\frac{\sqrt{nk}}{t_0} = k$ and

$$\frac{k}{t_0^2}\log\left(\frac{k}{t_0^2}\right) = \frac{k^2}{n}\log\left(\frac{k^2}{n}\right) \leqslant \frac{k}{\log n}\log\left(\frac{k^2}{n}\right) \leqslant k. \tag{3.24}$$

This implies that $N(P_F(T), \sqrt{n/k}B_F) \leqslant e^{ck}$. It follows that

$$|P_F(T)| \leqslant |C\sqrt{n/k}\, B_F|. \tag{3.25}$$

Therefore, there exists $\theta \in S_F$ such that

$$h_T(\theta) = h_{P_F(T)}(\theta) \leqslant C\sqrt{n/k}, \tag{3.26}$$

which implies

$$\|\langle \cdot, \theta\rangle\|_{2^i} \leqslant C\, 2^{i/2}\sqrt{n/k}\, L_K \tag{3.27}$$

for every $i = 1, 2, \dots, \lfloor \log_2 n\rfloor$. This easily implies (3.18).

(ii) If $n/\log n \leqslant k \leqslant n$ we choose $t_0 = \sqrt{\log n} \simeq \sqrt{\log k}$. Observe that $\frac{\sqrt{nk}}{t_0} = k\sqrt{\frac{n}{k\log n}} \leqslant k$ and

$$\frac{k}{t_0^2}\log\left(\frac{k}{t_0^2}\right) = \frac{k}{\log n}\log\left(\frac{k}{\log n}\right) \leqslant \frac{k}{\log n}\log\left(\frac{n}{\log n}\right) \leqslant k. \tag{3.28}$$

This implies that $N(P_F(T), \sqrt{\log n}B_F) \leqslant e^{ck}$ and, as in case (i), we see that

$$\|\langle\cdot,\theta\rangle\|_{2^i} \leqslant C\, 2^{i/2}\sqrt{\log n}\, L_K \tag{3.29}$$

for every $i = 1, 2, \ldots, \lfloor \log_2 n \rfloor$. The result follows. □

We close this Section with a sketch of the proof of an analogue of the estimate of Proposition 3.1 for $N(Z_q(K), t\sqrt{q}L_K B_2^n)$ for $t \in (0,1)$.

Proposition 3.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \leqslant q \leqslant n$ and $t \in (0,1)$, then*

$$N\left(Z_q(K), c_1 t\sqrt{q}L_K B_2^n\right) \leqslant \left(\frac{c_2}{t}\right)^n \tag{3.30}$$

and

$$N\left(Z_q(K), c_3 t\sqrt{q}B_2^n\right) \geqslant \left(\frac{c_4}{t}\right)^n, \tag{3.31}$$

where $c_i > 0$ are absolute constants.

Proof. The lower bound is a consequence of the estimate $|Z_q(K)|^{1/n} \geqslant c\sqrt{q}|B_2^n|^{1/n}$ (see [17]). Then, we write

$$N\left(Z_q(K), c_1 t\sqrt{q}B_2^n\right) \geqslant \frac{|Z_q(K)|}{|c_1 t\sqrt{q}B_2^n|} \geqslant \left(\frac{c_2}{t}\right)^n. \tag{3.32}$$

For the upper bound, we will use the fact (see [7, Section 3] for the idea of this construction) that there exists an isotropic convex body K_1 in $\mathbb{R}^n$ with the following properties:

(i) $N\left(Z_q(K), t\sqrt{q}L_K B_2^n\right) \leqslant N\left(Z_q(K_1), c_1 t\sqrt{q}B_2^n\right)$ for every $t > 0$.

(ii) $c_2\sqrt{q}B_2^n \subseteq Z_q(K_1)$ for all $1 \leqslant q \leqslant n$.

(iii) $|Z_q(K_1)|^{1/n} \leqslant c_3\sqrt{q/n}$ for all $1 \leqslant q \leqslant n$.

Therefore, for every $t \in (0,1)$ we have

$$\begin{aligned} N\left(Z_q(K), \frac{t}{2}\sqrt{q}L_K B_2^n\right) &\leqslant \frac{|Z_q(K_1) + t\sqrt{q}B_2^n|}{|t\sqrt{q}B_2^n|} \\ &\leqslant \frac{|cZ_q(K_1)|}{|t\sqrt{q}B_2^n|} \\ &\leqslant \left(\frac{c}{t}\right)^n, \end{aligned}$$

and (3.30) is proved. □

4 On the distribution of the ψ_2-norm

From Theorem 3.5 we can deduce a measure estimate for the set of directions which satisfy a given ψ_2-bound. We start with a simple lemma.

Lemma 4.1. *Let $1 \leqslant k \leqslant n$ and let A be a subset of S^{n-1} which satisfies $A \cap F \neq \emptyset$ for every $F \in G_{n,k}$. Then, for every $\varepsilon > 0$ we have*

$$\sigma(A_\varepsilon) \geqslant \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{k-1}, \tag{4.1}$$

where

$$A_\varepsilon = \big\{y \in S^{n-1} : \inf\{\|y-\theta\|_2 : \theta \in A\} \leqslant \varepsilon\big\}. \tag{4.2}$$

Proof. We write

$$\sigma(A_\varepsilon) = \int_{S^{n-1}} \chi_{A_\varepsilon}(y)\, d\sigma(y) = \int_{G_{n,k}} \int_{S_F} \chi_{A_\varepsilon}(y)\, d\sigma_F(y)\, d\mu_{n,k}(F), \tag{4.3}$$

and observe that, since $A \cap S_F \neq \emptyset$, the set $A_\varepsilon \cap S_F$ contains a cap $C_F(\varepsilon) = \{y \in S_F : \|y - \theta_0\|_2 \leqslant \varepsilon\}$ of Euclidean radius ε in S_F. It follows that

$$\int_{S_F} \chi_{A_\varepsilon}(y)\, d\sigma_F(y) \geqslant \sigma_F(C_F(\varepsilon)) \geqslant \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{k-1}, \tag{4.4}$$

by a well-known estimate on the area of spherical caps, and the result follows. □

Remark. As the proof of the Lemma shows, the strong assumption that $A \cap F \neq \emptyset$ for every $F \in G_{n,k}$ is not really needed for the estimate on $\sigma(A_\varepsilon)$. One can have practically the same lower bound for $\sigma(A_\varepsilon)$ under the weaker assumption that $A \cap F \neq \emptyset$ for every F in a subset $\mathcal{F}_{n,k}$ of $G_{n,k}$ with measure $\mu_{n,k}(\mathcal{F}_{n,k}) \geqslant c^{-k}$.

Theorem 4.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $\log^2 n \leqslant k \leqslant n$ there exists $A_k \subseteq S^{n-1}$ such that*

$$\sigma(A_k) \geqslant e^{-c_1 k \log k} \tag{4.5}$$

where $c_1 > 0$ is an absolute constant, and

$$\|\langle \cdot, y\rangle\|_{\psi_2} \leqslant C \max\left\{\sqrt{n/k}, \sqrt{\log n}\right\} L_K \tag{4.6}$$

for all $y \in A_k$.

Proof. We fix $\log^2 n \leqslant k \leqslant n/\log n$ and define A to be the set of $\theta \in S^{n-1}$ which satisfy (3.18). By Theorem 3.5 we have $A \cap S_F \neq \emptyset$ for every $F \in G_{n,k}$. Therefore, we can apply Lemma 4.1 with $\varepsilon = \frac{1}{\sqrt{k}}$. If $y \in A_\varepsilon$ then there exists $\theta \in A$ such that $\|y - \theta\|_2 \leqslant \varepsilon$, which implies

$$\|\langle \cdot, y-\theta\rangle\|_{\psi_2} \leqslant \left(\|\langle \cdot, y-\theta\rangle\|_\infty \|\langle \cdot, y-\theta\rangle\|_{\psi_1}\right)^{1/2} \leqslant c\sqrt{n}\varepsilon\, L_K, \tag{4.7}$$

if we take into account the well-known fact that $\|\langle\cdot,\theta\rangle\|_{\psi_1}\leqslant c\|\langle\cdot,\theta\rangle\|_1\leqslant cL_K$ (see [19]) and the fact that $\|\langle\cdot,\theta\rangle\|_\infty\leqslant(n+1)L_K$. It follows that

$$\begin{aligned}\|\langle\cdot,y\rangle\|_{\psi_2} &\leqslant \|\langle\cdot,\theta\rangle\|_{\psi_2}+\|\langle\cdot,y-\theta\rangle\|_{\psi_2}\\ &\leqslant \|\langle\cdot,\theta\rangle\|_{\psi_2}+c\sqrt{n/k}\,L_K.\end{aligned}$$

Since θ satisfies (3.18), we get (4.6) – with a different absolute constant C – for all $y\in A_k:=A_{1/\sqrt{k}}$. Finally, Lemma 4.1 shows that

$$\sigma(A_k)\geqslant\frac{1}{2}\left(\frac{1}{2\sqrt{k}}\right)^{k-1}\geqslant e^{-c_1k\log k},\tag{4.8}$$

which completes the proof in this case. A similar argument works for $k\geqslant n/\log n$: in this case, we apply Lemma 4.1 with $\varepsilon=\sqrt{\log n/n}$ and the measure estimate for A_k is the same. □

Proof of Theorem 1.1: Let $t\geqslant 1$ and consider the largest k for which $\sqrt{n/k}\geqslant t\sqrt{\log n}$. Then,

$$\frac{n}{t^2}\simeq k\log n\geqslant k\log k,\tag{4.9}$$

and hence, $e^{-c_1k\log k}\geqslant e^{-c_2n/t^2}$. Theorem 4.2 shows that

$$\psi_K(t)\geqslant\sigma(A_k)\geqslant e^{-c_2n/t^2}.\tag{4.10}$$

This proves our claim. □

5 On the mean width of L_q–centroid bodies

§5.1. Mean width of $Z_q(K)$. Let K be an isotropic convex body in $\mathbb{R}^n$. For every $q\leqslant q_*(K)$ we have

$$w(Z_q(K))\simeq w_q(Z_q(K))\simeq\sqrt{q/n}I_q(K)\leqslant c\sqrt{q}L_K.\tag{5.1}$$

Since $q_*(K)\geqslant c\sqrt{n}$, (5.1) holds at least for all $q\leqslant\sqrt{n}$. For $q\geqslant\sqrt{n}$, we may use the fact that $Z_q(K)\subseteq c(q/\sqrt{n})Z_{\sqrt{n}}(K)$ to write

$$w(Z_q(K))\leqslant c\frac{q}{\sqrt{n}}w(Z_{\sqrt{n}}(K))\leqslant c_1\frac{q}{\sqrt[4]{n}}L_K.\tag{5.2}$$

In other words, for all $q\geqslant 1$ we have

$$w(Z_q(K))\leqslant c\sqrt{q}L_K\left(1+\frac{\sqrt{q}}{\sqrt[4]{n}}\right).\tag{5.3}$$

Setting $q=n$ and taking into account (2.8) we get the general upper bound

$$w(K)\leqslant c_1w(Z_n(K))\leqslant c_2n^{3/4}L_K\tag{5.4}$$

for the mean width of K.

In the next Proposition we slightly improve these estimates, taking into account the radius of $Z_q(K)$ or K.

Proposition 5.1. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leqslant q \leqslant n/2$. Then,*

$$w(Z_q(K)) \leqslant c\sqrt{q}L_K \left(1 + \sqrt{R(Z_q(K))/\sqrt{n}L_K}\right). \tag{5.5}$$

In particular,

$$w(K) \leqslant c\sqrt{n}L_K \left(1 + \sqrt{R(K)/\sqrt{n}L_K}\right). \tag{5.6}$$

Proof. Recall that, for all $1 \leqslant q \leqslant n/2$,

$$I_{-q}(K) \simeq \sqrt{n/q}w_{-q}(Z_q(K)). \tag{5.7}$$

We first observe that, for every $t \geqslant 1$,

$$w_{-q/t^2}(Z_q(K)) \leqslant ct^2 w_{-q/t^2}(Z_{q/t^2}(K)) \simeq t^2\sqrt{\frac{q}{t^2n}}I_{-q/t^2}(K) \leqslant ct\sqrt{q}L_K. \tag{5.8}$$

Let $\delta \geqslant 1$. Recall that $d_*(C,\delta) = \max\{q \geq 1 : w(C) \leq \delta w_{-q}(C)\}$. We distinguish two cases:

(a) If $q \leqslant d_*(Z_q(K),\delta)$ then, by (5.7), we have that

$$w(Z_q(K)) \leqslant \delta w_{-q}(Z_q(K)) \simeq \delta\sqrt{q}I_{-q}(K)/\sqrt{n} \leqslant c\delta\sqrt{q}L_K. \tag{5.9}$$

(b) If $q \geqslant d_*(Z_q(K),\delta)$, we set $d := d_*(Z_q(K),\delta)$ and define $t \geqslant 1$ by the equation $q/t^2 = d$. Then, using (5.8), we have

$$w(Z_q(K)) \leqslant \delta w_{-d}(Z_q(K)) = \delta w_{-q/t^2}(Z_q(K)) \leqslant c\delta t\sqrt{q}L_K. \tag{5.10}$$

This gives the bound

$$w(Z_q(K)) \leqslant c\delta\frac{q}{\sqrt{d_*(Z_q(K),\delta)}}L_K. \tag{5.11}$$

Moreover, using the fact that

$$d_*(Z_q(K),c_2) \geqslant k_*(Z_q(K)) \simeq n\frac{w(Z_q(K))2}{R(Z_q(K))^2}, \tag{5.12}$$

we see that if if $q \geqslant c_1 d_*(Z_q(K),c_2)$ then

$$w(Z_q(K)) \leqslant c\frac{\sqrt{q}\sqrt{R(Z_q(K))}}{\sqrt[4]{n}}\sqrt{L_K}. \tag{5.13}$$

Choosing $\delta = 2$ and combining the estimates (5.9) and (5.13) we get (5.5). Setting $q = n$ and using (2.8) we obtain (5.6). □

Recall that K is called a ψ_α-body with constant b_α if

$$\|\langle \cdot, \theta\rangle\|_{\psi_\alpha} \leqslant b_\alpha \|\langle \cdot, \theta\rangle\|_1 \tag{5.14}$$

for all $\theta \in S^{n-1}$. If we assume that K is a ψ_α body for some $\alpha \in [1,2]$ then $R(Z_q(K)) \leqslant R(b_\alpha q^{1/\alpha} Z_2(K)) = b_\alpha q^{1/\alpha} L_K$, and Proposition 5.1 gives immediately the following.

Proposition 5.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. If K is a ψ_α-body with constant b_α for some $\alpha \in [1,2]$ then, for all $1 \leqslant q \leqslant n$,*

$$w(Z_q(K)) \leqslant c\sqrt{q} L_K \left(1 + \frac{\sqrt{b_\alpha} q^{\frac{1}{2\alpha}}}{\sqrt[4]{n}}\right) \tag{5.15}$$

and

$$w(K) \leqslant c\sqrt{b_\alpha} n^{\frac{\alpha+2}{4\alpha}} L_K. \tag{5.16}$$

§5.2. Mean width of $\Psi_2(K)$. As an application of Theorem 1.1 we can give the following estimate for the q-width of $\Psi_2(K)$ for negative values of q.

Proposition 5.3. *Let K be an isotropic convex body in $\mathbb{R}^n$ and $t \geqslant 1$. Then*

$$w_{-\frac{n}{t^2}}(\Psi_2(K)) \leqslant ct\sqrt{\log n} L_K. \tag{5.17}$$

Proof. Observe that, by Markov's inequality,

$$\sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leqslant \frac{1}{e} w_{-\frac{n}{t^2}}(\Psi_2(K))\}\right) \leqslant e^{-\frac{n}{t^2}}. \tag{5.18}$$

From Theorem 1.1 we know that

$$e^{-\frac{n}{t^2}} \leqslant \sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leqslant ct\sqrt{\log n} L_K\}\right), \tag{5.19}$$

for some absolute constant $c > 0$. This proves (5.17). □

We can also give an upper bound for the mean width of $\Psi_2(K)$:

Proposition 5.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$w(\Psi_2(K)) \leqslant c\sqrt[4]{n \log n} L_K. \tag{5.20}$$

Proof. Let $w := w(\Psi_2(K))$. Since $R(\Psi_2(K)) \leqslant c\sqrt{n} L_K$, using (2.18) we see that

$$d_*(\Psi_2(K)) \geqslant ck_*(\Psi_2(K)) \geqslant c\frac{w^2}{L_K^2}. \tag{5.21}$$

We choose t so that $\frac{n}{t^2} = c\frac{w^2}{L_K^2}$, i.e.

$$t = \frac{c\sqrt{n}L_K}{w} \geqslant 1. \tag{5.22}$$

Then, from Proposition 5.3 we see that

$$\begin{aligned} w &\leqslant cw_{-d_*}(\Psi_2(K)) \leqslant w_{-\frac{cw^2}{L_K^2}}(\Psi_2(K)) = w_{-\frac{n}{t^2}}(\Psi_2(K)) \\ &\leqslant c_1\frac{\sqrt{n}}{w}\sqrt{\log n}L_K^2, \end{aligned}$$

and (5.20) follows. □

Actually, we can remove the logarithmic term, starting with the next lemma:

Lemma 5.5. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n-1$. Then for every $F \in G_{n,k}$,*

$$P_F(\Psi_2(K)) \subseteq c\sqrt{n/k}\frac{L_K}{L_{\overline{B}_{k+1}(K,F)}}\Psi_2(\overline{B}_{k+1}(K,F)), \tag{5.23}$$

where $c > 0$ is an absolute constant.

Proof. Indeed, because of (2.21) and (2.22), for every $\theta \in S_F$ we can write

$$\begin{aligned} \frac{h_{\Psi_2(K)}(\theta)}{L_K} &\leqslant \sup_{1\leqslant q\leqslant k}\frac{h_{Z_q(K)}(\theta)}{\sqrt{q}L_K} + \sup_{k\leqslant q\leqslant n}\frac{h_{Z_q(K)}(\theta)}{\sqrt{q}L_K} \\ &= \sup_{1\leqslant q\leqslant k}\frac{h_{P_F(Z_q(K))}(\theta)}{\sqrt{q}L_K} + \sup_{k\leqslant q\leqslant n}\frac{h_{P_F(Z_q(K))}(\theta)}{\sqrt{q}L_K} \\ &\leqslant c_1\sup_{1\leqslant q\leqslant k}\frac{h_{Z_q(\overline{B}_{k+1}(K,F))}(\theta)}{\sqrt{q}L_{\overline{B}_{k+1}(K,F)}} + c_2\sup_{k\leqslant q\leqslant n}\frac{q}{k}\frac{h_{P_F(Z_k(K))}(\theta)}{\sqrt{q}L_K} \\ &= c_1\sup_{1\leqslant q\leqslant k}\frac{h_{Z_q(\overline{B}_{k+1}(K,F))}(\theta)}{\sqrt{q}L_{\overline{B}_{k+1}(K,F)}} + c_2\sup_{k\leqslant q\leqslant n}\sqrt{\frac{q}{k}}\frac{h_{Z_k(\overline{B}_{k+1}(K,F))}(\theta)}{\sqrt{k}L_{\overline{B}_{k+1}(K,F)}} \\ &\leqslant c_3\frac{h_{\Psi_2(\overline{B}_{k+1}(K,F))}(\theta)}{L_{\overline{B}_{k+1}(K,F)}} + c_4\sup_{k\leqslant q\leqslant n}\sqrt{\frac{q}{k}}\frac{h_{\Psi_2(\overline{B}_{k+1}(K,F))}(\theta)}{L_{\overline{B}_{k+1}(K,F)}} \\ &\leqslant c_5\sqrt{\frac{n}{k}}\frac{h_{\Psi_2(\overline{B}_{k+1}(K,F))}(\theta)}{L_{\overline{B}_{k+1}(K,F)}}. \end{aligned}$$

□

Proposition 5.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

$$w(\Psi_2(K)) \leqslant c\sqrt[4]{n}L_K. \tag{5.24}$$

Proof. Let $k=\sqrt{n}$. Using Lemma 5.5 we see that

$$\begin{aligned} w(\Psi_2(K)) &= \int_{G_{n,k}} w(P_F(\Psi_2(K)))d\mu_{n,k}(F) \\ &\leqslant c\sqrt{\frac{n}{k}}\int_{G_{n,k}} \frac{L_K}{L_{\overline{B}_{k+1}(K,F)}} w(\Psi_2(\overline{B}_{k+1}(K,F)))d\mu_{n,k}(F). \end{aligned}$$

Since $k=\sqrt{n}\leqslant q_*(K)$, we know that a "random" $\overline{B}_{k+1}(K,F)$ is "ψ_2" (see [8]), and the result follows. □

Applying Lemma 5.5 we can cover the case $1\leqslant k\leqslant \log^2 n$ in Theorem 3.5:

Corollary 5.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1\leqslant k\leqslant \log^2 n$ and every $F\in G_{n,k}$ there exists $\theta\in S_F$ such that*

$$\|\langle\cdot,\theta\rangle\|_{\psi_2}\leqslant C\sqrt{n/k}\sqrt{\log 2k}\,L_K, \tag{5.25}$$

where $C>0$ is an absolute constant. In fact, for a random $F\in G_{n,k}$ the term $\sqrt{\log 2k}$ is not needed in (5.25).

Proof. Let $1\leqslant k\leqslant \log^2 n$ and $F\in G_{n,k}$. Since $\overline{B}_{k+1}(K,F)$ is isotropic, Theorem 3.5(ii) shows that there exists $\theta\in S_F$ such that

$$h_{\Psi_2(\overline{B}_{k+1}(K,F))}(\theta)\leqslant c_1\sqrt{\log 2k}L_{\overline{B}_{k+1}(K,F)}. \tag{5.26}$$

Then, Lemma 5.5 shows that

$$\begin{aligned} \|\langle\cdot,\theta\rangle\|_{\psi_2} &\simeq h_{\Psi_2(K)}(\theta)=h_{P_F(\Psi_2(K))}(\theta) \\ &\leqslant c\sqrt{n/k}\frac{L_K}{L_{\overline{B}_{k+1}(K,F)}}h_{\Psi_2(\overline{B}_{k+1}(K,F))}(\theta)\leqslant C\sqrt{n/k}\,\sqrt{\log 2k}\,L_K. \end{aligned}$$

In fact, since $k\leqslant \log^2 n\leqslant q_*(K)$, for a random $F\in G_{n,k}$ we know that $\overline{B}_{k+1}(K,F))$ is a ψ_2-body (see [8]), and hence, $h_{\Psi_2(\overline{B}_{k+1}(K,F))}(\theta)\leqslant c_2L_{\overline{B}_{k+1}(K,F)}$ *for all* $\theta\in S_F$. Using this estimate instead of (5.26) we may remove the $\sqrt{\log 2k}$-term in (5.25) for a random $F\in G_{n,k}$. □

Proof of Proposition 1.3. Since $h_{\Psi_2(K)}$ is $\sqrt{n}L_K$-Lipschitz, we have that

$$\sigma\left(\{\theta\in S^{n-1}:h_{\Psi_2(K)}(\theta)-w(\Psi_2(K))\geqslant sw(\Psi_2(K))\}\right)\leqslant e^{-cns^2\left(\frac{w(\Psi_2(K))}{\sqrt{n}L_K}\right)^2}. \tag{5.27}$$

Let $u\geqslant 2w(\Psi_2(K))$. Then, $u=(1+s)w(\Psi_2(K))$ for some $s\geqslant 1$ and $sw(\Psi_2(K))\geqslant u/2$. From (5.27) it follows that

$$\sigma\left(\{\theta\in S^{n-1}:h_{\Psi_2(K)}(\theta)\geqslant u\}\right)\leqslant \exp\left(-cu^2/L_K^2\right). \tag{5.28}$$

If $t\geqslant c_1\sqrt[4]{n}/\sqrt{\log n}$, then Proposition 5.7 shows that $u=t\sqrt{\log n}L_K\geqslant 2w(\Psi_2(K))$. Then, we can apply (5.28) to get the result. □

The estimate of Proposition 1.3 holds true for all $t\geqslant cw(\Psi_2(K))/\sqrt{\log n}L_K$; this is easily checked from the proof. This shows that better lower bounds for $\psi_K(t)$ would follow from a better upper estimate for $w(\Psi_2(K))$ and vice versa.

6 On the mean width of isotropic convex bodies

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $2 \leqslant q \leqslant n$ we define

(6.1) $$k_*(q) = n \left(\frac{w(Z_q(K))}{R(Z_q(K))} \right)^2 .$$

Since $\|\langle \cdot, \theta \rangle\|_q \leqslant cqL_K$ for all $\theta \in S^{n-1}$, we have $R(Z_q(K)) \leqslant cqL_K$. Therefore,

(6.2) $$w(Z_q(K)) \leqslant cqL_K \, \frac{\sqrt{k_*(q)}}{\sqrt{n}}$$

Then, from (2.8) we see that

(6.3) $$w(K) \simeq w(Z_n(K)) \leqslant \frac{cn}{q} w(Z_q(K)) \leqslant c\sqrt{n}\sqrt{k_*(q)}\, L_K.$$

Define

(6.4) $$\rho_* = \rho_*(K) = \min_{2 \leqslant q \leqslant n} k_*(q).$$

Since q was arbitrary in (6.3), we get the following:

Proposition 6.1. *For every isotropic convex body K in $\mathbb{R}^n$ one has*

(6.5) $$w(K) \leqslant c\sqrt{n}\sqrt{\rho_*(K)}\, L_K.$$

Our next observation is the following: by the isoperimetric inequality on S^{n-1}, for every $q \geqslant 1$ one has

(6.6) $$\sigma \left(|\, \|\langle \cdot, \theta \rangle\|_q - w(Z_q) \,| \geqslant \frac{w(Z_q)}{2} \right) \leqslant \exp(-ck_*(q)) \leqslant \exp(-2c\rho_*)$$

where $c > 0$ is an absolute constant. Assume that $\log n \leqslant e^{c\rho_*}$. Then,

(6.7) $$\|\langle \cdot, \theta \rangle\| \simeq w(Z_q)$$

for all θ on a subset A_q of S^{n-1} of measure $\sigma(A_q) \geqslant 1 - \exp(-c\rho_*)$. Taking $q_i = 2^i$, $i \leqslant \log_2 n$ and setting $A = \bigcap A_{q_i}$, we have the following:

Lemma 6.2. *For every isotropic convex body K in $\mathbb{R}^n$ with $\rho_*(K) \geqslant C \log\log n$ one can find $A \subset S^{n-1}$ with $\sigma(A) \geqslant 1 - e^{-c\rho_*}$ such that*

(6.8) $$\|\langle \cdot, \theta \rangle\|_q \simeq w(Z_q)$$

for all $\theta \in A$ and all $2 \leqslant q \leqslant n$. In particular,

(6.9) $$\|\langle \cdot, \theta \rangle\|_{\psi_2} \simeq \max_{2 \leqslant q \leqslant n} \frac{w(Z_q)}{\sqrt{q}}$$

for all $\theta \in A$.

Lemma 6.2 implies that if $\rho_*(K)$ is "large" and $\|\langle\cdot,\theta\rangle\|_{\psi_2}$ is well-bounded on a "relatively large" subset of the sphere, then a similar bound holds true for "almost all" directions. As a consequence, we get a good bound for the mean width of K. The precise statement is the following.

Proposition 6.3. *Let K be an isotropic convex body in $\mathbb{R}^n$ which satisfies the following two conditions:*

1. $\rho_*(K) \geqslant C \log\log n$.
2. *For some $b_n > 0$ we have $\|\langle\cdot,\theta\rangle\|_{\psi_2} \leqslant b_n L_K$ for all θ in a set $B \subseteq S^{n-1}$ with $\sigma(B) > e^{-c\rho_*}$.*

Then,

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \leqslant C b_n L_K \tag{6.10}$$

for all θ in a set $A \subseteq S^{n-1}$ with $\sigma(A) > 1 - e^{-c\rho_*}$. Also,

$$w(Z_q(K)) \leqslant c\sqrt{q} b_n\, L_K \tag{6.11}$$

for all $2 \leqslant q \leqslant n$ and

$$w(K) \leqslant C\sqrt{n} b_n\, L_K. \tag{6.12}$$

Proof. We can find $u \in A \cap B$, where A is the set in Lemma 6.2. Since $u \in B$, we have

$$\|\langle\cdot,u\rangle\|_q \leqslant C_1\sqrt{q} b_n\, L_K \tag{6.13}$$

for all $2 \leqslant q \leqslant n$, and (6.8) shows that

$$w(Z_q(K)) \leqslant C_2\sqrt{q} b_n\, L_K \tag{6.14}$$

for all $2 \leqslant q \leqslant n$. Going back to (6.8) we see that if $\theta \in A$ then

$$\|\langle\cdot,\theta\rangle\|_q \leqslant c w(Z_q) \leqslant C_3\sqrt{q} b_n\, L_K \tag{6.15}$$

for all $2 \leqslant q \leqslant n$. For $q = n$ we get (6.12).

Finally, for every $\theta \in A$ we have

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \simeq \max_{2\leqslant q\leqslant n} \frac{\|\langle\cdot,\theta\rangle\|_q}{\sqrt{q}} \leqslant C b_n\, L_K. \tag{6.16}$$

This completes the proof. □

Propositions 6.1 and 6.3 provide a dichotomy. If $\rho_*(K)$ is small then we can use Proposition 6.1 to get an upper bound for $w(K)$. If $\rho_*(K)$ is large then we can use Proposition 6.3 provided that we have some sufficiently good lower bound for $\psi_K(t)$: what we have is

$$\psi_K(t) \geqslant e^{-c_1 n/t^2} \geqslant e^{-c\rho_*}, \tag{6.17}$$

if $t \simeq \sqrt{n/\rho_*}$. Therefore, we obtain the estimate

$$w(K) \leqslant C\sqrt{n\log n}\sqrt{n/\rho_*}\, L_K. \tag{6.18}$$

Combining the previous results, we deduce one more general upper bound for the mean width of K.

Theorem 6.4. *For every isotropic convex body K in $\mathbb{R}^n$ we have*

$$w(K) \leqslant C\sqrt{n}\min\left\{\sqrt{\rho_*}, \sqrt{n\log n/\rho_*}\right\} L_K, \tag{6.19}$$

where $c>0$ is an absolute constant.

The estimate in Theorem 6.4 depends on our knowledge for the behavior of $\psi_K(t)$; as it stands, it only recovers the $O(n^{3/4}L_K)$ bound for the mean width of K. Actually, the logarithmic term in (6.19) makes it slightly worse. However, we can remove this logarithmic term, starting with the following modification of Proposition 5.1.

Proposition 6.5. *Let K be an isotropic convex body in $\mathbb{R}^n$ and $1 \leqslant q \leqslant n$. Then,*

$$w(Z_q(K)) \leqslant c\sqrt{q}L_K\left(1+\sqrt{\frac{q}{k_*(Z_q(K))}}\right), \tag{6.20}$$

where $c>0$ is an absolute constant.

Proof. If $R(Z_q(K)) \leqslant c\sqrt{n}L_K$ then (6.20) is a direct consequence of (5.5). So, we assume that $R(Z_q(K)) \geqslant c\sqrt{n}L_K$. Then, writing (5.5) in the form

$$w(Z_q(K)) \leqslant c\frac{\sqrt{q}}{\sqrt[4]{n}}\sqrt{R(Z_q(K))}\sqrt{L_K}, \tag{6.21}$$

and taking into account the definition of $k_*(Z_q(K))$ we see that

$$\frac{R(Z_q(K))}{\sqrt{n}L_K} \leqslant c_1\frac{q}{k_*(Z_q(K))}, \tag{6.22}$$

and (6.20) follows from (5.5) again. □

Theorem 6.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$w(K) \leqslant c\sqrt{n}L_K\min\left\{\sqrt{\rho_*}, \sqrt{\frac{n}{\rho_*}}\right\}, \tag{6.23}$$

where $c>0$ is an absolute constant.

Proof. From Proposition 6.1 we know that

$$w(K) \leqslant c\sqrt{n}L_K\sqrt{\rho_*}. \tag{6.24}$$

Let q_0 satisfy $\rho_* = k_*(Z_{q_0}(K))$. From Proposition 6.5 and from (2.7) and (2.8) we have that, for all $1 \leqslant q \leqslant n$,

$$w(K) \leqslant c\frac{n}{q}w(Z_q(K)) \leqslant c_1\sqrt{n}L_K\left(\sqrt{\frac{n}{q}} + \sqrt{\frac{n}{k_*(Z_q(K))}}\right). \tag{6.25}$$

Recall that q_* is the parameter $q_*(K) := \max\{q \in [1,n] : k_*(Z_q(K)) \geqslant q\}$. We distinguish two cases.

(i) Assume that $q_0 \leqslant q_*$. Then we apply (6.25) for q_*; since $q_* = k_*(Z_{q_*}(K)) \geqslant \rho_*$, we get

$$w(K) \leqslant 2c_1\sqrt{n}L_K\sqrt{\frac{n}{q_*}} \leqslant 2c_1\sqrt{n}L_K\sqrt{\frac{n}{\rho_*}}. \tag{6.26}$$

(ii) Assume that $q_0 \geqslant q_*$. Then, $q_0 \geqslant k_*(Z_{q_0}(K)) = \rho_*$. Applying (6.25) for q_0, we get

$$w(K) \leqslant 2c_1\sqrt{n}L_K\sqrt{\frac{n}{k_*(Z_{q_0}(K))}} = 2c_1\sqrt{n}L_K\sqrt{\frac{n}{\rho_*}}. \tag{6.27}$$

In both cases, we have

$$w(K) \leqslant c\sqrt{n}L_K\sqrt{\frac{n}{\rho_*}}. \tag{6.28}$$

Combining (6.28) with (6.24) we get the result. □

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