Ψ_2-estimates for linear functionals on zonoids

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Abstract

Let K be a convex body in $\mathbb{R}^n$ with centre of mass at the origin and volume $|K| = 1$. We prove that if $K \subseteq \alpha\sqrt{n}B_2^n$ where B_2^n is the Euclidean unit ball, then there exists $\theta \in S^{n-1}$ such that

$$(*) \qquad \|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leq c\alpha \|\langle \cdot, \theta\rangle\|_{L_1(K)}$$

where $c > 0$ is an absolute constant. In other words, "every body with small diameter has ψ_2-directions". This criterion applies to the class of zonoids. In the opposite direction, we show that if an isotropic convex body K of volume 1 satisfies $(*)$ for every direction $\theta \in S^{n-1}$, then $K \subseteq C\alpha^2\sqrt{n}\log nB_2^n$, where $C > 0$ is an absolute constant.

1 Introduction

We shall work in $\mathbb{R}^n$ which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. The Euclidean norm $\langle x, x\rangle^{1/2}$ is denoted by $|\cdot|$. We write B_2^n for the Euclidean unit ball, S^{n-1} for the unit sphere, and σ for the rotationally invariant probability measure on S^{n-1}.

Throughout this note we assume that K is a convex body in $\mathbb{R}^n$ with volume $|K| = 1$ and centre of mass at the origin. Given $\alpha \in [1, 2]$, the Orlicz norm $\|f\|_{\psi_\alpha}$ of a bounded measurable function $f : K \to \mathbb{R}$ is defined by

$$(1.1) \qquad \|f\|_{\psi_\alpha} = \inf\left\{ t > 0 : \int_K \exp\left(\left(\frac{|f(x)|}{t}\right)^\alpha\right) dx \leq 2 \right\}.$$

It is not hard to check that

$$(1.2) \qquad \|f\|_{\psi_\alpha} \simeq \sup\left\{ \frac{\|f\|_p}{p^{1/\alpha}} : p \geq 1 \right\}.$$

Let $y \neq 0$ in $\mathbb{R}^n$. We say that K satisfies a **ψ_α-estimate with constant b_α in the direction of y** if

$$\|\langle\cdot, y\rangle\|_{\psi_\alpha} \leq b_\alpha \|\langle\cdot, y\rangle\|_1. \tag{1.3}$$

We say that K is a **ψ_α-body with constant b_α** if (1.3) holds for every $y \neq 0$.

It is easy to see that if K satisfies a ψ_α-estimate in the direction of y and if $T \in SL(n)$, then $T(K)$ satisfies a ψ_α-estimate (with the same constant) in the direction of $T^*(y)$. It follows that $T(K)$ is a ψ_α-body if K is a ψ_α-body. By Borell's lemma (see [17], Appendix III), every convex body K is a ψ_1-body with constant $b_1 = c$, where $c > 0$ is an absolute constant.

Estimates of this form are related to the hyperplane problem for convex bodies. Recall that a convex body K of volume 1 with centre of mass at the origin is called isotropic if there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.4}$$

for all $\theta \in S^{n-1}$. Every convex body K with centre of mass at the origin has an isotropic image under $GL(n)$ which is uniquely determined up to orthogonal transformations (for more information on the isotropic position, see [16]). It follows that the isotropic constant L_K is an invariant for the class $\{T(K) : T \in GL(n)\}$. The hyperplane problem asks if every convex body of volume 1 has a hyperplane section through its centre of mass with "area" greater than an absolute constant. An affirmative answer to this question is equivalent to the following statement: there exists an absolute constant $C > 0$ such that $L_K \leq C$ for every isotropic convex body K.

Bourgain [4] has proved that $L_K \leq c\sqrt[4]{n}\log n$ for every origin symmetric isotropic convex body K in $\mathbb{R}^n$ (the same estimate holds true for non-symmetric convex bodies as well; see [8] and [18]). Bourgain's argument shows that if K is a ψ_2-body with constant b_2, then $L_K \leq cb_2 \log n$ where $c > 0$ is an absolute constant. Examples of ψ_2-bodies are given by the ball and the cube in $\mathbb{R}^n$.

Alesker [1] has proved that the Euclidean norm satisfies a ψ_2-estimate: there exists an absolute constant $C > 0$ such that

$$\int_K \exp\left(\frac{|x|^2}{C^2 I_2^2}\right) dx \leq 2 \tag{1.5}$$

for every isotropic convex body K in $\mathbb{R}^n$, where $I_2^2 = \int_K |x|^2 dx$.

It is not clear if every isotropic convex body satisfies a good ψ_2-estimate for most directions $\theta \in S^{n-1}$; for a related conjecture, see [2]. On the other hand, to the best of our knowledge, even the existence of some good ψ_2-direction has not been verified in full generality. This would correspond to a sharpening of Alesker's result.

Bobkov and Nazarov [6] have recently proved that every 1-unconditional and isotropic convex body satisfies a ψ_2-estimate with constant c in the direction $y = (1, 1, \ldots, 1)$, where $c > 0$ is an absolute constant. The purpose of this note is to establish an analogous fact for zonoids.

Theorem 1.1 *There exists an absolute constant $C > 0$ with the following property: For every zonoid Z in $\mathbb{R}^n$ with volume $|Z| = 1$, there exists $\theta \in S^{n-1}$ such that*

$$\left(\int_Z |\langle x, \theta\rangle|^p dx\right)^{1/p} \leq C\sqrt{p}\int_Z |\langle x, \theta\rangle| dx$$

for every $p \geq 1$.

The proof of Theorem 1.1 is presented in Section 2. The argument shows that the same is true for every convex body in $\mathbb{R}^n$ which has a linear image of volume 1 with diameter of the order of $\sqrt{n}$ (we call these "bodies with small diameter"). In Section 3 we show that zonoids belong to this class.

In the opposite direction, we show that every ψ_2-isotropic convex body has small diameter. More precisely, in Section 4 we prove the following.

Theorem 1.2 *Let K be an isotropic convex body in $\mathbb{R}^n$. Assume that K is a ψ_2-body with constant b_2. Then,*

$$K \subseteq Cb_2^2\sqrt{n}\log n B_2^n,$$

where $C > 0$ is an absolute constant.

The letters c, c_1, c_2, c' etc. denote absolute positive constants, which may change from line to line. Wherever we write $a \simeq b$, this means that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. We refer the reader to the books [17], [20] and [22] for standard facts that we use in the sequel. We thank the referee for suggestions that improved the presentation and some estimates.

2 Bodies with small diameter

We say that a convex body K in $\mathbb{R}^n$ with centre of mass at the origin has "small diameter" if $|K| = 1$ and $K \subseteq \alpha\sqrt{n}B_2^n$, where α is "well bounded". Note that a convex body has a linear image with small diameter if and only if its polar body has bounded volume ratio. Our purpose is to show that bodies with small diameter have "good" ψ_2-directions.

Our first lemma follows by a simple computation.

Lemma 2.1 *For every $p \geq 1$ and every $x \in \mathbb{R}^n$,*

$$\left(\int_{S^{n-1}} |\langle x, \theta\rangle|^p \sigma(d\theta)\right)^{1/p} \simeq \frac{\sqrt{p}}{\sqrt{p+n}}|x|. \tag{2.1}$$

Proof: Observe that

$$\int_{B_2^n} |\langle x, y\rangle|^p dy = |B_2^n|\frac{n}{n+p}\int_{S^{n-1}} |\langle x, \theta\rangle|^p \sigma(d\theta).$$

On the other hand,

$$\begin{aligned}\int_{B_2^n} |\langle x,y\rangle|^p dy &= |x|^p \int_{B_2^n} |\langle e_1,y\rangle|^p dy \\ &= 2|B_2^{n-1}|\cdot |x|^p \int_0^1 t^p(1-t^2)^{(n-1)/2} dt \\ &= |B_2^{n-1}|\cdot |x|^p \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{p+n+2}{2}\right)}.\end{aligned}$$

Since $|B_2^k| = \pi^{k/2}/\Gamma\left(\frac{k+2}{2}\right)$, we get

$$\int_{S^{n-1}} |\langle x,\theta\rangle|^p \sigma(d\theta) = \frac{1}{\sqrt{\pi}}\frac{n+p}{n}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{p+n+2}{2}\right)}|x|^p.$$

The result follows from Stirling's formula. $\square$

Lemma 2.2 *Let K be a convex body in $\mathbb{R}^n$ with volume $|K|=1$ and centre of mass at the origin. Then,*

$$\sigma\left(\theta\in S^{n-1} : \int_K |\langle x,\theta\rangle|\, dx \geq c_1\right) \geq 1-2^{-n},$$

where $c_1>0$ is an absolute constant.

Proof: The Binet ellipsoid E of K is defined by

$$\|\theta\|_E^2 = \int_K \langle x,\theta\rangle^2 dx = \langle M_K\theta,\theta\rangle,$$

where $M_K = \left(\int_K x_i x_j dx\right)$ is the matrix of inertia of K (see [16]). It is easily checked that $\det M_K = \det M_{TK}$ for every $T\in SL(n)$, and this implies that

$$\int_{S^{n-1}} \|\theta\|_E^{-n}\sigma(d\theta) = \frac{|E|}{|B_2^n|} = (\det M_K)^{-1/2} = L_K^{-n}.$$

Then, Markov's inequality shows that

$$\sigma\left(\theta\in S^{n-1} : \|\theta\|_E \geq L_K/2\right) \geq 1-\frac{1}{2^n}.$$

Since $L_K\geq c$ and $\|\langle\cdot,\theta\rangle\|_1 \simeq \|\langle\cdot,\theta\rangle\|_2$ (see [16]), the result follows. $\square$

Lemma 2.3 *Let K be a convex body in $\mathbb{R}^n$ with volume $|K|=1$ and centre of mass at the origin. Assume that $K\subseteq \alpha\sqrt{n}B_2^n$. Then,*

$$\int_{S^{n-1}}\int_K \exp\left(\frac{|\langle x,\theta\rangle|}{c_2\alpha}\right)^2 dx\sigma(d\theta) \leq 2,$$

where $c_2>0$ is an absolute constant.

Proof: For every $s > 0$ we have

$$\int_{S^{n-1}} \int_K \exp\left(\frac{|\langle x,\theta\rangle|}{s}\right)^2 dx\sigma(d\theta) = 1 + \sum_{k=1}^{\infty} \frac{1}{k! s^{2k}} \int_K \int_{S^{n-1}} |\langle x,\theta\rangle|^{2k} \sigma(d\theta) dx.$$

From Lemma 2.1 we see that this is bounded by

$$1 + \sum_{k=1}^{\infty} \frac{1}{k! s^{2k}} \left(\frac{c \cdot 2k}{2k+n}\right)^k \int_K |x|^{2k} dx \leq 1 + \sum_{k=1}^{\infty} \left(\frac{c'\alpha}{s}\right)^{2k}$$

where $c, c' > 0$ are absolute constants. We conclude the proof taking $s = c_2\alpha$ where $c_2 = 2c'$. □

An application of Markov's inequality gives the following.

Corollary 2.1 *Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$ and centre of mass at the origin. Assume that $K \subseteq \alpha\sqrt{n}B_2^n$. Then, for every $A > 2$ we have*

$$\sigma\left(\theta \in S^{n-1} : \int_K \exp\left(\frac{|\langle x,\theta\rangle|}{c_2\alpha}\right)^2 dx < A\right) > 1 - \frac{2}{A}$$

where $c_2 > 0$ is the constant from Lemma 2.3. □

Theorem 2.1 *Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$ and centre of mass at the origin. Assume that $K \subseteq \alpha\sqrt{n}B_2^n$. There exists $\theta \in S^{n-1}$ such that*

$$\left(\int_K |\langle x,\theta\rangle|^p dx\right)^{1/p} \leq C\alpha\sqrt{p} \int_K |\langle x,\theta\rangle| dx$$

for every $p > 1$, where $C > 0$ is an absolute constant.

Proof: Choose $A = 4$. Using the inequality $e^z > z^k/k!$ $(z > 0)$, Lemma 2.2 and Corollary 2.1 we see that with probability greater than $\frac{1}{2} - \frac{1}{2^n}$ a direction $\theta \in S^{n-1}$ satisfies

$$\int_K |\langle x,\theta\rangle| dx \geq c_1 \quad \text{and} \quad \int_K \exp\left(\frac{|\langle x,\theta\rangle|}{c_2\alpha}\right)^2 dx < 4.$$

It follows that

$$\int_K |\langle x,\theta\rangle|^{2k} dx \leq 4k!(c_2\alpha)^{2k}$$

for every $k \geq 1$, and hence

$$\left(\int_K |\langle x,\theta\rangle|^{2k} dx\right)^{\frac{1}{2k}} \leq c\alpha\sqrt{2k} \leq \frac{c}{c_1}\alpha\sqrt{2k} \int_K |\langle x,\theta\rangle| dx.$$

This is the statement of the theorem for $p = 2k$. The general case follows easily. □

Remarks: (a) Bourgain's argument in [4] shows that L_K is bounded by a power of $\log n$ for every convex body K in $\mathbb{R}^n$ if the following statement holds true: If W is an isotropic convex body in $\mathbb{R}^n$ and $W \subseteq (\alpha\sqrt{n}L_W)B_2^n$, then W is a ψ_2-body with constant $O(\alpha^s)$. Lemma 2.3 shows that, under the same assumptions, "half" of the directions are ψ_2-directions for W, with constant $c\alpha$.

(b) It can be also easily proved that convex bodies with small diameter have large hyperplane sections (this can be verified in several other ways, but the argument below gives some estimate on the distribution of the volume of their $(n-1)$-dimensional sections).

Proposition 2.1 *Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$ and centre of mass at the origin. Assume that $K \subseteq \alpha\sqrt{n}B_2^n$. Then, for every $t > 0$ we have*

$$\sigma\left(\theta \in S^{n-1} : |K \cap \theta^{\perp}| \geq \frac{c_3}{t\alpha}\right) \geq 1 - 2e^{-t^2}.$$

where $c_3 > 0$ is an absolute constant.

Proof: Applying Jensen's inequality to Lemma 2.3, we get

$$\int_{S^{n-1}} \exp\left(\left(\frac{\int_K |\langle x, \theta\rangle|\, dx}{c_2\alpha}\right)^2\right) \sigma(d\theta) \leq 2.$$

Markov's inequality shows that

$$\sigma\left(\theta \in S^{n-1} : \int_K |\langle x, \theta\rangle|\, dx \geq c_2\alpha t\right) \leq 2e^{-t^2}$$

for every $t > 0$. On the other hand, it is a well-known fact (see [16] for the symmetric case) that if K has volume 1 and centre of mass at the origin, then

$$\int_K |\langle x, \theta\rangle|\, dx \simeq \frac{1}{|K \cap \theta^{\perp}|} \tag{2.2}$$

for every $\theta \in S^{n-1}$. This completes the proof. □

3 Positions of zonoids

We first introduce some notation and recall basic facts about zonoids. The support function of a convex body K is defined by $h_K(y) = \max_{x\in K}\langle x, y\rangle$ for all $y \neq 0$. The mean width of K is given by

$$w(K) = 2\int_{S^{n-1}} h_K(u)\sigma(du).$$

We say that K has minimal mean width if $w(K) \leq w(TK)$ for every $T \in SL(n)$.

Recall also the definition of the area measure σ_K of a convex body K: for every Borel $V \subseteq S^{n-1}$ we have

$$\sigma_K(V) = \nu\left(\{x \in \mathrm{bd}(K) : \text{ the outer normal to } K \text{ at } x \text{ is in } V\}\right),$$

where ν is the $(n-1)$-dimensional surface measure on K. It is clear that $\sigma_K(S^{n-1}) = A(K)$, the surface area of K. We say that K has minimal surface area if $A(K) \leq A(TK)$ for every $T \in SL(n)$.

A zonoid is a limit of Minkowski sums of line segments in the Hausdorff metric. Equivalently, a symmetric convex body Z is a zonoid if and only if its polar body is the unit ball of an n-dimensional subspace of an L_1 space; i.e. if there exists a positive measure μ (the supporting measure of Z) on S^{n-1} such that

$$\|x\|_{Z^\circ} = \frac{1}{2}\int_{S^{n-1}} |\langle x, y\rangle| \mu(dy). \tag{3.1}$$

The class of zonoids coincides with the class of projection bodies. Recall that the projection body ΠK of a convex body K is the symmetric convex body whose support function is defined by

$$h_{\Pi K}(\theta) = |P_\theta(K)|, \qquad \theta \in S^{n-1} \tag{3.2}$$

where $P_\theta(K)$ is the orthogonal projection of K onto $\theta^\perp$. From the integral representation

$$|P_\theta(K)| = \frac{1}{2}\int_{S^{n-1}} |\langle u, \theta\rangle| \, d\sigma_K(u) \tag{3.3}$$

which is easily verified in the case of a polytope and extends to any convex body K by approximation, it follows that the projection body of K is a zonoid whose supporting measure is σ_K. Moreover, if we denote by $\mathcal{C}_n$ the class of symmetric convex bodies and by $\mathcal{Z}$ the class of zonoids, Aleksandrov's uniqueness theorem shows that the Minkowski map $\Pi : \mathcal{C}_n \to \mathcal{Z}$ with $K \mapsto \Pi K$, is injective. Note also that $\mathcal{Z}$ is invariant under invertible linear transformations (in fact, $\Pi(TK) = (T^{-1})^*(\Pi K)$ for every $T \in SL(n)$) and closed in the Hausdorff metric. For more information on zonoids, see [22] and [5].

We shall see that three natural positions of a zonoid have small diameter in the sense of Section 2. The proof makes use of the isotropic description of such positions which allows the use of the Brascamp-Lieb inequality.

1. Lewis position: A result of Lewis [14] (see also [3]) shows that every zonotope Z has a linear image Z_1 (the "Lewis position" of Z) with the following property: there exist unit vectors $u_1, \ldots, u_m$ and positive real numbers $c_1, \ldots, c_m$ such that

$$h_{Z_1}(x) = \sum_{j=1}^m c_j |\langle x, u_j\rangle|$$

and

$$I = \sum_{j=1}^m c_j u_j \otimes u_j$$

where I denotes the identity operator in $\mathbb{R}^n$. Using the Brascamp-Lieb inequality, Ball proved in [3] that, under these conditions,

$$|Z_1^\circ| \leq \frac{2^n}{n!} \quad \text{and} \quad B_2^n \subseteq \sqrt{n} Z_1^\circ.$$

The reverse Santaló inequality for zonoids (see [21] and [13]) implies that

$$|Z_1| \geq 2^n \quad \text{and} \quad Z_1 \subseteq \sqrt{n} B_2^n. \tag{3.4}$$

This shows that

$$\mathrm{diam}(Z_1) \leq \sqrt{n} |Z_1|^{1/n}. \tag{A}$$

2. Lowner position: Assume that B_2^n is the ellipsoid of minimal volume containing a zonoid Z_2. Let Z_1 be the Lewis position of Z_2. Then,

$$\frac{|B_2^n|}{|Z_2|} \leq \frac{|\sqrt{n} B_2^n|}{|Z_1|}. \tag{3.5}$$

Now, (3.5) and (3.4) show that

$$\mathrm{diam}(Z_2) \leq 2 \leq |Z_1|^{1/n} \leq \sqrt{n} |Z_2|^{1/n}. \tag{B}$$

3. Minimal mean width position: Assume that $Z_3 = \Pi K$ is a zonoid of volume 1 which has minimal mean width. The results of [9] and [12] show that the area measure σ_K is isotropic, i.e.

$$\int_{S^{n-1}} \langle u, \theta \rangle^2 d\sigma_K(u) = \frac{A(K)}{n} \tag{3.6}$$

for every $\theta \in S^{n-1}$, where $A(K)$ is the surface area of K. Moreover, a result of Petty [19] shows that K has minimal surface area. Now, an application of the Cauchy-Schwarz inequality and (3.6) show that

$$h_{Z_3}(\theta) = \frac{1}{2} \int_{S^{n-1}} |\langle \theta, u \rangle| d\sigma_K(u) \leq \frac{A(K)}{2\sqrt{n}}$$

for every $\theta \in S^{n-1}$. We will use the following fact from [11]:

Lemma 3.1 *If K has minimal surface area, then*

$$A(K) \leq n |\Pi K|^{1/n}.$$

It follows that $h_{Z_3}(\theta) \leq \sqrt{n}/2$ for every $\theta \in S^{n-1}$. In other words,

$$\mathrm{diam}(Z_3) \leq \sqrt{n} |Z_3|^{1/n}. \tag{C}$$

The preceding discussion shows that zonoids have positions with small diameter. More precisely, we have the following statement.

Theorem 3.1 *Let Z be a zonoid in Lewis or Lowner or minimal mean width position. Then,*

$$\mathrm{diam}(Z) \leq \sqrt{n}|Z|^{1/n}. \qquad \Box$$

It follows that the results of Section 2 apply to the class of zonoids: every zonoid has ψ_2-directions in the sense of Theorem 1.1.

Remark: We do not know if isotropic zonoids have small diameter. One can check that their mean width is bounded by $c\sqrt{n}$ (it is of the smallest possible order).

4 Isotropic ψ_2-bodies have small diameter

The purpose of this last section is to show that a convex body is a ψ_2-body only if its isotropic position has small diameter. More precisely, we prove the following.

Theorem 4.1 *Let K be an isotropic convex body in $\mathbb{R}^n$. Assume that K is a ψ_2-body with constant b_2. Then,*

$$K \subseteq C b_2^2 \sqrt{n} \log n B_2^n,$$

where $C > 0$ is an absolute constant.

The proof will follow from two simple lemmas. The idea for the first one comes from [10].

Lemma 4.1 *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and centre of mass at the origin. Then, for every $\theta \in S^{n-1}$,*

$$\int_K |\langle x, \theta\rangle|^p dx \geq \frac{\Gamma(p+1)\Gamma(n)}{2e\Gamma(p+n+1)} \max\left\{h_K^p(\theta), h_K^p(-\theta)\right\}.$$

Proof: Consider the function $f_\theta(t) = |K \cap (\theta^\perp + t\theta)|$. Brunn's principle implies that $f_\theta^{1/(n-1)}$ is concave. It follows that

$$f_\theta(t) \geq \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0)$$

for all $t \in [0, h_K(\theta)]$. Therefore,

$$\begin{aligned}
\int_K |\langle x, \theta\rangle|^p dx &= \int_0^{h_K(\theta)} t^p f_\theta(t) dt + \int_0^{h_K(-\theta)} t^p f_{-\theta}(t) dt \\
&\geq \int_0^{h_K(\theta)} t^p \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0) dt \\
&\quad + \int_0^{h_K(-\theta)} t^p \left(1 - \frac{t}{h_K(-\theta)}\right)^{n-1} f_\theta(0) dt
\end{aligned}$$

$$
\begin{aligned}
&= f_\theta(0)\left(h_K^{p+1}(\theta)+h_K^{p+1}(-\theta)\right)\int_0^1 s^p(1-s)^{n-1}ds \\
&= \frac{\Gamma(p+1)\Gamma(n)}{\Gamma(p+n+1)} f_\theta(0)\left(h_K^{p+1}(\theta)+h_K^{p+1}(-\theta)\right) \\
&\geq \frac{\Gamma(p+1)\Gamma(n)}{2\Gamma(p+n+1)} f_\theta(0)\left(h_K(\theta)+h_K(-\theta)\right)\cdot \max\left\{h_K^p(\theta), h_K^p(-\theta)\right\}.
\end{aligned}
$$

Since K has its centre of mass at the origin, we have $\|f_\theta\|_\infty \leq e f_\theta(0)$ (see [15]), and hence

$$
1 = |K| = \int_{-h_K(-\theta)}^{h_K(\theta)} f_\theta(t)dt \leq e\left(h_K(\theta)+h_K(-\theta)\right) f_\theta(0).
$$

This completes the proof. $\square$

Lemma 4.2 *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and centre of mass at the origin. For every $\theta \in S^{n-1}$,*

$$
\|\langle \cdot, \theta\rangle\|_{\psi_2} \geq \frac{c\max\{h_K(\theta), h_K(-\theta)\}}{\sqrt{n}}
$$

where $c > 0$ is an absolute constant.

Proof: Let $\theta \in S^{n-1}$ and define

$$
I_p(\theta) := \left(\int_K |\langle x, \theta\rangle|^p dx\right)^{1/p}
$$

for every $p \geq 1$. Then, (1.2) shows that

$$
\|\langle \cdot, \theta\rangle\|_{\psi_2} \geq \frac{cI_n(\theta)}{\sqrt{n}}.
$$

From Lemma 4.1 we easily see that $I_n(\theta) \simeq \max\{h_K(\theta), h_K(-\theta)\}$ and the result follows. $\square$

Proof of Theorem 4.1: Since K is a ψ_2-body with constant b_2, Lemma 4.2 shows that

$$
\frac{ch_K(\theta)}{\sqrt{n}} \leq \|\langle \cdot, \theta\rangle\|_{\psi_2} \leq b_2\|\langle \cdot, \theta\rangle\|_1
$$

for every $\theta \in S^{n-1}$. Since K is isotropic, we have

$$
\|\langle \cdot, \theta\rangle\|_1 \leq L_K
$$

for every $\theta \in S^{n-1}$. Bourgain's argument in [4] (see also [7]) together with the ψ_2-assumption show that

$$
L_K \leq c' b_2 \log n.
$$

This implies that

$$K \subseteq C b_2^2 \sqrt{n} \log n B_2^n. \qquad \square$$

Theorem 4.1 shows that ψ_2-bodies belong to a rather restricted class (their polars have at most logarithmic volume ratio). It would be interesting to decide if zonoids are ψ_2-bodies or not.

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