Quermaßintegrals and asymptotic shape of random polytopes in an isotropic convex body

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Abstract

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $N > n$ consider the random polytope $K_N := \mathrm{conv}(\{\pm x_1, \ldots, \pm x_N\})$, where $x_1, \ldots, x_N$ are independent random points, uniformly distributed in K. We prove that if $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then the normalized quermaßintegrals

$$Q_k(K_N) = \left(\frac{1}{\omega_k}\int_{G_{n,k}} |P_F(K)|\, d\nu_{n,k}(F)\right)^{1/k}$$

of K_N satisfy the asymptotic formula $Q_k(K_N) \simeq L_K\sqrt{\log N}$ for all $1 \leqslant k \leqslant n$. From this fact, we obtain precise quantitative estimates on the asymptotic behaviour of basic geometric parameters of K_N.

1 Introduction

The aim of this work is to provide new information on the asymptotic shape of the random polytope

$$K_N = \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\} \tag{1.1}$$

spanned by N independent random points $x_1, \ldots, x_N$ which are uniformly distributed in an isotropic convex body K in $\mathbb{R}^n$. We fix $N > n$ and further exploit the idea of [9] to compare K_N with the L_q-centroid body $Z_q(K)$ of K for $q \simeq \log N$. [Recall that the L_q–centroid body $Z_q(K)$ of K has support function

$$h_{Z_q(K)}(x) = \|\langle \cdot, x\rangle\|_q := \left(\int_K |\langle y, x\rangle|^q dy\right)^{1/q}. \tag{1.2}$$

Background information on isotropic convex bodies and their L_q-centroid bodies is given in Section 2.]

This idea has its roots in previous works (see [11], [19] and [22]) on the behaviour of symmetric random ± 1–polytopes, the absolute convex hulls of random subsets of the discrete cube $D_2^n = \{-1,1\}^n$. These articles demonstrated that the absolute convex hull $D_N = \mathrm{conv}(\{\pm x_1,\dots,\pm x_N\})$ of N independent random points $x_1,\dots,x_N$ uniformly distributed over D_2^n has extremal behaviour – with respect to several geometric parameters – among all ±1–polytopes with N vertices, at every scale of n and $n < N \leqslant 2^n$. The main source of this information is the following estimate from [19] (which improves upon an analogous result from [11]): for all $N \geqslant (1+\delta)n$, where $\delta > 0$ can be as small as $1/\log n$, and for every $0<\beta<1$,

$$D_N \supseteq c\left(\sqrt{\beta\log(N/n)}B_2^n \cap Q_n\right) \tag{1.3}$$

with probability greater than $1-\exp(-c_1 n^\beta N^{1-\beta}) - \exp(-c_2 N)$, where B_2^n is the Euclidean unit ball and $Q_n = [-1/2,1/2]^n$ is the unit cube in $\mathbb{R}^n$.

In a sense, the model of D_N corresponds to the study of the geometry of a random polytope spanned by random points which are uniformly distributed in Q_n. Starting from the observation that $Z_q(Q_n) \simeq \sqrt{q}B_2^n \cap Q_n$, and hence (1.3) can be equivalently written in the form

$$D_N \supseteq cZ_{\beta\log(N/n)}(Q_n), \tag{1.4}$$

we proved in [9] that, in full generality, a precise analogue of (1.4) holds true for the random polytope K_N spanned by N independent random points $x_1,\dots,x_N$ uniformly distributed in an isotropic convex body K: for every $N \geqslant cn$, where $c>0$ is an absolute constant, and for every isotropic convex body K in $\mathbb{R}^n$, we have

$$K_N \supseteq c_1\, Z_q(K) \text{ for all } q \leqslant c_2\log(N/n), \tag{1.5}$$

with probability tending exponentially fast to 1 as $n,N\to\infty$.

The precise statement is given in Section 3, and it will play a main role in the present work. The inclusion is sharp; it is proved in [9] that K_N is "weakly sandwiched" between $Z_{q_i}(K)$ $(i=1,2)$, where $q_i \simeq \log N$, in the following sense. It can be easily checked that for every $\alpha>1$ one has

$$\mathbb{E}\left[\sigma(\{\theta : h_{K_N}(\theta) \geqslant \alpha h_{Z_q(K)}(\theta)\})\right] \leqslant N\alpha^{-q}, \tag{1.6}$$

and this implies that if $q \geqslant c_3\log(N/n)$ then, for most $\theta\in S^{n-1}$, one has $h_{K_N}(\theta) \leqslant c_4 h_{Z_q(K)}(\theta)$. It follows that several geometric parameters of K_N are controlled by the corresponding parameters of $Z_{[\log(N/n)]}(K)$. For example, in [9] the volume radius of a random K_N was determined for the full range of values of N: For every $cn \leqslant N \leqslant \exp(n)$, one has

$$\frac{c_5\sqrt{\log(N/n)}}{\sqrt{n}} \leq |K_N|^{1/n} \leqslant \frac{c_6 L_K\sqrt{\log(N/n)}}{\sqrt{n}} \tag{1.7}$$

with probability greater than $1-\frac{1}{N}$, where $c_5, c_6 > 0$ are absolute constants. Actually, combining the argument with a recent result of B. Klartag and E. Milman (see

[15]) one can see that in the range $N\in[cn,\exp(\sqrt{n})]$ the isotropic constant L_K of K may be inserted in the lower bound, thus leading to the asymptotic formula

$$|K_N|^{1/n}\simeq\frac{L_K\sqrt{\log(N/n)}}{\sqrt{n}}. \tag{1.8}$$

Our first result provides an extension of this formula to the full family of quermaßintegrals $W_{n-k}(K_N)$ of K_N. These are defined through Steiner's formula

$$|K+tB_2^n|=\sum_{k=0}^n\binom{n}{k}W_{n-k}(K)t^{n-k}, \tag{1.9}$$

where $W_{n-k}(K)$ is the mixed volume $V(K,k;B_2^n,n-k)$. We work with a normalized variant of $W_{n-k}(K)$: for every $1\leqslant k\leqslant n$ we set

$$Q_k(K)=\left(\frac{W_{n-k}(K)}{\omega_n}\right)^{1/k}=\left(\frac{1}{\omega_k}\int_{G_{n,k}}|P_F(K)|\,d\nu_{n,k}(F)\right)^{1/k}. \tag{1.10}$$

where the last equality follows from Kubota's integral formula (see Section 2 for background information on mixed volumes). In Section 3 we determine the expectation of $Q_k(K_N)$ for all values of k:

Theorem 1.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $n^2\leqslant N\leqslant\exp(cn)$ then for every $1\leqslant k\leqslant n$ we have*

$$\sqrt{\log N}\lesssim\mathbb{E}\left[Q_k(K_N)\right]\lesssim w(Z_{\log N}(K)). \tag{1.11}$$

In the range $n^2\leqslant N\leqslant\exp(\sqrt{n})$ we have an asymptotic formula: for every $1\leqslant k\leqslant n$,

$$\mathbb{E}\left[Q_k(K_N)\right]\simeq L_K\sqrt{\log N}. \tag{1.12}$$

We would like to comment here that all our estimates remain valid for $n^{1+\delta}\leqslant N\leqslant n^2$, where $\delta\in(0,1)$ is fixed, if we allow the constants to depend on δ. Working in the range $N\simeq n$ would require more delicate arguments. We chose to simplify the exposition; in fact, Proposition 3.1 (see Section 3) is proved for the range $cn\leqslant N\leqslant\exp(cn)$ and it is quite natural that similar extensions can be provided for most statements in this article (the interested reader may also consult [29] and [3]). Another comment is that in this paper we say that a random K_N satisfies a certain asymptotic formula (F) if this holds true with probability greater than $1-N^{-1}$, where the constants appearing in (F) are absolute positive constants.

A more careful analysis is carried out in Section 4, where we obtain the equivalence $Q_k(K_N)\simeq L_K\sqrt{\log N}$ with high probability for a random K_N, in the range $n^2\leqslant N\leqslant\exp(\sqrt{n})$.

Theorem 1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $n^2\leqslant N\leqslant\exp(\sqrt{n})$ then, with probability greater than $1-N^{-1}$ we have*

$$Q_k(K_N)\simeq L_K\sqrt{\log N} \tag{1.13}$$

for all $1\leqslant k\leqslant n$.

From Theorem 1.2 one can derive several geometric properties of a random K_N. In Section 4 we describe two of them, concerning the regularity of the covering numbers $N(K_N, \varepsilon B_2^n)$ and the size of random k-dimensional projections of K_N.

Theorem 1.3. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $n^2 \leqslant N \leqslant \exp(\sqrt{n})$.*
(i) *A random K_N satisfies with probability greater than $1-N^{-1}$ the entropy estimate*

$$\log N(K_N, c_1\varepsilon L_K \sqrt{\log N} B_2^n) \leqslant c_2 n \, \min\left\{\log\left(1+\frac{c_3}{\varepsilon}\right), \frac{1}{\varepsilon^2}\right\} \tag{1.14}$$

for every $\varepsilon > 0$, where c_1, c_2, $c_3 > 0$ are absolute constants.
(ii) *Moreover, a random K_N satisfies with probability greater than $1 - N^{-1}$ the following: for every $1 \leqslant k \leqslant n$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \simeq L_K \sqrt{\log N} \tag{1.15}$$

with probability greater than $1-e^{-ck}$ with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$.

Given $1 \leqslant k \leqslant n$, we also give upper bounds for the volume of the projection of a random K_N onto a *fixed* $F \in G_{n,k}$ and onto the *k-dimensional coordinate subspaces* of $\mathbb{R}^n$. These are valid provided that N is not too large, depending on k.

Theorem 1.4. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n$.*
(i) *For all $k < N \leqslant e^k$ and for every $F \in G_{n,k}$ we have*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \leqslant c L_K \sqrt{\log N} \tag{1.16}$$

with probability greater than $1 - N^{-1}$.
(ii) *For all $k < N \leqslant \exp(c_1\sqrt{k/\log k})$, a random K_N satisfies with probability greater than $1 - \exp(-c_2\sqrt{k/\log k})$ the following: for every $\sigma \subseteq \{1, \ldots, n\}$ with $|\sigma| = k$,*

$$\left(\frac{|P_\sigma(K_N)|}{\omega_k}\right)^{1/k} \leqslant c_3 L_K \log(en/k) \sqrt{\log N}, \tag{1.17}$$

where $c_i > 0$ are absolute constants.

In Section 5 we generalize a result of Mendelson, Pajor and Rudelson from [22] on the combinatorial dimension of the random polytope D_N. This is defined as follows: for a fixed orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and for every $\varepsilon > 0$, the (Vapnik-Chervonenkis) combinatorial dimension $\mathrm{VC}(K, \varepsilon)$ of a symmetric convex body K in $\mathbb{R}^n$ is the largest cardinality of a subset σ of $\{1, \ldots, n\}$ for which

$$\varepsilon Q_\sigma \subseteq P_\sigma(K), \tag{1.18}$$

where Q_σ is the unit cube in $\mathbb{R}^\sigma = \mathrm{span}\{e_i : i \in \sigma\}$ and P_σ denotes the orthogonal projection onto $\mathbb{R}^\sigma$. It is proved in [22] that a random D_N satisfies

$$\mathrm{VC}(D_N, \varepsilon) \simeq \min\left\{\frac{c\log(cN\varepsilon^2)}{\varepsilon^2}, n\right\}. \tag{1.19}$$

We extend this estimate to the more general class of random polytopes K_N where K is an isotropic convex body in $\mathbb{R}^n$ which is unconditional with respect to the basis $\{e_1, \dots, e_n\}$.

Theorem 1.5. *Let K be an unconditional isotropic convex body in $\mathbb{R}^n$. If $c_1 n \leqslant N \leqslant \exp(c_2 n)$ then a random K_N satisfies*

$$\mathrm{VC}(K_N, \varepsilon) \geqslant \min\left\{\frac{c_3\log(N/n)}{\varepsilon^2}, n\right\} \tag{1.20}$$

for every $\varepsilon \in (0,1)$.

2 Notation and background material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $1 \leqslant k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Similarly, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$. We also write $\overline{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\overline{A} := \frac{A}{|A|^{1/n}}$.

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We denote the class of convex bodies in $\mathbb{R}^n$ by $\mathcal{K}_n$. We say that C is symmetric if $-x \in C$ whenever $x \in C$. We say that C is centered if it has center of mass at the origin i.e. $\int_C \langle x, \theta\rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. For each $-\infty < q < \infty$, $q \neq 0$, we define the q-mean width of C by

$$w_q(C) := \left(\int_{S^{n-1}} h_C^q(\theta)\sigma(d\theta)\right)^{1/q}. \tag{2.1}$$

The mean width of C is the quantity $w(C) = w_1(C)$. The radius of C is defined as $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}. \tag{2.2}$$

A centered convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$ and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{2.3}$$

for every θ in the Euclidean unit sphere S^{n-1}. For every convex body K in $\mathbb{R}^n$ there exists an affine transformation T of $\mathbb{R}^n$ such that $T(K)$ is isotropic. Moreover, if we ignore orthogonal transformations, this isotropic image is unique, and hence, the isotropic constant L_K is an invariant of the affine class of K. We refer to [23] and [10] for more information on isotropic convex bodies.

2.1 Quermaßintegrals

The relation between volume and the operations of addition and multiplication of convex bodies by nonnegative reals is described by Minkowski's fundamental theorem: If $K_1, \ldots, K_m \in \mathcal{K}_n$, $m \in \mathbb{N}$, then the volume of $t_1K_1 + \cdots + t_mK_m$ is a homogeneous polynomial of degree n in $t_i \geqslant 0$:

$$|t_1K_1 + \cdots + t_mK_m| = \sum_{1\leqslant i_1,\ldots,i_n\leqslant m} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} \cdots t_{i_n}, \tag{2.4}$$

where the coefficients $V(K_{i_1}, \ldots, K_{i_n})$ can be chosen to be invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is called the mixed volume of the n-tuple $(K_{i_1}, \ldots, K_{i_n})$.

Steiner's formula is a special case of Minkowski's theorem; the volume of $K + tB_2^n$, $t > 0$, can be expanded as a polynomial in t:

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_{n-k}(K) t^{n-k}, \tag{2.5}$$

where $W_{n-k}(K) := V(K, k; B_2^n, n-k)$ is the $(n-k)$-th quermaßintegral of K. It will be convenient for us to work with a normalized variant of $W_{n-k}(K)$: for every $1 \leqslant k \leqslant n$ we set

$$Q_k(K) = \left(\frac{1}{\omega_k} \int_{G_{n,k}} |P_F(K)|\, d\nu_{n,k}(F)\right)^{1/k}. \tag{2.6}$$

Note that $Q_1(K) = w(K)$. Kubota's integral formula

$$W_{n-k}(K) = \frac{\omega_n}{\omega_k} \int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F) \tag{2.7}$$

shows that

$$Q_k(K) = \left(\frac{W_{n-k}(K)}{\omega_n}\right)^{1/k}. \tag{2.8}$$

The Aleksandrov-Fenchel inequality states that if K, L, $K_3, \ldots, K_n \in \mathcal{K}_n$, then

$$V(K, L, K_3, \ldots, K_n)^2 \geqslant V(K, K, K_3, \ldots, K_n)V(L, L, K_3, \ldots, K_n). \tag{2.9}$$

This implies that the sequence $(W_0(K), \ldots, W_n(K))$ is log-concave: we have

$$W_j^{k-i} \geqslant W_i^{k-j} W_k^{j-i} \tag{2.10}$$

if $0 \leqslant i < j < k \leqslant n$. Taking into account (2.8) we conclude that $Q_k(K)$ is a decreasing function of k. For the theory of mixed volumes we refer to [30].

2.2 L_q–centroid bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ and every $y \in \mathbb{R}^n$ we set

$$h_{Z_q(K)}(y) := \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}. \tag{2.11}$$

The L_q-centroid body $Z_q(K)$ of K is the centrally symmetric convex body with support function $h_{Z_q(K)}$. Note that K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in SL(n)$ then $Z_q(T(K)) = T(Z_q(K))$ for all $q \geqslant 1$. From Hölder's inequality it follows that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}(K, -K)$. Using Borell's lemma (see [24, Appendix III]), one can check that

$$Z_q(K) \subseteq c_1 \frac{q}{p} Z_p(K) \tag{2.12}$$

for all $1 \leqslant p < q$. In particular, if K is isotropic, then $R(Z_q(K)) \leqslant c_2 q L_K$. One can also check that if K is centered, then $Z_q(K) \supseteq c_3 K$ for all $q \geqslant n$ (a proof can be found in [25]). We will also use the fact that if K is isotropic, then

$$K \subseteq (n+1) L_K B_2^n \tag{2.13}$$

and hence

$$L_K B_2^n = Z_2(K) \subseteq Z_q(K) \subseteq Z_\infty(K) \subseteq (n+1) L_K B_2^n \tag{2.14}$$

for all $q \geqslant 2$. A proof of the first assertion is given in [14], while the second one is clear from Hölder's inequality.

Let C be a symmetric convex body in $\mathbb{R}^n$ and let $\|\cdot\|_C$ denote the norm induced on $\mathbb{R}^n$ by C. The parameter $k_*(C)$ is defined by

$$k_*(C) = n \frac{w(C)^2}{R(C)^2}. \tag{2.15}$$

It is known that, up to an absolute constant, $k_*(C)$ is the largest positive integer $k \leqslant n$ with the property that $\frac{1}{2}w(C)B_F \subseteq P_F(C) \subseteq 2w(C)B_F$ for most $F \in G_{n,k}$ (to be precise, with probability greater than $\frac{n}{n+k}$). The q-mean width $w_q(C)$ is equivalent to $w(C)$ as long as $q \leqslant k_*(C)$: it is proved in [18] that, for every symmetric convex body C in $\mathbb{R}^n$,

(i) If $1 \leqslant q \leqslant k_*(C)$ then $w(C) \leqslant w_q(C) \leqslant c_4 w(C)$.

(ii) If $k_*(C) \leqslant q \leqslant n$ then $c_5\sqrt{q/n}\, R(C) \leqslant w_q(C) \leqslant c_6\sqrt{q/n}\, R(C)$.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(K) := \left(\int_K \|x\|_2^q dx\right)^{1/q}. \tag{2.16}$$

In [26] and [27] it is proved that for every $1 \leqslant q \leqslant n/2$,

$$I_q(K) \simeq \sqrt{n/q}\, w_q(Z_q(K)) \text{ and } I_{-q}(K) \simeq \sqrt{n/q}\, w_{-q}(Z_q(K)). \tag{2.17}$$

Paouris introduced in [26] the parameter $q_*(K)$ as follows:

$$q_*(K) := \max\{q \leqslant n : k_*(Z_q(K)) \geqslant q\}. \tag{2.18}$$

Then, the main result of [27] states that, for every centered convex body K of volume 1 in $\mathbb{R}^n$, one has $I_{-q}(K) \simeq I_q(K)$ for every $1 \leqslant q \leqslant q_*(K)$. In particular, for all $q \leqslant q_*(K)$ one has $I_q(K) \leqslant c_7 I_2(K)$. If K is isotropic, one can check that $q_*(K) \geqslant c_8\sqrt{n}$, where $c_8 > 0$ is an absolute constant (for a proof, see [26]). Therefore,

$$I_q(K) \leqslant c_9\sqrt{n} L_K \text{ for every } q \leqslant \sqrt{n}. \tag{2.19}$$

When $q \simeq q_*(K)$, the result of [18] shows that $w(Z_q(K)) \simeq w_q(Z_q(K))$. Then, the following useful estimate is a direct consequence of (2.19) and (2.17).

Fact 2.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \leqslant q \leqslant q_*(K)$, then*

$$w(Z_q(K)) \simeq w_q(Z_q(K)) \simeq \sqrt{q}\, L_K. \tag{2.20}$$

In particular, this holds true for all $q \leqslant \sqrt{n}$.

Associated with any centered convex body $K \subset \mathbb{R}^n$ is a family of bodies which was introduced by Ball in [4] (see also [23]): to define them, let us consider a k-dimensional subspace F of $\mathbb{R}^n$ and its orthogonal subspace E. For every $\phi \in F \setminus \{0\}$ we set $E^+(\phi) = \{x \in \mathrm{span}\{E, \phi\} : \langle x, \phi\rangle \geqslant 0\}$. Ball proved that, for every $q \geqslant 0$, the function

$$\phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left(\int_{K\cap E^+(\phi)} \langle x, \phi\rangle^q dx\right)^{-\frac{1}{q+1}} \tag{2.21}$$

is the gauge function of a convex body $B_q(K,F)$ on F. In this article, we will need some facts about the relation of the bodies $B_q(K,F)$ with the L_q-centroid bodies $Z_q(K)$ and their projections. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ and if $1\leqslant k\leqslant n-1$ then, for every $F\in G_{n,k}$ and every $q\geqslant 1$, we have

$$P_F\big(Z_q(K)\big)=(k+q)^{1/q}|B_{k+q-1}(K,F)|^{\frac{1}{k}+\frac{1}{q}}Z_q\big(\overline{B}_{k+q-1}(K,F)\big). \tag{2.22}$$

and

$$|B_{k+q-1}(K,F)|^{\frac{1}{k}+\frac{1}{q}}\leqslant\frac{e(k+q)}{k}\left(\frac{1}{k+q}\right)^{1/q}\frac{1}{|K\cap F^{\perp}|^{1/k}}. \tag{2.23}$$

Also, for every $F\in G_{n,k}$ and every $q\geqslant 1$,

$$\begin{aligned}\frac{k}{e^2(k+q)}Z_q\big(\overline{B}_{k+1}(K,F)\big) &\subseteq Z_q\big(\overline{B}_{k+q-1}(K,F)\big)\\ &\subseteq e^2\frac{k+q}{k}Z_q\big(\overline{B}_{k+1}(K,F)\big).\end{aligned} \tag{2.24}$$

If K is isotropic, then

$$L_{\overline{B}_{k+1}(K,F)}\simeq|K\cap F^{\perp}|^{1/k}L_K. \tag{2.25}$$

For the proofs of these assertions we refer to [26] and [27].

3 Expectation of the Quermaßintegrals

In this Section we give the proof of Theorem 1.1. This will follow from the next proposition.

Proposition 3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $cn\leqslant N\leqslant\exp(cn)$ then for every $1\leqslant k\leqslant n$ we have*

$$c_1\sqrt{n}|Z_{\log(N/n)}(K)|^{1/n}\leqslant\mathbb{E}\left[Q_k(K_N)\right]\leqslant c_2w(Z_{\log N}(K)), \tag{3.1}$$

where $c_1,c_2>0$ are absolute constants.

Proof. We first recall the precise statements of the main results from [9] on the asymptotic shape of a random polytope with N vertices which are chosen independently and uniformly from an isotropic convex body.

Fact 3.2. *Let $\beta\in(0,1/2]$ and $\gamma>1$. If $N\geqslant N(\gamma,n)=c\gamma n$, where $c>0$ is an absolute constant, then, for every isotropic convex body K in $\mathbb{R}^n$ we have*

$$K_N\supseteq c_1Z_q(K)\ \textit{for all}\ q\leqslant c_2\beta\log(N/n), \tag{3.2}$$

with probability greater than $1-f(\beta,N,n)$, where $f(\beta,N,n)\to 0$ exponentially fast as n and N increase.

The upper bound obtained in [9] for $f(\beta, N, n)$ is

$$f(\beta,N,n) \leqslant \exp\left(-c_3 N^{1-\beta} n^{\beta}\right) + \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geqslant \gamma L_K \sqrt{N}), \tag{3.3}$$

where $\Gamma : \ell_2^n \to \ell_2^N$ is the random operator $\Gamma(y) = (\langle x_1, y\rangle, \dots \langle x_N, y\rangle)$ defined by the vertices $x_1, \dots, x_N$ of K_N. There are several known bounds for this last probability (see, for example, [21] or [13]). The best known estimate can be extracted from [1, Theorem 3.13]: one has $\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geqslant \gamma L_K \sqrt{N}) \leqslant \exp(-c_0 \gamma \sqrt{N})$ for all $N \geqslant c\gamma n$. Assuming that $\beta \leqslant 1/2$, one gets

$$f(\beta,N,n) \leqslant \exp(-c_4 \sqrt{n}). \tag{3.4}$$

Since $Q_k(\cdot)$ is decreasing in k, we immediately get

$$\mathbb{E}\left[Q_k(K_N)\right] \geqslant \mathbb{E}\left[Q_n(K_N)\right] = \mathbb{E}\left(\frac{|K_N|}{\omega_n}\right)^{1/n}. \tag{3.5}$$

Now, Fact 3.2 shows that

$$\mathbb{E}\left(\frac{|K_N|}{\omega_n}\right)^{1/n} \geqslant c_5 \left(\frac{|Z_{\log(N/n)}(K)|}{\omega_n}\right)^{1/n}, \tag{3.6}$$

where $c_5 > 0$ is an absolute constant. Combining (3.5) and (3.6) we get the left hand side inequality in (3.1).

We now turn our attention to the opposite direction. Let $N \geqslant n$. Observe that for every $\alpha > 0$ and $\theta \in S^{n-1}$, Markov's inequality shows that

$$\mathbb{P}(\alpha,\theta) := \mathbb{P}\left(\{x \in K : |\langle x, \theta\rangle| \geqslant \alpha \|\langle \cdot, \theta\rangle\|_q\}\right) \leqslant \alpha^{-q}, \tag{3.7}$$

and hence,

$$\begin{aligned} \mathbb{P}\left(h_{K_N}(\theta) \geqslant \alpha h_{Z_q(K)}(\theta)\right) &= \mathbb{P}\left(\max_{j \leqslant N} |\langle x_j, \theta\rangle| \geqslant \alpha \|\langle \cdot, \theta\rangle\|_q\right) \\ &\leqslant N\,\mathbb{P}(\alpha,\theta) \leqslant N\alpha^{-q}. \end{aligned} \tag{3.8}$$

Then, a standard application of Fubini's theorem shows that, for every $\alpha > 1$ one has

$$\mathbb{E}\left[\sigma(\theta : h_{K_N}(\theta) \geqslant \alpha h_{Z_q(K)}(\theta))\right] \leqslant N\alpha^{-q}. \tag{3.9}$$

Using the fact that $h_{K_N}(\theta) \leqslant h_{Z_\infty(K)}(\theta) \leqslant c_6 n L_K$, which follows from (2.14), we write

$$w(K_N) \leqslant \int_{A_N} h_{K_N}(\theta)\, d\sigma(\theta) + c_6 \sigma(A_N^c) n L_K, \tag{3.10}$$

where $A_N = \{\theta : h_{K_N}(\theta) \leqslant \alpha h_{Z_q(K)}(\theta)\}$. Then,

$$w(K_N) \leqslant \alpha \int_{A_N} h_{Z_q(K)}(\theta)\, d\sigma(\theta) + c_6 \sigma(A_N^c) n L_K, \tag{3.11}$$

and hence, by (3.9),

$$\mathbb{E}\left[w(K_N)\right] \leqslant \alpha w(Z_q(K)) + c_6 N n \alpha^{-q} L_K. \tag{3.12}$$

Since $w(Z_q(K)) \geqslant w(Z_2(K)) = L_K$, we get

$$\mathbb{E}\left[w(K_N)\right] \leqslant (\alpha + c_6 N n \alpha^{-q}) w(Z_q(K)). \tag{3.13}$$

Choosing $\alpha = e$ and $q = 2\log N$ we see that

$$\mathbb{E}\left[Q_1(K_N)\right] = \mathbb{E}\left[w(K_N)\right] \leqslant c_7\, w(Z_{2\log N}(K)) \leqslant c_8\, w(Z_q(K)) \tag{3.14}$$

for all $q \geqslant \log N$, taking into account the fact that $Z_{2\log N}(K) \subseteq cZ_{\log N}(K) \subseteq Z_q(K)$ by (2.12). Since $Q_k(K)$ is decreasing in k, we get

$$\mathbb{E}\left[Q_k(K_N)\right] \leqslant \mathbb{E}\left[Q_1(K_N)\right] \leqslant c_9\, w(Z_{\log N}(K)), \tag{3.15}$$

for all $1 \leqslant k \leqslant n$, where $c_9 > 0$ is an absolute constant. This completes the proof of the proposition. $\square$

For the proof of Theorem 1.1 we combine Proposition 3.1 with the known bounds for $|Z_q(K)|$; the first one follows from the results of [26] and [15], while the second one was obtained in [20].

Fact 3.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \leqslant q \leqslant \sqrt{n}$ then*

$$|Z_q(K)|^{1/n} \simeq \sqrt{q/n}\, L_K, \tag{3.16}$$

while if $\sqrt{n} \leqslant q \leqslant n$ then

$$c_9 \sqrt{q/n} \leqslant |Z_q(K)|^{1/n} \leqslant c_{10} \sqrt{q/n}\, L_K. \tag{3.17}$$

Proof of Theorem 1.1. We first assume that $n^2 \leqslant N \leqslant \exp(\sqrt{n})$. From (3.16) we have

$$|Z_{\log N}(K)|^{1/n} \geqslant c_{11} \sqrt{\log N/n}\, L_K, \tag{3.18}$$

and from Fact 2.1 we have

$$w(Z_{\log N}(K)) \leqslant c_{12} \sqrt{\log N}\, L_K. \tag{3.19}$$

Therefore, (3.1) takes the form

$$\mathbb{E}\left[Q_k(K_N)\right] \simeq \sqrt{\log N}\, L_K, \tag{3.20}$$

as claimed. In the case $\exp(\sqrt{n}) \leqslant N \leqslant \exp(cn)$, we use (3.1) and the left hand side inequality from (3.17). It follows that

$$c_{13} \sqrt{\log N} \leqslant \mathbb{E}\left[Q_k(K_N)\right] \leqslant c_2 w(Z_{\log N}(K)), \tag{3.21}$$

for every $1 \leqslant k \leqslant n$, and the proof is complete. $\square$

4 The range $n^2 \leqslant N \leqslant \exp(\sqrt{n})$

Next, we prove Theorem 1.2 on the quermaßintegrals of a random K_N in the range $n^2 \leqslant N \leqslant \exp(\sqrt{n})$. The precise statement is the following.

Theorem 4.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then a random K_N satisfies, with probability greater than $1 - N^{-1}$,*

$$Q_k(K_N) \leqslant c_1 L_K \sqrt{\log N} \tag{4.1}$$

for all $1 \leqslant k \leqslant n$ and, with probability greater than $1 - \exp(-\sqrt{n})$,

$$Q_k(K_N) \geqslant c_2 L_K \sqrt{\log N} \tag{4.2}$$

for all $1 \leqslant k \leqslant n$, where $c_1, c_2 > 0$ are absolute constants.

Proof. Let $n^2 \leqslant N \leqslant \exp(\sqrt{n})$. For the proof of (4.2) recall that, with probability greater than $1 - \exp(-\sqrt{n})$ a random K_N contains $c_3 Z_{\log N}(K)$. Then, using (3.5), (3.6) and the volume estimate from Fact 3.3 we see that any such K_N satisfies

$$Q_k(K_N) \geqslant Q_n(K_N) \geqslant c_3\sqrt{n}|Z_{\log N}(K)|^{1/n} \geqslant c_4 L_K \sqrt{\log N} \tag{4.3}$$

for all $1 \leqslant k \leqslant n$.

For the proof of (4.1) we need two Lemmas.

Lemma 4.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $n^2 \leqslant N \leqslant \exp(cn)$ and for every $q \geqslant \log N$ and $r \geqslant 1$, we have*

$$\int_{S^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(K)}(\theta)}\, d\sigma(\theta) \leqslant (c_1 r)^q \tag{4.4}$$

with probability greater than $1 - r^{-q}$, where $c_1 > 0$ is an absolute constant.

Proof. We have assumed that K is isotropic and hence, from (2.14) and (2.14), we have that $K_N \subseteq \mathrm{conv}(K, -K) \subseteq (n+1)L_K B_2^n$ and $Z_q(K) \supseteq Z_2(K) = L_K B_2^n$. This implies that $h_{K_N}(\theta) \leqslant (n+1) h_{Z_q(K)}(\theta)$ for all $\theta \in S^{n-1}$. We write

$$\begin{aligned}&\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta) \\ &\qquad = \int_0^{n+1} q t^{q-1} \sigma\Big(\theta : \, h_{K_N}(\theta) \geqslant t h_{Z_q(K)}(\theta)\Big)\, dt.\end{aligned} \tag{4.5}$$

We fix $\alpha > 1$ (to be chosen) and estimate the expectation over K^N: using (3.9) we get

$$\begin{aligned}\mathbb{E}\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta)\right) &\leqslant \alpha^q + \int_\alpha^{n+1} q t^{q-1} N t^{-q}\, dt \\ &\leqslant \alpha^q + qN \log\left(\frac{n+1}{\alpha}\right).\end{aligned} \tag{4.6}$$

We choose $\alpha = e$; if $q \geqslant \log N$, then

$$\mathbb{E}\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta)\right) \leqslant c_1^q \tag{4.7}$$

for some absolute constant $c_1 > 0$. Markov's inequality shows that, for every $r \geqslant 1$,

$$\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta) \leqslant (c_1 r)^q \tag{4.8}$$

with probability greater than $1 - r^{-q}$. □

Lemma 4.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $n^2 \leqslant N \leqslant \exp(cn)$ and for every $q \geqslant \log N$ and $r \geqslant 1$, we have*

$$w(K_N) \leqslant c_1 r w_q(Z_q(K)) \tag{4.9}$$

with probability greater than $1 - r^{-q}$.

Proof. Using Hölder's inequality and the Cauchy-Schwarz inequality, we write

$$\begin{aligned} [w(K_N)]^q &\leqslant \left(\int_{S^{n-1}} h_{K_N}(\theta)^{q/2} d\sigma(\theta)\right)^2 \\ &\leqslant [w_q(Z_q(K))]^q \int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta). \end{aligned} \tag{4.10}$$

Lemma 4.2 shows that if $q \geqslant \log N$ and $r \geqslant 1$, then

$$\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta) \leqslant (c_1 r)^q, \tag{4.11}$$

and hence

$$w(K_N) \leqslant c_1 r\, w_q\big(Z_q(K)\big) \tag{4.12}$$

with probability greater than $1 - r^{-q}$. □

We can now prove (4.1): we have assumed that $\log N \lesssim \sqrt{n}$. We choose $q = \log N$ and $r = e$. Then, Lemma 4.3 and Fact 2.1 show that

$$w(K_N) \leqslant c\, w_{\log N}\big(Z_{\log N}(K)\big) \simeq w\big(Z_{\log N}(K)\big) \leqslant c_1 L_K \sqrt{\log N} \tag{4.13}$$

with probability greater than $1 - N^{-1}$. Since $Q_k(K_N) \leqslant w(K_N)$ for all $1 \leqslant k \leqslant n$, the proof is complete. □

Note. Theorem 1.2 and Fact 3.2 show that if $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then a random K_N has – with probability greater than $1 - N^{-1}$ – the next two properties:

(P1) $K_N \supseteq c_1 Z_{\log N}(K)$ and

(P2) $Q_k(K_N) \simeq L_K \sqrt{\log N}$ for all $1 \leqslant k \leqslant n$.

In the next two subsections we derive the two claims of Theorem 1.3 from (P1) and (P2).

4.1 Regularity of the covering numbers

Recall that if K and L are nonempty sets in $\mathbb{R}^n$, then the covering number $N(K,L)$ of K by L is defined to be the smallest number of translates of L whose union covers K. If K is a convex body and L is a symmetric convex body in $\mathbb{R}^n$ then a standard volume argument shows that

$$2^{-n}\frac{|K+L|}{|L|} \leqslant N(K,L) \leqslant 2^n\frac{|K+L|}{|L|}. \tag{4.14}$$

The next Proposition concerns the covering numbers of a random K_N by multiples of the Euclidean unit ball; in particular, it provides a proof for Theorem 1.3 (i).

Proposition 4.4. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $n^2 \leqslant N \leqslant \exp(\sqrt{n})$. Then, a random K_N satisfies the entropy estimate*

$$\log N(K_N, c_1\varepsilon L_K\sqrt{\log N}B_2^n) \leqslant c_2 n \min\left\{\log\left(1+\frac{c_3}{\varepsilon}\right), \frac{1}{\varepsilon^2}\right\} \tag{4.15}$$

for every $\varepsilon > 0$, where c_1, c_2, $c_3 > 0$ are absolute constants. Moreover, if $0 < \varepsilon \leq 1$ we have that

$$c_4 n \log\frac{c_5}{\varepsilon} \leqslant \log N(K_N, c_6\varepsilon L_K\sqrt{\log N}B_2^n) \leqslant c_7 n \log\frac{c_8}{\varepsilon}, \tag{4.16}$$

for suitable absolute constants c_i, $i = 4,\dots,8$.

Proof. We will give estimates for the covering numbers $N(K_N, \varepsilon r_{n,N}B_2^n)$, where K_N satisfies (P1) and (P2), and

$$r_{n,N} := \left(\frac{|K_N|}{\omega_n}\right)^{1/n} \simeq L_K\sqrt{\log N} \tag{4.17}$$

is the volume radius of K_N. Using the right hand side inequality of (4.14), we write

$$N(K_N, \varepsilon r_{n,N}B_2^n) \leqslant 2^n \frac{\left|\frac{1}{\varepsilon r_{n,N}}K_N + B_2^n\right|}{\omega_n}. \tag{4.18}$$

Now, by Steiner's formula,

$$\frac{\left|\frac{1}{\varepsilon r_{n,N}}K_N + B_2^n\right|}{\omega_n} = \sum_{k=0}^{n}\binom{n}{k}Q_k^k(K_N)\frac{1}{\varepsilon^k r_{n,N}^k}. \tag{4.19}$$

and, using the fact that $Q_k(K_N) \simeq r_{n,N}$ by (P2), we get

$$\frac{\left|\frac{1}{\varepsilon r_{n,N}}K_N + B_2^n\right|}{\omega_n} \leqslant \sum_{k=0}^{n}\binom{n}{k}\left(\frac{c}{\varepsilon}\right)^k = \left(1+\frac{c}{\varepsilon}\right)^n. \tag{4.20}$$

Going back to (4.18) we see that

$$\log N(K_N, \varepsilon r_{n,N} B_2^n) \leqslant c_1 n \log\Big(1+\frac{c_2}{\varepsilon}\Big) \tag{4.21}$$

for suitable absolute constants c_1, $c_2 > 0$. A second upper bound can be given by Sudakov's inequality $\log N(K, tB_2^n) \leqslant cnw^2(K)/t^2$ (see e.g. [28]). Since $w(K_N) \simeq r_{n,N}$, we immediately get

$$\log N(K_N, \varepsilon r_{n,N} B_2^n) \leqslant \frac{cn}{\varepsilon^2} \tag{4.22}$$

for all $\varepsilon > 0$. This proves (4.15).

A lower bound on the covering numbers can also be obtained for the case where $0 < \varepsilon \leq 1$. For this we can use the lower bound on the volume of K_N from equation (1.7) or (1.8) depending on whether $\log N \leq \sqrt{n}$ or not. For example, in the case where the latter inequality holds we have

$$N(K_N, \varepsilon r_{n,N} B_2^n)^{1/n} \geq \left(\frac{|K_N|}{|\varepsilon r_{n,N} B_2^n|}\right)^{1/n} = \frac{1}{\varepsilon}. \tag{4.23}$$

Hence, $\log N(K_N, \varepsilon r_{n,N} B_2^n) \geq n \log(1/\varepsilon)$. □

4.2 Random projections of K_N

Next, we show that if K_N has properties (P1) and (P2) then the volume radius of a random projection $P_F(K_N)$ onto $F \in G_{n,k}$ is completely determined by n, k and N; this is the content of Theorem 1.3 (ii).

Proposition 4.5. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $n^2 \leqslant N \leqslant \exp(\sqrt{n})$. Then, a random K_N satisfies with probability greater than $1 - N^{-1}$ the following: for every $1 \leqslant k \leqslant n$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \simeq L_K \sqrt{\log N} \tag{4.24}$$

with probability greater than $1 - e^{-ck}$ with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$.

Proof. The upper bound is a corollary of Theorem 1.2. We know that if $\log N \leqslant \sqrt{n}$ then K_N satisfies (P2) with probability greater than $1 - N^{-1}$; in particular,

$$Q_k(K_N) = \left(\frac{1}{\omega_k}\int_{G_{n,k}} |P_F(K_N)|\, d\nu_{n,k}(F)\right)^{1/k} \lesssim L_K \sqrt{\log N} \tag{4.25}$$

for all $1 \leqslant k \leqslant n$. Applying Markov's inequality we get the following.

Fact 4.6. *If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then K_N satisfies, with probability greater than $1 - N^{-1}$, the following: for every $1 \leqslant k \leqslant n$ and every $t \geqslant 1$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \leqslant c_1 t \sqrt{\log N} L_K \tag{4.26}$$

with probability greater than $1 - t^{-k}$ with respect to $\nu_{n,k}$.

For the lower bound we use (P1). Integrating in polar coordinates we have

$$\begin{aligned} \int_{G_{n,k}} \frac{|P_F^{\circ}(K_N)|}{\omega_k}\, d\nu_{n,k}(F) &= \int_{G_{n,k}} \int_{S_F} \frac{1}{h^k_{P_F(K_N)}(\theta)} d\sigma_F(\theta)\, d\nu_{n,k}(F) \\ &= \int_{G_{n,k}} \int_{S_F} \frac{1}{h^k_{K_N}(\theta)} d\sigma_F(\theta)\, d\nu_{n,k}(F) \\ &\leqslant \left(\int_{G_{n,k}} \int_{S_F} \frac{1}{h^n_{K_N}(\theta)} d\sigma_F(\theta)\, d\nu_{n,k}(F)\right)^{k/n} \\ &= \left(\int_{S^{n-1}} \frac{1}{h^n_{K_N}(\theta)} d\sigma(\theta)\right)^{k/n} \\ &= \left(\frac{|K_N^{\circ}|}{\omega_n}\right)^{k/n}. \end{aligned} \tag{4.27}$$

By the Blaschke–Santaló inequality and the inclusion $K_N \supseteq Z_{c_2 \log N}(K)$, we get

$$\left(\frac{|K_N^{\circ}|}{\omega_n}\right)^{k/n} \leqslant \left(\frac{\omega_n}{|K_N|}\right)^{k/n} \leqslant \left(\frac{\omega_n}{|Z_{c_2 \log N}(K)|}\right)^{k/n}. \tag{4.28}$$

Now, we use the fact that if $q \leqslant \sqrt{n}$ then $\left(\frac{|Z_q(K)|}{\omega_n}\right)^{1/n} \geqslant c_3 \sqrt{q} L_K$ to conclude that

$$\int_{G_{n,k}} \frac{|P_F^{\circ}(K_N)|}{\omega_k}\, d\nu_{n,k}(F) \leqslant \left(\frac{c_4}{\sqrt{\log N} L_K}\right)^k. \tag{4.29}$$

From Markov's inequality we obtain an upper bound for the volume radius of a random $P_F^{\circ}(K_N)$ and the reverse Santaló inequality shows the following.

Fact 4.7. *If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then K_N satisfies, with probability greater than $1 - N^{-1}$, the following: for every $1 \leqslant k \leqslant n$ and every $t \geqslant 1$,*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \geqslant \frac{c_5 L_K \sqrt{\log N}}{t} \tag{4.30}$$

with probability greater than $1 - t^{-k}$ with respect to $\nu_{n,k}$.

Fact 4.6 and Fact 4.7 prove the Proposition. □

Remark 4.8. Making use of [16, Proposition 3.1] one can actually prove that if $k \leqslant n/4$ (or, more generally, $k \leqslant \lambda n$ for some $\lambda \in (0,1)$) then most k-dimensional projections of K_N contain a ball of radius $L_K\sqrt{\log N}$: one has

$$P_F(K_N) \supseteq \frac{c_6}{t} L_K \sqrt{\log N} B_F \tag{4.31}$$

with probability greater than $1 - t^{-k}$ with respect to $\nu_{n,k}$. This in turn shows that (4.30) is satisfied by $P_F(K_N)$. We omit the details.

4.3 Coordinate projections of K_N

In this subsection we prove Theorem 1.4. The first claim of the Theorem is proved in the next Proposition: it gives an estimate on the size of the projection of a random K_N onto a fixed subspace F in $G_{n,k}$.

Proposition 4.9. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n$. For all $k < N \leqslant e^k$ and for every $F \in G_{n,k}$ we have*

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \leqslant cL_K\sqrt{\log N} \tag{4.32}$$

with probability greater than $1 - N^{-1}$.

Proof. Fix $F \in G_{n,k}$. Since $h_{P_F(Z_q(K))}(\theta) = h_{Z_q(K)}(\theta)$ and $\langle P_F(x), \theta\rangle = \langle x, \theta\rangle$ for all $\theta \in S_F$ and all $x \in K$, arguing as in Lemma 4.2 we can show that if $q \geqslant \log N$ then a random K_N satisfies

$$\int_{S_F} \frac{h^q_{P_F(K_N)}(\theta)}{h^q_{P_F(Z_q(K))}(\theta)}\, d\sigma_F(\theta) \leqslant c_1^q. \tag{4.33}$$

Now, applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\Big[w_{-q/2}\Big(P_F\big(Z_q(K)\big)\Big)\Big]^{-q} &= \left(\int_{S_F} \frac{1}{h^{q/2}_{P_F(Z_q(K))}(\theta)}\, d\sigma_F(\theta)\right)^2 \\
&\leqslant \left(\int_{S_F} \frac{1}{h^q_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\right)\left(\int_{S_F} \frac{h^q_{P_F(K_N)}(\theta)}{h^q_{P_F(Z_q(K))}}\, d\sigma_F(\theta)\right) \\
&\leqslant w_{-q}\big(P_F(K_N)\big)^{-q} c_1^q,
\end{aligned}$$

and hence, if $q \geqslant \log N$ we have

$$w_{-q}(P_F(K_N)) \leqslant c_1 s w_{-q/2}(P_F(Z_q(K))) \tag{4.34}$$

with probability greater than $1 - s^{-q}$.

Assume that $q \leqslant k$. Using Hölder's inequality and taking polars in the subspace F, we get

$$\begin{aligned}(4.35)\qquad \left(\frac{|(P_F(K_N))^\circ|}{|B_2^k|}\right)^{1/k} &= \left(\int_{S_F} \frac{1}{h^k_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\right)^{1/k} \\ &\geqslant \left(\int_{S_F} \frac{1}{h^q_{P_F(K_N)}(\theta)}\, d\sigma_F(\theta)\right)^{1/q} \\ &= w_{-q}(P_F(K_N))^{-1}.\end{aligned}$$

Applying the Blaschke-Santaló inequality on F, we see that

$$(4.36)\qquad |P_F(K_N)|^{1/k} \leqslant \frac{c_2}{\sqrt{k}} w_{-q}(P_F(K_N))$$

for a suitable absolute constant $c_2 > 0$. Then, (4.34) shows that

$$(4.37)\qquad |P_F(K_N)|^{1/k} \leqslant \frac{c_3 s}{\sqrt{k}} w_{-q/2}(P_F(Z_q(K)))$$

with probability greater than $1 - s^{-q}$ for $\log N \leqslant q \leqslant k$. From (2.22) we know that

$$(4.38)\qquad P_F(Z_q(K)) = (k+q)^{1/q} |B_{k+q-1}(K,F)|^{\frac{1}{k}+\frac{1}{q}} Z_q\big(\overline{B}_{k+q-1}(K,F)\big),$$

and using (2.24) we get $Z_q\big(\overline{B}_{k+q-1}(K,F)\big) \subseteq c_4 Z_{q/2}\big(\overline{B}_{k+1}(K,F)\big)$ for a new absolute constant $c_4 > 0$. Hence, with probability greater than $1 - s^{-q}$, if $\log N \leqslant q \leqslant k$ we get

$$(4.39)\qquad |P_F(K_N)|^{1/k} \leqslant \frac{c_5 s}{\sqrt{k}} (k+q)^{\frac{1}{q}} |B_{k+q-1}(K,F)|^{\frac{1}{k}+\frac{1}{q}} w_{-\frac{q}{2}}(Z_{\frac{q}{2}}(\overline{B}_{k+1}(K,F))).$$

But $\overline{B}_{k+1}(K,F)$ is easily checked to be isotropic, and from (2.17) and (2.19) we have

$$\begin{aligned}(4.40)\qquad w_{-q/2}\Big(Z_{q/2}\big(\overline{B}_{k+1}(K,F)\big)\Big) &\leqslant c_6 \frac{\sqrt{q}}{\sqrt{k}} I_{-q/2}\big(\overline{B}_{k+1}(K,F)\big) \\ &\leqslant c_7 \sqrt{q} L_{\overline{B}_{k+1}(K,F)}.\end{aligned}$$

From (2.23) and (2.25) we have

$$(4.41)\qquad L_{\overline{B}_{k+1}(K,F)} \leqslant c_8 |K \cap F^\perp|^{1/k} L_K$$

and

$$(4.42)\qquad (k+q)^{1/q} |B_{k+q-1}(K,F)|^{\frac{1}{k}+\frac{1}{k}} |K \cap F^\perp| \leqslant e\frac{k+q}{k} \leqslant 2e$$

for $q \leqslant k$. Going back to (4.39) we conclude that

$$(4.43)\qquad |P_F(K_N)|^{1/k} \leqslant c L_K \frac{\sqrt{q}}{\sqrt{k}}$$

with probability greater than $1 - s^{-q}$ for all q satisfying $\log N \leqslant q \leqslant k$. Choosing $q = \log N$ for $N \leqslant e^k$ we get the result. $\square$

In the previous result, F may be one of the k-dimensional coordinate subspaces of $\mathbb{R}^n$. Using a recent result from [2] we can get a uniform estimate of the same order on the size of all projections of a random K_N onto k-dimensional coordinate subspaces of $\mathbb{R}^n$. This is the second claim of Theorem 1.4.

Proposition 4.10. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n$. For all $k < N \leqslant \exp(c_1\sqrt{k/\log k})$, a random K_N satisfies with probability greater than $1 - \exp(-c_2\sqrt{k/\log k})$ the following: for every $\sigma \subseteq \{1, \dots, n\}$ with $|\sigma| = k$,*

$$\left(\frac{|P_\sigma(K_N)|}{\omega_k}\right)^{1/k} \leqslant c_3 L_K \log(en/k)\sqrt{\log N}, \tag{4.44}$$

where $c_i > 0$ are absolute constants.

Proof. Let $1 \leqslant k \leqslant n$. It is proved in [2, Theorem 1.1] that, for every $t \geqslant 1$,

$$\mathbb{P}\left(\max_{|\sigma|=k} \|P_\sigma(x)\|_2 \geqslant c_1 t L_K \sqrt{k} \log\left(\frac{en}{k}\right)\right) \leqslant \exp\left(-\frac{t\sqrt{k}\log\left(\frac{en}{k}\right)}{\sqrt{\log(ek)}}\right). \tag{4.45}$$

Assume that $N \leqslant \exp(c_2\sqrt{k/\log k})$. Then, with probability greater than $1 - \exp(-c_3\sqrt{k/\log k})$, we have that N random points $x_1, \dots, x_N$ from K satisfy the following: for every $\sigma \subseteq \{1, \dots, n\}$ and for every $1 \leqslant i \leqslant N$,

$$\|P_\sigma(x_i)\|_2 \leqslant c_4 L_K \sqrt{k} \log\left(\frac{en}{k}\right). \tag{4.46}$$

Now, we recall a well-known volume bound that was obtained independently in [5], [8] and [12]: if $z_1, \dots, z_N \in \mathbb{R}^k$ and $\max \|z_i\|_2 \leqslant \alpha$, then

$$|\mathrm{conv}(\{z_1, \dots, z_N\})|^{1/k} \leqslant \frac{c_5\alpha\sqrt{\log N}}{k}. \tag{4.47}$$

In our case, this implies that, for every σ with $|\sigma| = k$,

$$\left(\frac{|P_\sigma(K_N)|}{\omega_k}\right)^{1/k} \leqslant c_6 L_K \log(en/k)\sqrt{\log N}, \tag{4.48}$$

as claimed. $\square$

5 Combinatorial dimension in the unconditional case

In this Section we assume that K is an unconditional isotropic convex body in $\mathbb{R}^n$: it is symmetric and the standard orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ is a 1-unconditional basis for $\|\cdot\|_K$: for every choice of real numbers $t_1, \dots, t_n$ and every choice of signs $\varepsilon_j = \pm 1$,

$$\big\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\big\|_K = \big\|t_1 e_1 + \cdots + t_n e_n\big\|_K. \tag{5.1}$$

It is known that the isotropic constant of K satisfies $L_K \simeq 1$. Moreover, Bobkov and Nazarov have proved that $K \supseteq c_2 Q_n$, where $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ (see [7]).

We will use the fact that the family of L_q-centroid bodies of the cube Q_n is extremal for this class of convex bodies (the argument is due to R. Latała).

Lemma 5.1. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$Z_q(K) \supseteq c Z_q(Q_n) \tag{5.2}$$

for all $q \geq 1$, where $c > 0$ is an absolute constant.

Proof. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be independent and identically distributed ± 1 random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. For every $\theta \in S^{n-1}$, by the unconditionality of K, Jensen's inequality and the contraction principle, one has

$$\begin{aligned}
\|\langle \cdot, \theta\rangle\|_{L^q(K)} &= \left(\int_K \left|\sum_{i=1}^n \theta_i x_i\right|^q dx\right)^{1/q} \\
&= \left(\int_\Omega \int_K \left|\sum_{i=1}^n \theta_i \varepsilon_i |x_i|\right|^q dx\, d\mathbb{P}(\varepsilon)\right)^{1/q} \\
&\geq \left(\int_\Omega \left|\sum_{i=1}^n \theta_i \varepsilon_i \int_K |x_i|\, dx\right|^q d\mathbb{P}(\varepsilon)\right)^{1/q} \\
&\geq \left(\int_\Omega \left|\sum_{i=1}^n t_i \theta_i \varepsilon_i\right|^q d\mathbb{P}(\varepsilon)\right)^{1/q} \\
&\geq \left(\int_{Q_n} \left|\sum_{i=1}^n t_i \theta_i y_i\right|^q dy\right)^{1/q} = \|\langle \cdot, (t\theta)\rangle\|_{L^q(Q_n)},
\end{aligned} \tag{5.3}$$

where $t_i = \int_K |x_i|\, dx \simeq L_K \simeq 1$ and $t\theta = (t_1\theta_1, \ldots, t_n\theta_n)$. Recall that

$$\|\langle \cdot, \theta\rangle\|_{L^q(Q_n)} \simeq \sum_{j \leq q} \theta_j^* + \sqrt{q} \left(\sum_{q < j \leq n} (\theta_j^*)^2\right)^{1/2} \tag{5.4}$$

(see [6]). Since $t_i \simeq 1$ for all $i = 1, \ldots, n$, we get that

$$\|\langle \cdot, \theta\rangle\|_{L^q(K)} \geq \|\langle \cdot, (t\theta)\rangle\|_{L^q(Q_n)} \geq c\|\langle \cdot, \theta\rangle\|_{L^q(Q_n)}, \tag{5.5}$$

and this proves the lemma. $\square$

Since $Z_q(Q_n) \simeq \sqrt{q} B_2^n \cap Q_n$, from Fact 3.1 we immediately get the following.

Proposition 5.2. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. If $c_1 n \leqslant N \leqslant \exp(c_2 n)$ and if $K_N = \mathrm{conv}\{x_1,\ldots,x_N\}$ is a random polytope spanned by N independent random points $x_1,\ldots,x_N$ uniformly distributed in K, then for every $\sigma \subseteq \{1,\ldots,n\}$ we have*

$$P_\sigma(K_N) \supseteq c_1\left(\sqrt{\log(N/n)}B_\sigma \cap Q_\sigma\right) \tag{5.6}$$

with probability $1-o_n(1)$.

Proof of Theorem 1.5. Let $\varepsilon \in (0,1)$. For every $\sigma \subseteq \{1,\ldots,n\}$ with $|\sigma| = k$ we have $Q_\sigma \subseteq \sqrt{k}B_\sigma$, and hence

$$P_\sigma(K_N) \supseteq c_1 \min\left\{\frac{\sqrt{\log(N/n)}}{\sqrt{k}}, 1\right\} Q_\sigma \supseteq \varepsilon Q_\sigma, \tag{5.7}$$

provided that

$$\varepsilon \leqslant \frac{c_2\sqrt{\log(N/n)}}{\sqrt{k}}. \tag{5.8}$$

This shows that

$$\mathrm{VC}(K_N,\varepsilon) \geqslant \min\left\{\frac{c_3\log(N/n)}{\varepsilon^2}, n\right\}, \tag{5.9}$$

which is the lower bound in Theorem 1.5. □

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References

[1] R. Adamczak, A. E. Litvak, A. Pajor and N. Tomczak-Jaegermann, *Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles*, J. Amer. Math. Soc. **23** (2010), No. 2, 535-561.

[2] R. Adamczak, R. Łatała, A. E. Litvak, A. Pajor and N. Tomczak-Jaegermann, *Tail estimates for norms of sums of log-concave random vectors*, Preprint.

[3] D. Alonso-Gutiérrez and J. Prochno, *On the Gaussian behavior of marginals and the mean width of random polytopes*, Preprint, arXiv:1205.6174v1.

[4] K. M. Ball, *Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$*, Studia Math. **88** (1988), 69-84.

[5] I. Bárány and Z. Füredi, *Approximation of the sphere by polytopes having few vertices*, Proc. Amer. Math. Soc. **102** (1988), 651-659.

[6] F. Barthe, O. Guédon, S. Mendelson and A. Naor, *A probabilistic approach to the geometry of the ℓ_p^n-ball*, Ann. Prob. **33** (2005), 480-513.

[7] S. G. Bobkov and F. L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 53-69.

[8] B. Carl and A. Pajor, *Gelfand numbers of operators with values in a Hilbert space*, Invent. Math. **94** (1988), 479-504.

[9] N. Dafnis, A. Giannopoulos and A. Tsolomitis, *Asymptotic shape of a random polytope in a convex body*, J. Funct. Anal. **257** (2009), 2820-2839.

[10] A. Giannopoulos, *Notes on isotropic convex bodies*, Warsaw (2003).

[11] A. Giannopoulos and M. Hartzoulaki, *Random spaces generated by vertices of the cube*, Discrete Comput. Geom. **28** (2002), 255-273.

[12] E. D. Gluskin, *Extremal properties of ortogonal parallelepipeds and their applications to the geometry of Banach spaces*, Mat. Sb. (N.S.) **136** (1988), 85-96.

[13] O. Guédon and M. Rudelson, *L_p-moments of random vectors via majorizing measures*, Adv. Math. **208** (2007), 798-823.

[14] R. Kannan, L. Lovasz and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), 541-559.

[15] B. Klartag and E. Milman, *Centroid Bodies and the Logarithmic Laplace Transform-A Unified Approach*, J. Funct. Anal. **262** (2012), 10-34.

[16] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky theorem*, Israel J. Math. **157** (2007), 193-207.

[17] R. Łatała, *Order statistics and concentration of ℓ_r-norms for log-concave vectors*, J. Funct. Anal. **261** (2011), 681-696.

[18] A. E. Litvak, V. D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), 95-124.

[19] A. E. Litvak, A. Pajor, M. Rudelson and N. Tomczak-Jaegermann, *Smallest singular value of random matrices and geometry of random polytopes*, Adv. Math. **195** (2005), 491-523.

[20] E. Lutwak, D. Yang and G. Zhang, *L^p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111-132.

[21] S. Mendelson and A. Pajor, *On singular values of matrices with independent rows*, Bernoulli **12** (2006), 761-773.

[22] S. Mendelson, A. Pajor and M. Rudelson, *The geometry of random $\{-1,1\}$-polytopes*, Discrete Comput. Geom. **34** (2005), 365-379.

[23] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1376** (1989), 64-104.

[24] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. **1200** (1986), Springer, Berlin.

[25] G. Paouris, *ψ_2-estimates for linear functionals on zonoids*, Lecture Notes in Mathematics **1807** (2003), 211-222.

[26] G. Paouris, *Concentration of mass in convex bodies*, Geometric and Functional Analysis **16** (2006), 1021-1049.

[27] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. **364** (2012), 287-308.

[28] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics **94** (1989).

[29] P. Pivovarov, *On determinants and the volume of random polytopes in isotropic convex bodies*, Geometriae Dedicata, **149**, (2010), 45-58.

[30] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its applications, vol. 44. Cambridge University Press, 1993.

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