A remark on the slicing problem

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Abstract

The purpose of this article is to describe a reduction of the slicing problem to the study of the parameter $I_1(K, Z_q^{\circ}(K)) = \int_K \|\langle \cdot, x\rangle\|_{L_q(K)} dx$. We show that an upper bound of the form $I_1(K, Z_q^{\circ}(K)) \leqslant C_1 q^s \sqrt{n} L_K^2$, with $1/2 \leqslant s \leqslant 1$, leads to the estimate

$$L_n \leqslant \frac{C_2 \sqrt[4]{n} \log n}{q^{\frac{1-s}{2}}},$$

where $L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\}$.

1 Introduction

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, it is *centered*, i.e. its center of mass is at the origin, and if its inertia matrix is a multiple of the identity. The last property is equivalent to the existence of a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.1}$$

for every θ in the Euclidean unit sphere S^{n-1}. It is not hard to see that for every convex body K in $\mathbb{R}^n$ there exists an affine transformation T of $\mathbb{R}^n$ such that $T(K)$ is isotropic. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, one may define the isotropic constant L_K as an invariant of the affine class of K.

The isotropic constant is closely related to the hyperplane conjecture (also known as the slicing problem) which asks if there exists an absolute constant $c > 0$ such that $\max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \geqslant c$ for every convex body K of volume 1 in $\mathbb{R}^n$ with center of mass at the origin. This is because, by Brunn's principle, for any convex body K in $\mathbb{R}^n$ and any $\theta \in S^{n-1}$, the function $t \mapsto |K \cap (\theta^{\perp} + t\theta)|^{\frac{1}{n-1}}$ is concave on its support, and this is enough to show that

$$\int_K \langle x, \theta\rangle^2 dx \simeq |K \cap \theta^{\perp}|^{-2}. \tag{1.2}$$

Using this relation one can check that an affirmative answer to the slicing problem is equivalent to the following statement: "There exists an absolute constant $C > 0$

such that $L_K \leqslant C$ for every convex body K". We refer to the article [13] of Milman and Pajor for background information about isotropic convex bodies.

It is known that $L_K \geqslant L_{B_2^n} \geqslant c > 0$ for every convex body K in $\mathbb{R}^n$ (we use the letters c, c_1, C etc. to denote absolute constants). In the opposite direction, let us write L_n for the maximum of all isotropic constants of convex bodies in $\mathbb{R}^n$,

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\}. \tag{1.3}$$

Bourgain first proved in [4] that $L_n \leqslant c\sqrt[4]{n}\log n$ and, a few years ago, Klartag [8] obtained the estimate $L_n \leqslant c\sqrt[4]{n}$ (see also [9] for a second proof of this bound).

The purpose of this article is to describe a reduction of the slicing problem (or, equivalently, the question whether L_n can be bounded by a quantity independent of the dimension n), to the study of the parameter

$$I_1(K, Z_q^\circ(K)) = \int_K \|\langle\cdot, x\rangle\|_{L_q(K)} dx \tag{1.4}$$

for isotropic convex bodies K. Generally, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then for every symmetric convex body C in $\mathbb{R}^n$ and for every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(K, C) := \left(\int_K \|x\|_C^q dx\right)^{1/q}. \tag{1.5}$$

The notation $I_1(K, Z_q^\circ(K))$ is then justified by the fact that $\|\langle\cdot, x\rangle\|_{L_q(K)}$ is the norm induced on $\mathbb{R}^n$ by the polar body $Z_q^\circ(K)$ of the L_q-centroid body of K (see the next section for background information on L_q-centroid bodies).

Our reduction can be viewed as a continuation of Bourgain's approach to the slicing problem in [4]: the bound $O(\sqrt[4]{n}\log n)$ followed from the inequality

$$nL_K^2 \leqslant I_1\big(K, (T(K))^\circ\big), \tag{1.6}$$

after obtaining an upper bound for the quantity $I_1\big(K, (T(K))^\circ\big)$, where $T \in SL(n)$ is a symmetric, positive definite matrix such that the mean width of $T(K)$ satisfies the estimate $w(T(K)) \leqslant c\sqrt{n}\log n$ (the existence of such a position for K is guaranteed by Pisier's estimate on the norm of the Rademacher projection; see [19]). In Section 4 we prove the following statement:

Theorem 1.1. *There exists an absolute constant $\rho \in (0,1)$ with the following property: given $\kappa, \tau \geqslant 1$, for every $n \geqslant n_0(\tau)$ and every isotropic convex body K in $\mathbb{R}^n$ which satisfies the following entropy estimate:*

$$\log N(K, tB_2^n) \leqslant \frac{\kappa n^2 \log^2 n}{t^2} \quad \textit{for all } t \geqslant \tau\sqrt{n\log n}, \tag{1.7}$$

we have that, if $q \geqslant 2$ satisfies

$$2 \leqslant q \leqslant \rho^2 n \textit{ and } I_1(K, Z_q^\circ(K)) \leqslant \rho n L_K^2, \tag{1.8}$$

then

$$L_K^2 \leqslant C\kappa\sqrt{\frac{n}{q}}\log^2 n \,\max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}. \tag{1.9}$$

Theorem 1.1 can lead to an upper bound for L_n, provided that there exist (κ,τ)-*regular* isotropic convex bodies in $\mathbb{R}^n$, i.e. bodies which satisfy the entropy estimate (1.7) for a pair of constants κ,τ, and at the same time have maximal isotropic constant, i.e. $L_K \simeq L_n$. The existence of such bodies is essentially established by [5, Theorem 5.7]. In Section 5 we give a self-contained proof of this fact; see Theorem 5.1.

Observe that, for every isotropic convex body K in $\mathbb{R}^n$, we have that both conditions in (1.8) are satisfied with $q = 2$, since $I_1(K, Z_2^\circ(K)) \leqslant \sqrt{n}L_K^2$. Therefore, Theorem 1.1 will give us that

$$L_K^2 \leqslant C_1\sqrt{n}\log^2 n \tag{1.10}$$

for any such body which is regular. Theorem 5.1 then guarantees that, for some absolute constants κ,τ and $\delta > 0$, there exists a (κ,τ)-regular isotropic convex body K in $\mathbb{R}^n$ with $L_K \geqslant \delta L_n$, and hence (1.10) leads us to Bourgain's bound again: $L_n \leqslant C_2\sqrt[4]{n}\log n$.

However, the behaviour of $I_1(K, Z_q^\circ(K))$ may allow us to use much larger values of q. In Section 3 we discuss upper and lower bounds for this quantity. For every isotropic convex body K in $\mathbb{R}^n$ we have some simple general estimates:

(i) For every $2 \leqslant q \leqslant n$,

$$c_1 \max\left\{\sqrt{n}L_K^2, \sqrt{qn}, R(Z_q(K))L_K\right\} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c_2 q\sqrt{n}L_K^2.$$

(ii) If $2 \leqslant q \leqslant \sqrt{n}$, then

$$c_1 \max\left\{\sqrt{n}L_K^2, \sqrt{qn}L_K\right\} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c_2 q\sqrt{n}L_K^2.$$

Any improvement of the exponent of q in the upper bound $I_1(K, Z_q^\circ(K)) \leqslant cq\sqrt{n}L_K^2$ would lead to an estimate $L_n \leqslant Cn^\alpha$ with $\alpha < \frac{1}{4}$. It seems plausible that one could even have $I_1(K, Z_q^\circ(K)) \leqslant c\sqrt{qn}L_K^2$, at least when q is small, say $2 \leqslant q \ll \sqrt{n}$. Some evidence is given by the following facts:

(iii) If K is an unconditional isotropic convex body in $\mathbb{R}^n$, then

$$c_1\sqrt{qn} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c_2\sqrt{qn}\log n$$

for all $2 \leqslant q \leqslant n$.

(iv) If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $2 \leqslant q \leqslant \sqrt{n}$, there exists a set $A_q \subseteq O(n)$ with $\nu(A_q) \geqslant 1 - e^{-q}$ such that $I_1(K, Z_q^\circ(U(K))) \leqslant c_3\sqrt{qn}\,L_K^2$ for all $U \in A_q$.

The proofs of (i)-(iv) are given in Section 3.

We can make a final observation about the reduction of Theorem 1.1 on the basis that there exist (κ,τ)-regular isotropic convex bodies K in $\mathbb{R}^n$ with $L_K \geqslant \delta L_n$ (where $\kappa,\tau,\delta > 0$ are absolute constants) which, at the same time, have "small diameter": they satisfy $K \subseteq \gamma\sqrt{n}L_K\, B_2^n$, where $\gamma > 0$ is an absolute constant (see Theorem 5.9). In Section 6, we show that then it is enough to study the parameter $I_1(K, Z_q^\circ(K))$ within the class $\mathcal{IK}_{\rm sd}$ of isotropic convex bodies which are $O(\gamma)$-close to the Euclidean ball D_n of volume 1 and have uniformly bounded isotropic constant. The precise statement which we prove is the following: if we have an isotropic symmetric convex body K in $\mathbb{R}^n$ satisfying $K \subseteq \gamma\sqrt{n}L_K\, B_2^n$, then we can find an isotropic symmetric convex body C such that $L_C \leqslant c_1$, $c_2 D_n \subseteq C \subseteq c_3\gamma D_n$, and

$$\frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2} \leqslant c_4 \frac{I_1(C, Z_q^\circ(C))}{\sqrt{qn}} \tag{1.11}$$

for all $1 \leqslant q \leqslant n$, where $c_1, c_2, c_3, c_4 > 0$ are absolute constants.

2 Notation and preliminaries

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. Let $k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with nonempty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if it has center of mass at the origin, i.e. $\int_C \langle x, \theta\rangle\, dx = 0$ for every $\theta \in S^{n-1}$. The support function of a convex body C is defined by

$$h_C(y) := \max\{\langle x, y\rangle : x \in C\}, \tag{2.1}$$

and the mean width of C is

$$w(C) := \int_{S^{n-1}} h_C(\theta)\sigma(d\theta). \tag{2.2}$$

For each $-\infty < q < \infty$, $q \neq 0$, we define the q-mean width of C by

$$w_q(C) := \left(\int_{S^{n-1}} h_C^q(\theta)\sigma(d\theta) \right)^{1/q}. \tag{2.3}$$

The radius of C is the quantity $R(C) := \max\{\|x\|_2 : x \in C\}$. Also, if the origin is an interior point of C, the polar body C° of C is defined as follows:

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}. \tag{2.4}$$

Finally, we write $\overline{C}$ for the homothetic image of volume 1 of a convex body $C \subseteq \mathbb{R}^n$, i.e. $\overline{C} := \frac{C}{|C|^{1/n}}$.

Recall that if A and B are nonempty sets in $\mathbb{R}^n$, then the covering number $N(A,B)$ of A by B is defined to be the smallest number of translates of B whose union covers A. In this paper, B will usually be a multiple of the Euclidean ball: in those cases we also require that the centres of the translates of B are taken from the set A; one can easily check that this additional requirement does not crucially affect our entropy estimates.

2.1 L_q-centroid bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ and every $y \in \mathbb{R}^n$ we set

$$h_{Z_q(K)}(y) := \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}. \tag{2.5}$$

The L_q-centroid body $Z_q(K)$ of K is the centrally symmetric convex body with support function $h_{Z_q(K)}$. Note that K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in GL(n)$ with $\det T = \pm 1$, then $Z_p(T(K)) = T(Z_p(K))$. From Hölder's inequality it follows that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. Using Borell's lemma (see [15, Appendix III]), one can check that inverse inclusions also hold:

$$Z_q(K) \subseteq \beta_1 q Z_1(K), \tag{2.6}$$

and more generally,

$$Z_q(K) \subseteq \beta_2 \frac{q}{p} Z_p(K) \tag{2.7}$$

for all $1 \leqslant p < q$. In particular, if K is isotropic, then $R(Z_q(K)) \leqslant \beta_1 q L_K$. One can also check that if K is centered, then $Z_q(K) \supseteq \beta_3 K$ for all $q \geqslant n$ (see [16]). All the constants $\beta_i, \overline{\beta}_j$ that appear in this section are absolute positive constants which may be used again in the arguments of the next sections.

Let C be a symmetric convex body in $\mathbb{R}^n$ and let $\|\cdot\|_C$ denote the norm induced on $\mathbb{R}^n$ by C. The parameter $k_*(C)$ is the largest positive integer $k \leqslant n$ with the property that the measure of $F \in G_{n,k}$ for which we have $\frac{1}{2}w(C)B_F \subseteq P_F(C) \subseteq 2w(C)B_F$ is greater than $\frac{n}{n+k}$. It is known that

$$\beta_4 n \frac{w(C)^2}{R(C)^2} \leqslant k_*(C) \leqslant \beta_5 n \frac{w(C)^2}{R(C)^2}. \tag{2.8}$$

The q-mean width $w_q(C)$ is equivalent to $w(C)$ as long as $q \leqslant k_*(C)$. Litvak, Milman and Schechtman proved in [11] that, for every symmetric convex body C in $\mathbb{R}^n$,

(i) If $1 \leqslant q \leqslant k_*(C)$ then $w(C) \leqslant w_q(C) \leqslant \beta_6 w(C)$.

(ii) If $k_*(C) \leqslant q \leqslant n$ then $\beta_7 \sqrt{q/n}\, R(C) \leqslant w_q(C) \leqslant \beta_8 \sqrt{q/n}\, R(C)$.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Recall that, for every symmetric convex body C in $\mathbb{R}^n$ and for every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(K, C) := \left(\int_K \|x\|_C^q dx \right)^{1/q}. \tag{2.9}$$

When $C = B_2^n$, we write $I_q(K) := I_q(K, B_2^n)$ for simplicity. In [17] and [18] it is proved that for every $1 \leqslant q \leqslant n/2$,

$$I_q(K) \simeq \sqrt{n/q}\, w_q(Z_q(K)) \text{ and } I_{-q}(K) \simeq \sqrt{n/q}\, w_{-q}(Z_q(K)). \tag{2.10}$$

The parameter $q_*(K)$ is also defined by

$$q_*(K) := \max\{q \leqslant n : k_*(Z_q(K)) \geqslant q\}. \tag{2.11}$$

Then, the main result of [18] states that, for every centered convex body K of volume 1 in $\mathbb{R}^n$, one has $I_{-q}(K) \simeq I_q(K)$ for every $1 \leqslant q \leqslant q_*(K)$. In particular, for all $q \leqslant q_*(K)$ one has $I_q(K) \leqslant \beta_9 I_2(K)$. If K is isotropic, one can check that $q_*(K) \geqslant c\sqrt{n}$, where $c > 0$ is an absolute constant (for a proof, see [17]). Therefore,

$$I_q(K) \leqslant \beta_{10} \sqrt{n} L_K \quad \text{for every } q \leqslant \sqrt{n}. \tag{2.12}$$

In particular, from (2.10) and (2.12) we see that $w(Z_q(K)) \simeq w_q(Z_q(K)) \simeq \sqrt{q} L_K$ for all $q \leqslant \sqrt{n}$.

2.2 The bodies $B_q(K, F)$

Another family of convex bodies associated with a centered convex body $K \subset \mathbb{R}^n$ was introduced by Ball in [1] (see also [13]): to define them, let us consider a k-dimensional subspace F of $\mathbb{R}^n$ and its orthogonal subspace E. For every $\phi \in F \setminus \{0\}$ we denote by $E^+(\phi)$ the halfspace $\{x \in \mathrm{span}\{E, \phi\} : \langle x, \phi\rangle \geqslant 0\}$. Ball proved that, for every $q \geqslant 0$, the function

$$\phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left(\int_{K \cap E^+(\phi)} \langle x, \phi\rangle^q dx \right)^{-\frac{1}{q+1}} \tag{2.13}$$

is the gauge function of a convex body $B_q(K, F)$ on F. Several properties of these bodies can be found in [1], [13] and also in [17], [18]. In Section 5, we will make use of only two of those:

(i) Let $K \subset \mathbb{R}^n$ be isotropic, let $1 \leqslant k < n$ and let $F \in G_{n,k}$. Then the body $\overline{B}_{k+1}(K,F)$ is *almost isotropic*, namely it has (by definition) volume 1, and we can write $\overline{B}_{k+1}(K,F) \simeq T(\overline{B}_{k+1}(K,F))$ where $T(\overline{B}_{k+1}(K,F))$ is an isotropic (in the regular sense) linear image of $\overline{B}_{k+1}(K,F)$. In addition,

$$|K \cap F^{\perp}|^{1/k} \simeq \frac{L_{B_{k+1}(K,F)}}{L_K}. \tag{2.14}$$

(ii) Let K, F and $k < n$ be as above and consider any $p \in [1,k]$. Then

$$Z_p\big(\overline{B}_{k+1}(K,F)\big) \simeq |K \cap F^{\perp}|^{1/k} P_F(Z_p(K)). \tag{2.15}$$

2.3 Two related lemmas

We close this section with two lemmas that will be used in the sequel; they reveal some properties of the support function of the L_q-centroid bodies of a convex body with respect to subsets or certain integrals of maxima.

Lemma 2.1. *Let K be a convex body of volume 1 in $\mathbb{R}^n$, and consider any points $z_1, z_2, \ldots, z_N \in \mathbb{R}^n$. If $q \geqslant 1$ and $p \geqslant \max\{\log N, q\}$, then*

$$\left(\int_K \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle|^q dx\right)^{1/q} \leqslant \overline{\beta}_1 \max_{1\leqslant i\leqslant N} h_{Z_p(K)}(z_i), \tag{2.16}$$

where $\overline{\beta}_1 > 0$ is an absolute constant.

Proof. Let $p \geqslant \max\{\log N, q\}$ and $\theta \in S^{n-1}$. Markov's inequality shows that

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant e^3 h_{Z_p(K)}(\theta)\}| \leqslant e^{-3p}. \tag{2.17}$$

Since $x \mapsto |\langle x, \theta\rangle|$ is a seminorm, from Borell's lemma (see [15, Appendix III]) we get that

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant e^3 t h_{Z_p(K)}(\theta)\}| \leqslant (1 - e^{-3p}) \left(\frac{e^{-3p}}{1 - e^{-3p}}\right)^{\frac{t+1}{2}} \leqslant e^{-pt} \tag{2.18}$$

for every $t \geqslant 1$. We set $S := e^3 \max_{1\leqslant i\leqslant N} h_{Z_p(K)}(z_i)$. Then, for every $t \geqslant 1$ we have that

$$\begin{aligned} |\{x \in K : \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle| \geqslant St\}| &\leqslant \sum_{i=1}^N |\{x \in K : |\langle x, z_i\rangle| \geqslant e^3 t h_{Z_p(K)}(z_i)\}| \\ &\leqslant N e^{-pt}. \end{aligned}$$

It follows that

$$\begin{aligned}
\int_K \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle|^q dx &= q\int_0^\infty s^{q-1}|\{x\in K : \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle| \geqslant s\}|\, ds \\
&\leqslant S^q + q\int_S^\infty s^{q-1}|\{x\in K : \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle| \geqslant s\}|\, ds \\
&= S^q\left(1 + q\int_1^\infty t^{q-1}|\{x\in K : \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle| \geqslant St\}|\, dt\right) \\
&\leqslant S^q\left(1 + qN\int_1^\infty t^{q-1}e^{-pt}dt\right) \\
&= S^q\left(1 + \frac{qN}{p^q}\int_p^\infty t^{q-1}e^{-t}dt\right) \\
&\leqslant S^q\left(1 + \frac{qN}{p^q}e^{-p}p^q\right) \\
&\leqslant (3S)^q,
\end{aligned}$$

where we have also used the fact that, for every $p \geqslant q \geqslant 1$,

$$\int_p^\infty t^{q-1}e^{-t}dt \leqslant e^{-p}p^q. \tag{2.19}$$

This finishes the proof (with $\overline{\beta}_1 = 3e^3$). □

Remark 2.2. It is a well-known fact (see e.g. [6, Proposition 2.5.1]) that

$$\int_K \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle| dx \leqslant C_1 \log N \max_{1\leqslant i\leqslant N} h_{Z_1(K)}(z_i). \tag{2.20}$$

Through a variant of the argument in [6], and using (2.19) as well, one can also show that for $q \leqslant \log N$,

$$\left(\int_K \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle|^q dx\right)^{1/q} \leqslant C_2 \log N \max_{1\leqslant i\leqslant N} h_{Z_1(K)}(z_i). \tag{2.21}$$

Now, both inequalities can be directly deduced from Lemma 2.1 combined with (2.6), however the lemma provides additional information on how well the quantities $\left(\int_K \max_i |\langle x, z_i\rangle|^q dx\right)^{1/q}$ and $\max_i \left(\int_K |\langle x, z_i\rangle|^q dx\right)^{1/q} \equiv \max_i h_{Z_q(K)}(z_i)$ can be compared: for $q \leqslant \log N$, using also (2.7), we have that

$$\left(\int_K \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle|^q dx\right)^{1/q} \leqslant \overline{\beta}_1 \max_{1\leqslant i\leqslant N} h_{Z_{\log N}(K)}(z_i) \leqslant C\frac{\log N}{q}\max_{1\leqslant i\leqslant N} h_{Z_q(K)}(z_i), \tag{2.22}$$

whereas for $q > \log N$,

$$\left(\int_K \max_{1\leqslant i\leqslant N}|\langle x, z_i\rangle|^q dx\right)^{1/q} \simeq \max_{1\leqslant i\leqslant N}\left(\int_K |\langle x, z_i\rangle|^q dx\right)^{1/q}. \tag{2.23}$$

We now turn our attention to the L_q-centroid bodies of subsets of K.

Lemma 2.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant q, r \leqslant n$. There exists an absolute constant $\overline{\beta}_2 > 0$ such that if A is a subset of K with $|A| \geqslant 1 - e^{-\overline{\beta}_2 q}$, then*

$$Z_p(K) \subseteq 2Z_p(\overline{A}) \tag{2.24}$$

for all $1 \leqslant p \leqslant q$. Also, for the opposite inclusion, it suffices to have $|A| \geqslant 2^{-\frac{r}{2}}$ to conclude that

$$Z_p(\overline{A}) \subseteq 2Z_p(K) \tag{2.25}$$

for all $r \leqslant p \leqslant n$.

Proof. Let $\theta \in S^{n-1}$. Note that

$$h_{Z_p(\overline{A})}(\theta) = \left(\int_{\overline{A}} |\langle x, \theta \rangle|^p dx \right)^{1/p} = \frac{1}{|A|^{\frac{1}{p}+\frac{1}{n}}} \left(\int_A |\langle x, \theta \rangle|^p dx \right)^{1/p}. \tag{2.26}$$

We first prove (2.25): since $A \subseteq K$ and assuming that $|A| \geqslant 2^{-\frac{r}{2}}$, we have

$$\begin{aligned} h_{Z_p(K)}(\theta) &= \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{1/p} \geqslant \left(\int_A |\langle x, \theta \rangle|^p dx \right)^{1/p} \\ &\geqslant 2^{-\frac{r}{2p} - \frac{r}{2n}} \left(\int_{\overline{A}} |\langle x, \theta \rangle|^p dx \right)^{1/p} \geqslant \frac{1}{2} h_{Z_p(\overline{A})}(\theta) \end{aligned}$$

for all $r \leqslant p \leqslant n$. On the other hand, assuming that $|A| \geqslant 1 - e^{-\overline{\beta}_2 q}$ and using the fact that $\|\langle \cdot, \theta \rangle\|_{2p} \leqslant 2\beta_2 \|\langle \cdot, \theta \rangle\|_p$, we have

$$\begin{aligned} \int_K |\langle x, \theta \rangle|^p dx &= \int_A |\langle x, \theta \rangle|^p dx + \int_{K \setminus A} |\langle x, \theta \rangle|^p dx \\ &\leqslant |A|^{1+\frac{p}{n}} \int_{\overline{A}} |\langle x, \theta \rangle|^p dx + |K \setminus A|^{1/2} \left(\int_K |\langle x, \theta \rangle|^{2p} dx \right)^{1/2} \\ &\leqslant \int_{\overline{A}} |\langle x, \theta \rangle|^p dx + e^{-\overline{\beta}_2 q/2} (2\beta_2)^p \int_K |\langle x, \theta \rangle|^p dx \\ &\leqslant \int_{\overline{A}} |\langle x, \theta \rangle|^p dx + \frac{1}{2} \int_K |\langle x, \theta \rangle|^p dx \end{aligned}$$

for every $p \leqslant q$, if $\overline{\beta}_2 > 0$ is chosen large enough. This proves (2.24). $\square$

3 Simple estimates for $I_1(K, Z_q^\circ(K))$

In this section we give some upper and lower bounds for $I_1(K, Z_q^\circ(K))$ which hold true for every isotropic convex body K in $\mathbb{R}^n$ and any $1 \leqslant q \leqslant n$. In fact, our arguments are quite direct and make use of estimates for simple parameters of the bodies $Z_q(K)$, such as their radius or their volume, so that it is straightforward to reach analogous upper and lower bounds for $I_1(K, Z_q^\circ(M))$ in the more general case when K and M are not necessarily the same isotropic convex body.

Since $h_{Z_q(K)}(x) \leqslant R(Z_q(K))\|x\|_2$, we have that

$$I_1(K, Z_q^\circ(K)) \leqslant R(Z_q(K)) \int_K \|x\|_2\, dx \leqslant R(Z_q(K))\sqrt{n}L_K, \tag{3.1}$$

which, in combination with the fact that $R(Z_q(K)) \leqslant \beta_1 q L_K$ (a direct consequence of (2.6)), leads to the bound

$$I_1(K, Z_q^\circ(K)) \leqslant \beta_1 q\sqrt{n}L_K^2. \tag{3.2}$$

More generally, we have that

$$I_1(K, Z_q^\circ(M)) \leqslant R(Z_q(M)) \int_K \|x\|_2\, dx \leqslant \beta_1 q\sqrt{n}L_K L_M. \tag{3.3}$$

However, in the case that M is an orthogonal transformation of K, the next lemma shows that the average of the quantity $I_1(K, Z_q^\circ(M))$ can be bounded much more effectively than in (3.3).

Lemma 3.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $2 \leqslant q \leqslant n$,*

$$\left(\int_{O(n)} I_1^q\big(K, Z_q^\circ(U(K))\big)\, d\nu(U)\right)^{1/q} \leqslant C\sqrt{q/n}\, I_q^2(K), \tag{3.4}$$

where $C > 0$ is an absolute constant.

Proof. We write

$$\begin{aligned}
\int_{O(n)} I_1^q\big(K, Z_q^\circ(U(K))\big)\, d\nu(U) &\leqslant \int_{O(n)} I_q^q\big(K, Z_q^\circ(U(K))\big)\, d\nu(U) \\
&= \int_{O(n)} \int_K \int_{U(K)} |\langle x, y\rangle|^q dy\, dx\, d\nu(U) \\
&= \int_K \int_K \int_{O(n)} |\langle x, Uy\rangle|^q d\nu(U)\, dy\, dx \\
&= \int_K \int_K \|y\|_2^q \int_{S^{n-1}} |\langle x, \theta\rangle|^q d\sigma(\theta)\, dy\, dx \\
&= c_{n,q}^q \int_K \int_K \|y\|_2^q \|x\|_2^q dy\, dx \\
&= c_{n,q}^q I_q^{2q}(K),
\end{aligned}$$

where $c_{n,q} \simeq \sqrt{q/n}$. □

Recall that in the case that K is isotropic, one has from [17] that $I_q(K) \simeq \max\{\sqrt{n}L_K, R(Z_q(K))\}$. Then, Lemma 3.1 shows that, for every $2 \leqslant q \leqslant n$,

$$\left(\int_{O(n)} I_1^q\big(K, Z_q^\circ(U(K))\big)\, d\nu(U)\right)^{1/q} \leqslant C_1 \max\{\sqrt{qn}, q^2\sqrt{q/n}\}\, L_K^2, \tag{3.5}$$

where $C_1 > 0$ is an absolute constant. Therefore, for every $2 \leqslant q \leqslant \sqrt{n}$, there exists a set $A_q \subseteq O(n)$ with $\nu(A_q) \geqslant 1 - e^{-q}$ such that $I_1\big(K, Z_q^\circ(U(K))\big) \leqslant C_2\sqrt{qn}\, L_K^2$ for all $U \in A_q$. It is thus conceivable that there are properties of the bodies $Z_q(K)$ which we can exploit to also bound $I_1(K, Z_q^\circ(K))$ more effectively than in (3.1) and (3.2).

We now pass to lower bounds; we will present three simple arguments. For the first one we do not have to assume that K or M are in the isotropic position, only that they are centered and have volume 1: from [13, Corollary 2.2.a] we have that

$$I_1(K, Z_q^\circ(M)) = \int_K h_{Z_q(M)}(x)\, dx \geqslant \frac{n}{n+1}\, \frac{1}{|Z_q^\circ(M)|^{1/n}}. \tag{3.6}$$

Then, by the Blaschke–Santaló inequality, we get that

$$I_1(K, Z_q^\circ(M)) \geqslant c_1 n |Z_q(M)|^{1/n} \geqslant c_2\sqrt{qn}L_M \tag{3.7}$$

for all $2 \leqslant q \leqslant \sqrt{n}$, because $|Z_q(M)|^{1/n} \geqslant c_3\sqrt{q/n}\, L_M$ for this range of values of q by a recent result of Klartag and E. Milman (see [9]). When $\sqrt{n} \leqslant q \leqslant n$, we have the weaker lower bound $|Z_q(M)|^{1/n} \geqslant c_4\sqrt{q/n}$, which is due to Lutwak, Yang and Zhang (see [12]). It follows that $I_1(K, Z_q^\circ(M)) \geqslant c_5\sqrt{qn}$ for this range of values of q.

For the second argument, we require that K is isotropic and we write

$$\begin{aligned} I_1(K, Z_q^\circ(M)) &= \int_K h_{Z_q(M)}(x)\, dx = \int_K \max_{z \in Z_q(M)} |\langle x, z\rangle|\, dx \\ &\geqslant \max_{z \in Z_q(M)} \int_K |\langle x, z\rangle|\, dx \geqslant c \max_{z \in Z_q(M)} \|z\|_2 L_K \\ &= c\, R(Z_q(M)) L_K. \end{aligned} \tag{3.8}$$

Finally, if M is isotropic as well, we can use Hölder's inequality to get

$$\begin{aligned} I_1(K, Z_q^\circ(M)) &= \int_K h_{Z_q(M)}(x)\, dx \\ &\geqslant \int_K h_{Z_2(M)}(x)\, dx = \int_K \|x\|_2 L_M\, dx \geqslant c\sqrt{n}L_K L_M. \end{aligned} \tag{3.9}$$

All the estimates presented above are gathered in the next proposition.

Proposition 3.2. *Let K and M be isotropic convex bodies in $\mathbb{R}^n$. For every $2 \leqslant q \leqslant n$,*

$$c_1 \max\left\{\sqrt{n}L_K L_M, \sqrt{qn}, R(Z_q(M))L_K\right\} \leqslant I_1(K, Z_q^\circ(M)) \leqslant c_2 q\sqrt{n}L_K L_M. \tag{3.10}$$

In addition, if $2 \leqslant q \leqslant \sqrt{n}$ *then*

$$c_1 \max\left\{\sqrt{n}L_KL_M, \sqrt{qn}L_M\right\} \leqslant I_1(K, Z_q^\circ(M)) \leqslant c_2 q\sqrt{n}L_KL_M. \tag{3.11}$$

The situation is more or less clear in the unconditional case. Recall that a convex body K in $\mathbb{R}^n$ is called unconditional if it is symmetric with respect to all coordinate hyperplanes (for some orthonormal basis of $\mathbb{R}^n$). Then, it is easy to check that one can bring K to the isotropic position by applying an operator which is diagonal with respect to this basis. It is also not hard to prove that the isotropic constant of K satisfies $L_K \simeq 1$. The upper bound follows from the Loomis–Whitney inequality; see also [2]. It is known (from [3]) that, for every $q \geqslant 2$, one has $h_{Z_q(K)}(y) \leqslant c\sqrt{qn}\|y\|_\infty$, where $c > 0$ is an absolute constant. This leads us to the estimates

$$c_1\sqrt{qn} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c\sqrt{qn}\int_K \|x\|_\infty dx \leqslant c_2\sqrt{qn}\log n \tag{3.12}$$

for all $2 \leqslant q \leqslant n$ (the same estimates hold true for the quantity $I_1(K, Z_q^\circ(M))$ when M is too an unconditional isotropic convex body).

4 The reduction

Let $\kappa, \tau > 0$. Throughout this paper, we say that an isotropic convex body K in $\mathbb{R}^n$ is (κ, τ)-regular if

$$\log N(K, tB_2^n) \leqslant \frac{\kappa n^2 \log^2 n}{t^2} \text{ for all } t \geqslant \tau\sqrt{n\log n}. \tag{4.1}$$

The purpose of this section is to present a reduction of the slicing problem to the study of the quantity $I_1(K, Z_q^\circ(K))$ for (κ, τ)-regular isotropic convex bodies: we show that any upper bound for $I_1(K, Z_q^\circ(K))$ immediately leads to an upper bound for the isotropic constant L_K of a regular convex body K. Note that the dependence seems to be nontrivial, in the sense that using the simple estimates of Section 3 we can already recover the currently known bound for L_K with a loss of a logarithmic factor, while a small (although not necessarily easy) improvement to those estimates will also result in new bounds for L_K. In a sense, we will have fully presented our reduction by the end of the next section, where we provide a self-contained proof of the fact that there exist regular isotropic convex bodies K in $\mathbb{R}^n$ with $L_K \simeq L_n$. First, let us see how the quantity $I_1(K, Z_q^\circ(K))$ and the isotropic constant of a regular convex body K are connected.

Theorem 4.1. *There exists an absolute constant* $\rho \in (0,1)$ *with the following property: given* $\kappa, \tau \geqslant 1$*, for every* $n \geqslant n_0(\tau)$ *and every* (κ, τ)*-regular isotropic convex body* K *in* $\mathbb{R}^n$ *we have that, if* $q \geqslant 2$ *satisfies*

$$2 \leqslant q \leqslant \rho^2 n \text{ and } I_1(K, Z_q^\circ(K)) \leqslant \rho n L_K^2, \tag{4.2}$$

then

$$L_K^2 \leqslant C\kappa\sqrt{\frac{n}{q}}\log^2 n\,\max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}. \tag{4.3}$$

Proof. We define a convex body W in $\mathbb{R}^n$, setting

$$W := \{x \in K : h_{Z_q(K)}(x) \leqslant C_1 I_1(K, Z_q^\circ(K))\}, \tag{4.4}$$

where $C_1 = e^{2\overline{\beta}_2}$ and $\overline{\beta}_2 > 0$ is the constant which appears in Lemma 2.3. From Markov's inequality we have that $|W| \geqslant 1 - e^{-2\overline{\beta}_2}$ and also trivially that $|W| \geqslant 2^{-1} \geqslant 2^{-\frac{q}{2}}$ (as long as $\overline{\beta}_2 \geqslant 1$). Then we set

$$K_1 := \overline{W}. \tag{4.5}$$

Applying both cases of Lemma 2.3 to the set W with $p = 2$, we see that

$$\frac{1}{2}Z_2(K_1) \subseteq Z_2(K) \subseteq 2Z_2(K_1). \tag{4.6}$$

This implies that

$$\frac{1}{4}L_K^2 = \frac{1}{4}\int_K \langle x, \theta\rangle^2 dx \leqslant \int_{K_1} \langle x, \theta\rangle^2 dx \leqslant 4\int_K \langle x, \theta\rangle^2 dx = 4L_K^2 \tag{4.7}$$

for every $\theta \in S^{n-1}$, and hence

$$\frac{nL_K^2}{4} \leqslant \sum_{i=1}^n \int_{K_1} \langle x, e_i\rangle^2 dx = \int_{K_1} \|x\|_2^2 dx \leqslant 4nL_K^2. \tag{4.8}$$

We also have

$$K_1 = |W|^{-1/n} W \subseteq 2W \subseteq 2K, \tag{4.9}$$

thus for every $x \in K_1$ we have $x/2 \in W$, and using (2.25) of Lemma 2.3 again, we can write

$$h_{Z_q(K_1)}(x) \leqslant 2h_{Z_q(K)}(x) = 4h_{Z_q(K)}(x/2) \leqslant 4C_1 I_1(K, Z_q^\circ(K)). \tag{4.10}$$

Finally,

$$\log N(K_1, tB_2^n) \leqslant \log N(2K, tB_2^n) \leqslant \frac{4\kappa n^2 \log^2 n}{t^2}, \tag{4.11}$$

for all $t \geqslant 2\tau\sqrt{n\log n}$. We now write

$$nL_K^2 \leqslant 4\int_{K_1} \|x\|_2^2 dx \leqslant 4\int_{K_1} \max_{z\in K_1} |\langle x, z\rangle|\, dx. \tag{4.12}$$

(4.11) tells us that for every $t \geqslant 2\tau\sqrt{n\log n}$, we can find $z_1, \dots, z_{N_t} \in K_1$ such that $K_1 \subseteq \bigcup_{i=1}^{N_t}(z_i + tB_2^n)$, and $|N_t| \leqslant \exp\left(\frac{4\kappa n^2\log^2 n}{t^2}\right)$. It follows that

$$\max_{z\in K_1}|\langle x, z\rangle| \leqslant \max_{1\leqslant i\leqslant N_t}|\langle x, z_i\rangle| + \max_{w\in tB_2^n}|\langle x, w\rangle| = \max_{1\leqslant i\leqslant N_t}|\langle x, z_i\rangle| + t\|x\|_2, \tag{4.13}$$

and hence

$$\begin{aligned} nL_K^2 &\leqslant 4\int_{K_1}\max_{1\leqslant i\leqslant N_t}|\langle x, z_i\rangle|dx + 4t\int_{K_1}\|x\|_2 dx \\ &\leqslant 4\int_{K_1}\max_{1\leqslant i\leqslant N_t}|\langle x, z_i\rangle|dx + 8t\sqrt{n}L_K. \end{aligned} \tag{4.14}$$

We choose

$$t_0^2 = 16C_2\kappa \max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}\frac{n^{3/2}}{\sqrt{q}}\log^2 n, \tag{4.15}$$

where $C_2 = 16C_1\beta_2\overline{\beta}_1$ with β_2 the constant appearing in (2.7) and $\overline{\beta}_1$ the constant from Lemma 2.1. With this choice of t_0, we have

$$t_0^2 \geqslant 16C_2\kappa\sqrt{\frac{n}{q}}\,n\log^2 n \geqslant \frac{16C_2\kappa}{\rho}\,n\log^2 n, \tag{4.16}$$

as long as q satisfies (4.2), and

$$t_0^2 \geqslant 16C_2\kappa\,\frac{I_1(K, Z_q^\circ(K))}{qL_K^2}\,n\log^2 n. \tag{4.17}$$

From (4.16) it is clear that

$$t_0^2 \geqslant 16C_2\kappa\frac{n\log^2 n}{\rho} \geqslant 4\tau^2 n\log n, \tag{4.18}$$

provided that $n \geqslant n_0(\tau, \kappa, \rho)$, so the argument above, leading up to (4.14), remains valid for $t = t_0$. We also set $p_0 := \frac{4\kappa n^2\log^2 n}{t_0^2}$. Observe that $p_0 \geqslant q$ (as long as q is assumed to satisfy (4.2)), if ρ is chosen properly: indeed, we have $\max\left\{1, \frac{I_1(K,Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\} \leqslant \rho\sqrt{n/q}$, and hence

$$t_0^2 \leqslant 16C_2\kappa\rho\frac{n^2\log^2 n}{q}. \tag{4.19}$$

If we choose $\rho < 1/(4C_2)$, then we have

$$p_0 = \frac{4\kappa n^2\log^2 n}{t_0^2} \geqslant \frac{4\kappa n^2 q\log^2 n}{16C_2\kappa\rho n^2\log^2 n} = \frac{q}{4C_2\rho} \geqslant q. \tag{4.20}$$

We therefore see that, using Lemma 2.1 with $q'=1$, we can write
(4.21)

$$\int_{K_1}\max_{1\leqslant i\leqslant N_{t_0}}|\langle x,z_i\rangle|dx\leqslant\overline{\beta}_1\max_{1\leqslant i\leqslant N_{t_0}}h_{Z_{p_0}(K_1)}(z_i)\leqslant\overline{\beta}_1\beta_2\frac{p_0}{q}\max_{1\leqslant i\leqslant N_{t_0}}h_{Z_q(K_1)}(z_i).$$

Combining the above with (4.14), (4.10) and the definition of C_2, we get

$$nL_K^2\leqslant C_2\frac{p_0}{q}I_1(K,Z_q^\circ(K))+8t_0\sqrt{n}L_K. \tag{4.22}$$

Also, from (4.17) and the definition of p_0, we have

$$C_2\frac{p_0}{q}I_1(K,Z_q^\circ(K))=\frac{4C_2\kappa I_1(K,Z_q^\circ(K))}{qt_0^2}\,n^2\log^2 n\leqslant\frac{1}{4}\,nL_K^2. \tag{4.23}$$

Therefore, (4.22) becomes

$$nL_K^2\leqslant C_3t_0\sqrt{n}L_K. \tag{4.24}$$

This gives us that

$$L_K^2\leqslant C_4\frac{t_0^2}{n}=C\kappa\max\left\{1,\frac{I_1(K,Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}\sqrt{\frac{n}{q}}\log^2 n, \tag{4.25}$$

as we desired. □

5 Regular convex bodies with maximal isotropic constant

Recall that $L_n:=\max\{L_K : K \text{ is isotropic in}\,\mathbb{R}^n\}$. In order to be able to use the argument of the previous section to bound L_n, we need to establish the existence of (κ,τ)-regular convex bodies, namely bodies satisfying (4.1), whose isotropic constant is as "close" to L_n as possible. The following theorem, formulated in the more general setting of log-concave measures, was proven in [5].

Theorem 5.1. *There exist absolute constants κ,τ and $\delta>0$ such that, for every $n\in\mathbb{N}$, there exists an isotropic convex body K in $\mathbb{R}^n$ with the following properties:*

(i) $L_K\geqslant\delta L_n$.

(ii) $\log N(K,tB_2^n)\leqslant\frac{\kappa n^2\log^2 n}{t^2}$, *for all* $t\geqslant\tau\sqrt{n\log n}$.

For the reader's convenience, we will give an outline of the proof in the setting of convex bodies. First, we recall the following theorem by Pisier which will be used in several steps of the argument (see [19] for a proof in the symmetric case; this can easily be extended to the general case):

Theorem 5.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $\alpha \in (0,2)$ there exists an ellipsoid $\mathcal{E}_\alpha$ with $|\mathcal{E}_\alpha| = 1$ such that, for every $t \geqslant 1$,*

$$\log N(K, t\mathcal{E}_\alpha) \leqslant \frac{\kappa(\alpha)}{t^\alpha} n, \tag{5.1}$$

where $\kappa(\alpha) > 0$ is a constant depending only on α.

Remark 5.3. One can take $\kappa(\alpha) \leqslant \frac{\kappa_1}{2-\alpha}$, where $\kappa_1 > 0$ is an absolute constant. An ellipsoid $\mathcal{E}_\alpha$ which satisfies (5.1) is called an α-regular M-ellipsoid for K.

Secondly, let us gather some useful facts about ellipsoids in $\mathbb{R}^n$ that we are going to need for the proof of Theorem 5.1 (proofs for these facts can be found in [5], [10] and [22]).

Lemma 5.4. *Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^n$, then $\mathcal{E} = T(B_2^n)$ for some $T \in GL(n)$. We denote the eigenvalues of the matrix $\sqrt{T^*T}$ by $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$ (recall that T^*T is a symmetric, positive definite matrix). Then, for all $1 \leqslant k \leqslant n-1$,*

$$\max_{F \in G_{n,k}} |\mathcal{E} \cap F| = \max_{F \in G_{n,k}} |P_F(\mathcal{E})| = \omega_k \prod_{i=1}^{k} \lambda_i \tag{5.2}$$

and

$$\min_{F \in G_{n,k}} |\mathcal{E} \cap F| = \min_{F \in G_{n,k}} |P_F(\mathcal{E})| = \omega_k \prod_{i=n-k+1}^{n} \lambda_i. \tag{5.3}$$

Also, if the dimension n is even, we can find a subspace $F \in G_{n,n/2}$ such that $P_F(\mathcal{E}) = \lambda_{n/2} B_F$ $(= \lambda_{n/2} B_2^n \cap F)$.

In view of the last part of Lemma 5.4, we choose to restrict ourselves to the cases that the dimension n is even, $n = 2m$ for some $m \geqslant 1$, and prove Theorem 5.1 for those. However, as we will see in Remark 5.7, it is not hard to then extend the theorem to all dimensions.

Proof of Theorem 5.1. We start with an isotropic convex body K_1 with $L_{K_1} \geqslant \delta_1 L_{2m}$, where $\delta_1 \in (0,1)$. Then, one has the following upper bound for the volume of sections of K_1.

Lemma 5.5. *For every k-codimensional subspace E of $\mathbb{R}^{2m}$, $|K_1 \cap E|^{1/k} \leqslant c_1(\delta_1)$, where $c_1(\delta_1) > 0$ depends only on δ_1.*

Proof. Let E be a k-codimensional subspace of $\mathbb{R}^{2m}$, and denote its orthogonal subspace by F. We consider the body $B_{k+1}(K_1, F)$, a convex body in the subspace F defined as in Subsection 2.2, and we recall that

$$c_1 \frac{L_{B_{k+1}(K_1,F)}}{L_{K_1}} \leqslant |K_1 \cap E|^{1/k} \leqslant c_2 \frac{L_{B_{k+1}(K_1,F)}}{L_{K_1}} \tag{5.4}$$

for some absolute constants c_1, c_2 independent of m or k. On the other hand, it is not hard to check that if $k \leqslant j$ then $L_k \leqslant c_3 L_j$ (see e.g. [6, Theorem 4.2.2]). Thus,

$$L_{B_{k+1}(K_1,F)} \leqslant L_k \leqslant c_3 L_{2m} = (c_3/\delta_1) L_{K_1}, \tag{5.5}$$

and the lemma follows with $c_1(\delta_1) = c_2 c_3/\delta_1$. □

We will now invoke Pisier's theorem to also give a lower bound for the volume of m-dimensional sections of K_1 that contain its barycenter.

Lemma 5.6. *For every $F \in G_{2m,m}$ we have $|K_1 \cap F|^{1/m} \geqslant c_2(\delta_1)$, where $c_2(\delta_1) > 0$ depends only on δ_1.*

Proof. We consider an α-regular M–ellipsoid $\mathcal{E}_\alpha$ for K_1 (for the proof of this lemma we could have fixed $\alpha = 1$; however, some steps of this more general argument will be needed again later). Set $t_\alpha = \max\{1, [\kappa(\alpha)]^{1/\alpha}\}$. Then,

$$|P_F(K_1)| \leqslant N(K_1, t_\alpha \mathcal{E}_\alpha)|P_F(t_\alpha \mathcal{E}_\alpha)| \leqslant e^{2m}|P_F(t_\alpha \mathcal{E}_\alpha)| \tag{5.6}$$

for every $F \in G_{2m,m}$. We also need the Rogers-Shephard inequality (see [20]) for both K_1 and $\mathcal{E}_\alpha$: since $|K_1| = |\mathcal{E}_\alpha| = 1$, we know that

$$1 = c_1 \leqslant |K_1 \cap F|^{1/m} |P_{F^\perp}(K_1)|^{1/m} \leqslant c_2, \tag{5.7}$$

and similar estimates hold true for $\mathcal{E}_\alpha$ (see [21] or [14] for the left hand side inequality). The idea of the argument is the following: inequality (5.7) helps us relate the volume of m-dimensional sections of K_1 (or $\mathcal{E}_\alpha$) to that of m-dimensional projections of K_1 (or $\mathcal{E}_\alpha$ respectively); an upper bound for the former will give us a lower bound for the latter and vice versa. Also, inequality (5.6) allows us to compare the maximum (or minimum) volume of the m-dimensional projections of K_1 to the maximum (or minimum) volume of the corresponding projections of $\mathcal{E}_\alpha$. However, as we recalled in Lemma 5.4, the maximum volume of the m-dimensional projections of an ellipsoid is the same as the maximum volume of its m-dimensional sections, so we can use inequalities (5.6) and (5.7) once more to get from upper bounds for the volume of sections of K_1 to lower bounds.

We now give the precise argument: combining (5.7) with the conclusion of Lemma 5.5, we see that $\min_{F \in G_{2m,m}} |P_{F^\perp}(K_1)|^{1/m} \geqslant c_3(\delta_1)$. We then get from (5.6) that $\min_{F \in G_{2m,m}} |P_{F^\perp}(t_\alpha \mathcal{E}_\alpha)|^{1/m} \geqslant c_4(\delta_1)$. Now, using (5.7) for $\mathcal{E}_\alpha$ we get $|\mathcal{E}_\alpha \cap F|^{1/m} \leqslant c_5(\delta_1) t_\alpha$ for every $F \in G_{2m,m}$. But from (5.2) we have that

$$\max_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)|^{1/m} = \max_{F \in G_{2m,m}} |\mathcal{E}_\alpha \cap F|^{1/m} \leqslant c_5(\delta_1) t_\alpha. \tag{5.8}$$

Using (5.6) once again, we get $|P_F(K_1)|^{1/m} \leqslant c_6(\delta_1) t_\alpha^2$ for every $F \in G_{2m,m}$. Inserting this estimate into (5.7), we see that $|K_1 \cap F|^{1/m} \geqslant c_7(\delta_1)/t_\alpha^2$ for every $F \in G_{2m,m}$. We may choose $\alpha = 1$ now, and complete the proof with $c_2(\delta_1) = c_7(\delta_1)/t_1^2$. □

Conclusion of the proof of Theorem 5.1. Let $\alpha \in (1,2)$ and let $\mathcal{E}_\alpha$ be an α-regular M–ellipsoid for K. Recall that $|\mathcal{E}_\alpha| = 1$. Also, if $\mathcal{E}_\alpha = T(B_2^n) = T(B_2^{2m})$, let $\lambda_1 \geqslant \cdots \geqslant \lambda_{2m} > 0$ be the eigenvalues of the matrix $\sqrt{T^*T}$; observe from Lemma 5.4 that

$$|B_2^m| \prod_{i=m+1}^{2m} \lambda_i = \min_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)| \leqslant \max_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)| = |B_2^m| \prod_{i=1}^{m} \lambda_i. \tag{5.9}$$

Using (5.6) and the conclusion of Lemma 5.6, we get

$$\begin{aligned} |B_2^m|^{1/m} \lambda_m &\geqslant \min_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)|^{1/m} \geqslant \frac{e^{-2}}{t_\alpha} \min_{F \in G_{2m,m}} |P_F(K_1)|^{1/m} \\ &\geqslant \frac{e^{-2}}{t_\alpha} \min_{F \in G_{2m,m}} |K_1 \cap F|^{1/m} \geqslant \frac{c_8(\delta_1)}{t_\alpha}, \end{aligned} \tag{5.10}$$

and hence

$$\lambda_m \geqslant \frac{c_9(\delta_1)}{t_\alpha} \sqrt{n}. \tag{5.11}$$

In a similar way, using (5.8), we see that $|B_2^m|^{1/m} \lambda_m \leqslant \max_{F \in G_{2m,m}} |P_F(\mathcal{E}_\alpha)|^{1/m} \leqslant c_5(\delta_1) t_\alpha$, and hence $\lambda_m \leqslant c_{10}(\delta_1) t_\alpha \sqrt{n}$. But from the last part of Lemma 5.4 we know that there exists a subspace $F_0 \in G_{2m,m}$ such that $P_{F_0}(\mathcal{E}_\alpha) = \lambda_m B_{F_0}$, therefore,

$$\frac{c_9(\delta_1)}{t_\alpha} \sqrt{n} B_{F_0} \subseteq P_{F_0}(\mathcal{E}_\alpha) \subseteq c_{10}(\delta_1) t_\alpha \sqrt{n} B_{F_0}. \tag{5.12}$$

Let $W := \overline{B}_{m+1}(K_1, F_0)$ and $K := W \times U(W)$, where $U \in O(2m)$ satisfies $U(F_0) = F_0^\perp$. Since W is almost isotropic and $L_{U(W)} = L_W$, from [6, Lemma 1.6.6] we see that $K = W \times U(W)$ is an almost isotropic convex body in $\mathbb{R}^n \equiv \mathbb{R}^{2m}$ with $L_K = L_W$. We will show that K satisfies (i) and (ii); the same conclusion will then immediately follow (perhaps with slightly different constants for property (ii)) for any isotropic linear image $T(K)$ of K satisfying $T(K) \simeq K$.

Proof of (i): Since $L_K = L_W$, from (5.4) we get

$$L_K = L_W \geqslant c_2^{-1} L_{K_1} |K_1 \cap F_0^\perp|^{1/m} \geqslant c_2^{-1} c_2(\delta_1) L_{K_1} \geqslant \delta L_n, \tag{5.13}$$

where $\delta = \delta_1 c_2(\delta_1)/c_2$. For the last two inequalities we have used Lemma 5.6 and the fact that $L_{K_1} \geqslant \delta_1 L_n$.

Proof of (ii): Using the fact that $N(A \times A, B \times B) \leqslant N(A,B)^2$ for any two nonempty sets A, B, and also the fact that $B_2^m \times B_2^m \subseteq \sqrt{2} B_2^{2m}$, we may write for any $s > 0$,

$$N\big(K, s\sqrt{2n} B_2^n\big) \leqslant N\big(W \times U(W), s\sqrt{n}(B_{F_0} \times B_{F_0^\perp})\big) \leqslant N\big(W, s\sqrt{n} B_{F_0}\big)^2. \tag{5.14}$$

From (2.15) we know that

$$Z_m(\overline{B}_{m+1}(K_1,F_0)) \simeq |K_1\cap F_0^{\perp}|^{1/m} P_{F_0}(Z_m(K_1)), \tag{5.15}$$

therefore, using Lemmas 5.5, 5.6 and the fact that $\mathrm{conv}(C,-C)\simeq Z_m(C)$ for every centered convex body C of volume 1 in F_0 or in $\mathbb{R}^n$, we get

$$\begin{aligned}\mathrm{conv}(W,-W) &\simeq Z_m(\overline{B}_{m+1}(K_1,F_0)) \simeq |K_1\cap F_0^{\perp}|^{1/m} P_{F_0}(Z_m(K_1))\\ &\simeq_{\delta_1} P_{F_0}(\mathrm{conv}(K_1,-K_1)).\end{aligned} \tag{5.16}$$

But then, recalling also (5.12), we have for every $r>0$,

$$\begin{aligned}N(W,c_{10}(\delta_1)t_\alpha r\sqrt{n}B_{F_0}) &\leqslant N\big(\mathrm{conv}(W,-W),c_{10}(\delta_1)t_\alpha r\sqrt{n}B_{F_0}\big)\\ &\leqslant N\big(\mathrm{conv}(W,-W),rP_{F_0}(\mathcal{E}_\alpha)\big)\\ &\leqslant N\big(c_{11}(\delta_1)P_{F_0}(\mathrm{conv}(K_1,-K_1)),rP_{F_0}(\mathcal{E}_\alpha)\big)\\ &\leqslant N\big(c_{11}(\delta_1)\mathrm{conv}(K_1,-K_1),r\mathcal{E}_\alpha\big)\\ &\leqslant N\big(K_1-K_1,c_{12}(\delta_1)r(\mathcal{E}_\alpha-\mathcal{E}_\alpha)\big)\\ &\leqslant N(K_1,c_{13}(\delta_1)r\mathcal{E}_\alpha)^2\end{aligned} \tag{5.17}$$

(note that for the last two inequalities we have also used that $\mathcal{E}_\alpha$ is convex and symmetric, so $\mathcal{E}_\alpha-\mathcal{E}_\alpha=2\mathcal{E}_\alpha$, that K_1 is convex and contains the origin, so $\mathrm{conv}(K_1,-K_1)\subset K_1-K_1$, as well as the fact that $N(A-A,B-B)\leqslant N(A,B)^2$). It follows that

$$N(K,t\sqrt{n}\,B_2^n)\leqslant N\left(K_1,\frac{c_{13}(\delta_1)t}{\sqrt{2}c_{10}(\delta_1)t_\alpha}\,\mathcal{E}_\alpha\right)^4 \tag{5.18}$$

for every $t>0$. Since $\mathcal{E}_\alpha$ is an α-regular M–ellipsoid for K_1, it remains to consider large enough $t\geqslant\tau(\delta_1,\alpha)$, where

$$\tau(\delta_1,\alpha):=\sqrt{2}c_{10}(\delta_1)t_\alpha/c_{13}(\delta_1)=t_\alpha/c_{14}(\delta_1), \tag{5.19}$$

to deduce from (5.1) and (5.18) that

$$\log N(K,t\sqrt{n}\,B_2^n)\leqslant 4\log N\left(K_1,\frac{c_{14}(\delta_1)t}{t_\alpha}\,\mathcal{E}_\alpha\right)\leqslant\frac{4\kappa(\alpha)t_\alpha^\alpha}{c_{14}^\alpha(\delta_1)}\frac{n}{t^\alpha}. \tag{5.20}$$

Choosing $\alpha=2-\frac{1}{\log n}$, we have $\kappa(\alpha)\leqslant\kappa_1\log n$ and $t^\alpha\simeq t^2$ as long as, say, $t\leqslant n^2$. This completes the proof. $\square$

Remark 5.7. Now that we have proven the existence of an isotropic body K in $\mathbb{R}^{2m}$ which has properties (i) and (ii) of Theorem 5.1, we can easily prove the existence of such bodies in $\mathbb{R}^{2m-1}$ as well: just note that for every subspace $F\in G_{2m,2m-1}$ we have that $2L_K\leqslant|K\cap F^{\perp}|\leqslant 2R(K)$. Combining this with the properties

(2.14), (2.15) for the almost isotropic convex body $\overline{B}_{2m}(K,F)$ in the $(2m-1)$–dimensional subspace F, we get that

$$L_{\overline{B}_{2m}(K,F)} \simeq |K\cap F^{\perp}|^{\frac{1}{2m-1}} L_K \simeq L_K \geqslant \delta L_{2m} \geqslant \frac{\delta}{c_3} L_{2m-1}, \tag{5.21}$$

and also that

$$\overline{B}_{2m}(K,F) \simeq Z_{2m-1}\big(\overline{B}_{2m}(K,F)\big) \simeq |K\cap F^{\perp}|^{\frac{1}{2m-1}} P_F(Z_{2m-1}(K)) \simeq P_F(K). \tag{5.22}$$

Since for every $t>0$, $N(P_F(K),tB_F) = N\big(P_F(K),tP_F(B_2^{2m})\big) \leqslant N(K,tB_2^{2m})$, we conclude that the body $\overline{B}_{2m}(K,F)$ will also satisfy properties (i) and (ii) of Theorem 5.1 with perhaps slightly different, but still independent of the dimension, constants κ,τ and δ.

In the statement of Theorem 5.1, we can add one more property about the radius of the body K that we look for: we can require that $R(K)\leqslant \gamma\sqrt{n}L_K$ where $\gamma>0$ is an absolute constant. The first step towards this is to use Bourgain's argument [4] which reduces the slicing problem to the study of bodies of small diameter; one can prove the following fact (see e.g. [6, Proposition 2.3.1]).

Lemma 5.8. *There exists an isotropic convex body K_1 in $\mathbb{R}^n$ with $L_{K_1}\geqslant \delta_1 L_n$ and $R(K_1)\leqslant \gamma_1\sqrt{n}L_{K_1}$, where $\delta_1,\gamma_1>0$ are absolute constants.*

Then, we can repeat the proof of Theorem 5.1 starting with the body $K_1\subset\mathbb{R}^n=\mathbb{R}^{2m}$ given by Lemma 5.8. One has now that $R(W)\leqslant c(\delta_1)\gamma_1\sqrt{n}L_{K_1}$: to see this, write

$$\begin{aligned} R(W) = R\big(\overline{B}_{m+1}(K_1,F_0)\big) &\leqslant c_1|K_1\cap F_0^{\perp}|^{1/m} R\big(P_{F_0}\big(Z_m(K_1)\big)\big)\\ &\leqslant c_2(\delta_1)R\big(\mathrm{conv}(K_1,-K_1)\big) = c_2(\delta_1)R(K_1)\leqslant c_2(\delta_1)\gamma_1\sqrt{n}L_{K_1}. \end{aligned} \tag{5.23}$$

It is also easy to check that $R(K)=R(W\times U(W))\simeq R(W)$, hence $R(K)\leqslant\gamma\sqrt{n}L_K$ for some absolute constant $\gamma>0$. Similarly for the odd dimensions, we see that for every $F\in G_{2m,2m-1}$,

$$R(\overline{B}_{2m}(K,F)) \simeq R(P_F(K)) \leqslant R(K) \leqslant c\gamma\sqrt{2m-1}\,L_{\overline{B}_{2m}(K,F)}, \tag{5.24}$$

where we have made use of (5.21), (5.22). Thus, we can state the following version of Theorem 5.1.

Theorem 5.9. *There exist absolute constants κ,τ,γ and $\delta>0$ such that, for every $n\in\mathbb{N}$, there can be found an isotropic convex body K in $\mathbb{R}^n$ with $R(K)\leqslant\gamma\sqrt{n}L_K$, $L_K\geqslant\delta L_n$, and the property that*

$$\log N(K,tB_2^n) \leqslant \frac{\kappa n^2\log^2 n}{t^2} \quad \textit{for all } t\geqslant \tau\sqrt{n\log n}.$$

Definition 5.10. Let $\mathcal{IK}(\kappa,\tau,\gamma,\delta)$ denote the class of isotropic convex bodies whose existence is established in Theorem 5.9. Let $\rho > 0$ be the absolute constant in Theorem 4.1. Then, we define $A(n,\kappa,\tau,\gamma,\delta)$ to be the set of all $q \in [2,\rho^2 n]$ for which there exists $K \in \mathcal{IK}(\kappa,\tau,\gamma,\delta)$ such that $I_1(K, Z_q^\circ(K)) \leqslant \rho n L_K^2$. Observe that already, by (3.2), $A(n,\kappa,\tau,\gamma,\delta)$ can be shown to contain an interval of the form $[2, c\sqrt{n}]$ where $c > 0$ is an absolute constant. Clearly, any improvement to the upper bound in (3.2) will automatically give us that $A(n,\kappa,\tau,\gamma,\delta)$ contains an even larger part of $[2,\rho^2 n]$. For those q we set

$$B(q) = \inf\left\{ \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn} L_K^2} : K \in \mathcal{IK}(\kappa,\tau,\gamma,\delta)\right\}. \tag{5.25}$$

Then, Theorem 4.1 implies the following: for every $q \in A(n,\kappa,\tau,\gamma,\delta)$,

$$\delta^2 L_n^2 \leqslant C\kappa\sqrt{n/q}\log^2 n \max\{1, B(q)\}. \tag{5.26}$$

In other words, we have:

Theorem 5.11. *There exist absolute constants κ, τ, γ and $\delta > 0$ such that, for every $n \in \mathbb{N}$,*

$$L_n^2 \leqslant \min\left\{ \frac{C\kappa}{\delta^2}\sqrt{n/q}\log^2 n \max\{1, B(q)\} : q \in A(n,\kappa,\tau,\gamma,\delta)\right\}. \tag{5.27}$$

The estimate $L_n \leqslant c\sqrt[4]{n}\log n$ is a direct consequence of Theorem 5.11: observe that $B(2) \simeq 1$.

6 Isotropic convex bodies of small diameter

In [7, Section 3] it is proven that for every isotropic convex body K there exists a second isotropic convex body C with bounded isotropic constant and the "same behaviour" as K with respect to linear functionals.

Theorem 6.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. There exists an isotropic convex body C in $\mathbb{R}^n$ with the following properties:*

(i) $L_C \leqslant c_1$.

(ii) $c_2 Z_q(C) \subseteq \frac{Z_q(K)}{L_K} + \sqrt{q} B_2^n \subseteq c_3 Z_q(C)$ *for all* $1 \leqslant q \leqslant n$.

(iii) $c_4 I_q(C, W) \leqslant \frac{I_q(K,W)}{L_K} + I_q(D_n, W) \leqslant c_5 I_q(C, W)$ *for all* $1 \leqslant q \leqslant n$ *and every symmetric convex body* W *in* $\mathbb{R}^n$.

The constants c_i, $i = 1, \ldots, 5$ are absolute positive constants.

The body C is defined as the "convolution" of K with a multiple of B_2^n. If we also assume that K is symmetric, then using the fact that $Z_n(C) \simeq C$ and $Z_n(K) \simeq K$, we see that

$$C \simeq \frac{K}{L_K} + D_n. \tag{6.1}$$

From the previous section, we know that for our purposes it is enough to study the quantity $I_1(K, Z_q^\circ(K))$ in the cases that K is an isotropic symmetric convex body of small diameter; that is, we can assume that $R(K) \leqslant \gamma\sqrt{n}L_K$ for some $\gamma \simeq 1$. The next proposition, which makes use of Theorem 6.1, shows us that it even suffices to consider isotropic convex bodies which are $c(\gamma)$-isomorphic to a ball.

Proposition 6.2. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$ with $R(K) \leqslant \gamma\sqrt{n}L_K$. Then, there exists an isotropic symmetric convex body C such that:*

(i) $L_C \leqslant c_6$,

(ii) $c_7 D_n \subseteq C \subseteq c_8\gamma D_n$, *and*

(iii) $I_1(K, Z_q^\circ(K)) \leqslant c_9 I_1(C, Z_q^\circ(C))L_K^2$ *for all* $1 \leqslant q \leqslant n$,

where $c_6, c_7, c_8, c_9 > 0$ *are absolute constants.*

Proof. We will use the fact that $w_q(Z_q(K)) \simeq \sqrt{q/n}I_q(K)$, and hence

$$c_1\sqrt{q}L_K \leqslant w_q(Z_q(K)) \leqslant \gamma\sqrt{q}L_K \tag{6.2}$$

for all $1 \leqslant q \leqslant n$. We consider the body C defined by Theorem 6.1. It is clear that $L_C \leqslant c_6$ for some absolute constant $c_6 > 0$. Since $\frac{1}{L_K}Z_q(K) \subseteq c_3 Z_q(C)$, we have $c_3 L_K Z_q^\circ(K) \supseteq Z_q^\circ(C)$, and hence

$$I_1(C, Z_q^\circ(C)) \geqslant I_1(C, c_3 L_K Z_q^\circ(K)) = \frac{1}{c_3 L_K} I_1(C, Z_q^\circ(K)). \tag{6.3}$$

Applying the inequality $c_5 I_1(C, W) \geqslant \frac{I_1(K,W)}{L_K}$ with $W = Z_q^\circ(K)$, we get

$$I_1(C, Z_q^\circ(C)) \geqslant c_9 \frac{I_1(K, Z_q^\circ(K))}{L_K^2}, \tag{6.4}$$

with $c_9 = (c_3c_5)^{-1}$ Finally, from (6.1) and the fact that $\frac{K}{L_K} \subseteq \gamma\sqrt{n}B_2^n$, we see that $c_7 D_n \subseteq C \subseteq c_8\gamma D_n$. □

In view of this result, we can give one more version of the "reduction theorem" of Section 4.

Definition 6.3. Let $\mathcal{IK}_{\rm sd}(\gamma)$ denote the class of isotropic convex bodies that satisfy

(i) $L_C \leqslant c_6$ and

(ii) $c_7 D_n \subseteq C \subseteq c_8\gamma D_n$,

where $c_i > 0$ are absolute constants (e.g. the ones in Proposition 6.2). For every $2 \leqslant q \leqslant n$, set

$$\Gamma(q) = \sup\left\{ \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}} : K \in \mathcal{IK}_{\rm sd}(\gamma) \right\}.$$

Then, Theorem 5.11 and Proposition 6.2 imply the following:

Theorem 6.4. *There exist absolute constants κ, τ, γ and $\delta > 0$ such that, for every $n \in \mathbb{N}$,*

$$L_n^2 \leqslant \min\left\{\frac{C\kappa}{\delta^2}\sqrt{n/q}\log^2 n\Gamma(q) : q \in A(n,\kappa,\tau,\gamma,\delta)\right\}. \tag{6.5}$$

In other words, studying the behaviour of $\frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}}$ within the class $\mathcal{IK}_{\rm sd}(\gamma)$ is enough in order to understand the behaviour of the parameter $B(q)$ as well as whether that behaviour can lead to improved upper bounds for L_n.

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References

[1] K. M. Ball, *Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$*, Studia Math. **88** (1988), 69–84.

[2] S. G. Bobkov and F. L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 53–69.

[3] S. G. Bobkov and F. L. Nazarov, *Large deviations of typical linear functionals on a convex body with unconditional basis*, Stochastic Inequalities and Applications, Progr. Probab. 56, Birkhauser, Basel, 2003, 3–13.

[4] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1469** (1991), 127–137.

[5] N. Dafnis, G. Paouris, *Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture*, Journal of Functional Analysis **258** (2010), 1933–1964.

[6] A. Giannopoulos, *Notes on isotropic convex bodies*, Warsaw University Notes (2003).

[7] A. Giannopoulos, G. Paouris and P. Valettas, *On the existence of subgaussian directions for log-concave measures*, Contemporary Mathematics **545** (2011), 103–122.

[8] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274–1290.

[9] B. Klartag and E. Milman, *Centroid Bodies and the Logarithmic Laplace Transform - A Unified Approach*, arXiv:1103.2985v1.

[10] B. Klartag and V. D. Milman, *Rapid Steiner Symmetrization of most of a convex body and the slicing problem*, Combin. Probab. Comput. **14**, no. 5-6 (2005) 829–843.

[11] A. Litvak, V. D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), 95–124.

[12] E. Lutwak, D. Yang and G. Zhang, *L^p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

[13] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1376** (1989), 64–104.

[14] V.D. Milman, A. Pajor, *Entropy and Asymptotic Geometry of Non-Symmetric Convex Bodies*, Advances in Mathematics, **152** (2000), 314–335.

[15] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. **1200** (1986), Springer, Berlin.

[16] G. Paouris, *ψ_2-estimates for linear functionals on zonoids*, Lecture Notes in Mathematics **1807** (2003), 211–222.

[17] G. Paouris, *Concentration of mass in convex bodies*, Geometric and Functional Analysis **16** (2006), 1021–1049.

[18] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. (2011), DOI:10.1090/S0002-9947-2011-05411-5.

[19] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics **94** (1989).

[20] C. A. Rogers and G. C. Shephard, *Convex bodies associated with a given convex body*, J. London Soc. **33** (1958), 270–281.

[21] J. Spingarn, *An inequality for sections and projections of a convex set*, Proc. Amer. Math. Soc. **118** (1993), 1219–1224.

[22] C. Zong, *Strange phenomena in convex and discrete geometry*, Universitext, Springer (2003).

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