Euclidean regularization in John's position

G. Chasapis and A. Giannopoulos

Abstract

Starting from a result of D. Fresen we propose a Euclidean regularization method which provides short proofs of several facts on the local structure of a symmetric convex body in John's position. In particular, we obtain a simple proof of the isomorphic version of Dvoretzky's theorem as well as a new isomorphic version of the global form of Dvoretzky's theorem.

1 Introduction

The purpose of this note is to propose a method based on "Euclidean regularization" which provides short proofs of several facts on the local structure of finite dimensional normed spaces, including the isomorphic Dvoretzky theorem of V. Milman and Schecthman and an isomorphic version of the global version of Dvoretzky's theorem due to Bourgain, Lindenstrauss and V. Milman.

Let $X=(\mathbb{R}^n,\|\cdot\|)$ be an n-dimensional normed space. We write K for the unit ball $\{x\in\mathbb{R}^n:\|x\|\leqslant 1\}$ of X, and define

$$M:=\int_{S^{n-1}}\|x\|\,d\sigma_n(x)\ \text{ and }\ b:=\max\{\|x\|:x\in S^{n-1}\},\tag{1.1}$$

where S^{n-1} is the Euclidean unit sphere and σ denotes the rotationally invariant probability measure on S^{n-1}. The parameters M and b play a crucial role in the local theory of finite dimensional normed spaces. This was first demonstrated by V. Milman [17] in his sharp version of the classical Dvoretzky theorem [5] on approximately Euclidean sections of high-dimensional symmetric convex bodies: for any $\varepsilon\in(0,1)$ and any

$$k\leqslant k_X(\varepsilon)=c_1\varepsilon^2[\log^{-1}(2/\varepsilon)]n(M/b)^2,$$

one can find a subspace F of $\mathbb{R}^n$ with dimension $\dim(F)=k$ such that

$$(1+\varepsilon)^{-1}M\|x\|_2\leqslant\|x\|\leqslant M(1+\varepsilon)\|x\|_2\tag{1.2}$$

is satisfied for every $x\in F$, where $\|\cdot\|_2$ is the Euclidean norm. Moreover, this holds for all F in a subset $A_{n,k}\subset G_{n,k}$ of measure $\nu_{n,k}(A_{n,k})\geqslant 1-\exp(-c_2\varepsilon^2k)$, where $G_{n,k}$ is the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,k}$. Choosing $\varepsilon=1/2$ we get:

Theorem 1.1 (V. Milman)**.** *Let $X=(\mathbb{R}^n,\|\cdot\|)$ be an n-dimensional normed space. If $k\leqslant c_0n(M/b)^2$, where $c_0>0$ is an absolute constant, then we can find a subset $A_{n,k}\subset G_{n,k}$ of measure $\nu_{n,k}(A_{n,k})\geqslant 1-\exp(-\bar{c}_0k)$ such that for any subspace $F\in A_{n,k}$ and for any $x\in F$ one has*

$$\frac{2}{3}M\|x\|_2\leqslant\|x\|\leqslant\frac{3}{2}M\|x\|_2.\tag{1.3}$$

Theorem 1.1 states that the Banach-Mazur distance (in fact, the geometric distance) $d_{\rm BM}(K\cap F,B_2^n\cap F)$ between $K\cap F$ and $B_2^n\cap F$ is bounded by 3 for a random $F\in G_{n,k}$ provided that $k\leqslant k(X):=c_0n(M/b)^2$. The parameter $k(X)$, which is completely determined by M and b, will be called the "Dvoretzky dimension" of X (see [8], [19] and [1] for a variety of results in which it appears naturally and plays an essential role). It is natural to ask what can be said when k is larger than $k(X)$ i.e. if $k(X)\leqslant k\leqslant n$. The "isomorphic Dvoretzky theorem" of V. Milman and Schechtman provides an exact answer:

Theorem 1.2 (V. Milman-Schechtman). *There exist absolute constants $C_1, C_2 > 0$ with the following property: for every $n \geqslant 1$, for every $C_1 \log n \leqslant k \leqslant n$ and for every n-dimensional normed space X, there exists a k-dimensional subspace Y of X such that*

$$d_{\rm BM}(Y, \ell_2^k) \leqslant C_2 \frac{\sqrt{k}}{\sqrt{\log\left(1+\frac{n}{k}\right)}}. \tag{1.4}$$

A slightly weaker version of Theorem 1.2 was first obtained by V. Milman and Schechtman in [18]. The precise statement above was established by the same authors in [20] and by Guédon in [12] with a different approach. Extensions of the result to the not necessarily symmetric case were given by Gordon, Guédon and Meyer in [11] and by Litvak and Tomczak-Jaegermann in [14]. Note that the estimate is sharp: if $X = \ell_\infty^n$ then, for any $k \geqslant \log n$ and any k-dimensional subspace Y of X one has $d_{\rm BM}(Y, \ell_2^k) \leqslant \delta \frac{\sqrt{k}}{\sqrt{\log\left(1+\frac{n}{k}\right)}}$, where $\delta > 0$ is an absolute constant. This was first observed by Figiel and Johnson [7], where a slightly weaker result is obtained, and the full statement was proved, independently, by Carl and Pajor in [4] and by Gluskin in [10].

We provide a short proof of Theorem 1.2 which is based on the following idea: given an n-dimensional normed space X and an integer $1 \leqslant k \leqslant n$ it is enough to find a space Y such that $k(Y) \geqslant k$ and $d_{\rm BM}(X, Y) \leqslant f(n/k)$ where $f(n/k)$ is as small as possible. To this end, we exploit a recent result of Fresen [9] on "Euclidean regularization" of a convex body in John's position (see Section 2 for background information).

Theorem 1.3 (Fresen). *Let K be a symmetric convex body in $\mathbb{R}^n$ such that B_2^n is the maximal volume ellipsoid of K. For every $t \in [\alpha, \beta\sqrt{n}]$, where $\alpha, \beta > 0$ are absolute constants, there exists a symmetric convex body K_t such that $d(K, K_t) \leqslant t$ and*

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant c_1 \frac{t^2}{n} \log\left(\frac{c_2 n}{t^2}\right), \tag{1.5}$$

where $c_1, c_2 > 0$ are absolute constants.

We believe that Theorem 1.3 can be useful in many situations and the main purpose of this note is to illustrate its use. As a second application, we revisit the global form of Dvoretzky's theorem. The next theorem of Bourgain, Lindenstrauss and Milman [3] (see also [22] for the dependence on ε) shows that the average of (roughly) $(b/M)^2$ rotations of the polar body of a symmetric convex body K is an isomorphic Euclidean ball.

Theorem 1.4 (Bourgain-Lindenstrauss-Milman). *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space. For any $\varepsilon \in (0, 1/2)$ and for any integer*

$$k \geqslant \frac{c_3}{\varepsilon^2}\left(\frac{b}{M}\right)^2,$$

a random choice of k orthogonal transformations $U_1, \dots, U_k \in O(n)$ satisfies

$$\frac{M}{1+\varepsilon}\|x\|_2 \leqslant \frac{1}{k}\sum_{i=1}^k \|U_i(x)\| \leqslant M(1+\varepsilon)\|x\|_2, \tag{1.6}$$

for every $x \in \mathbb{R}^n$, and hence,

$$d_{\rm G}\left(\frac{1}{k}\sum_{i=1}^k U_i^*(K^\circ), B_2^n\right) \leqslant (1+\varepsilon)^2, \tag{1.7}$$

with probability greater than $1 - \exp(-c\varepsilon^2 nk(M/b)^2)$, where $c > 0$ is an absolute constant.

We present an isomorphic version of this result. In the spirit of Theorem 1.2 we fix $k \geqslant 2$, we consider $U_1, \ldots, U_k \in O(n)$ and ask for the typical Banach-Mazur distance between $\frac{1}{k}\sum_{i=1}^k U_i^*(K^\circ)$ and B_2^n. Using Theorem 1.3 we get:

Theorem 1.5. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that the Euclidean unit ball B_2^n is the maximal volume ellipsoid of K. For every $k \leqslant \delta n/\log(n+1)$, where $\delta \in (0,1)$ is an absolute constant, a random k-tuple of orthogonal transformations $U_1, \ldots, U_k \in O(n)$ satisfies with probability greater than $1-\exp(-c_4 n)$*

$$d_{\mathrm{G}}\left(\frac{1}{k}\sum_{i=1}^k U_i^*(K^\circ), B_2^n\right) \leqslant C_3\sqrt{\frac{n}{k\log k}}, \tag{1.8}$$

where $C_3, c_4 > 0$ are absolute constants.

The restriction $k \leqslant n/\log(n+1)$ in Theorem 1.5 is natural. Note that if K is in John's position then $\left(\frac{b}{M}\right)^2 \leqslant \frac{Cn}{\log(n+1)}$ (this is a consequence of the Dvoretzky-Rogers lemma, see Section 2). Therefore, if $k \geqslant n/\log(n+1)$ we have

$$d_{\mathrm{G}}\left(\frac{1}{k}\sum_{i=1}^k U_i^*(K^\circ), B_2^n\right) \leqslant C$$

for a random k-tuple $U_1, \ldots, U_k \in O(n)$.

Theorem 1.5 may be viewed as a global version of Theorem 1.2 and, to the best of our knowledge, it has not appeared before. We give the proofs of both theorems in Section 3. Regarding the sharpness of the estimate in (1.8) we show (see Remark 3.3) that if $K = [-1,1]^n$ is the unit cube then, *for any k-tuple* $U_1, \ldots, U_k \in O(n)$,

$$d_{\mathrm{G}}\left(\frac{1}{k}\sum_{i=1}^k U_i^*(K^\circ), B_2^n\right) \geqslant c\sqrt{\frac{n}{k^2\log k}}, \tag{1.9}$$

where $c > 0$ is an absolute constant.

In Section 4 we study the diameter of a random k-dimensional section of a symmetric convex body K in $\mathbb{R}^n$ which is in John's position. We provide a new proof of the following result of Litvak, Mankiewicz and Tomczak-Jaegermann from [15].

Theorem 1.6 (Litvak-Mankiewicz-Tomczak). *Let K be a symmetric convex body in $\mathbb{R}^n$ such that the Euclidean unit ball B_2^n is the maximal volume ellipsoid of K. For every $k(K) \leqslant k \leqslant n$ we have that a subspace $F \in G_{n,k}$ satisfies*

$$c_5\sqrt{\frac{n}{k}}\, B_2^n \cap F \subseteq K \cap F \subseteq c_6\sqrt{\frac{n}{\log\left(1+\frac{n}{k}\right)}}\, B_2^n \cap F \tag{1.10}$$

with probability greater than $1-\exp(-c_7 k)$, where $c_5, c_6, c_7 > 0$ are absolute constants.

Note that Theorem 1.6 is stronger than Theorem 1.2. When $k(K) \leqslant k \leqslant n$ it provides the same upper bound for the Banach-Mazur distance $d_{\mathrm{BM}}(K \cap F, B_2^n \cap F)$ between a random k-dimensional section of K and the Euclidean ball, while if $1 \leqslant k \leqslant k(K)$ we anyway have that a random k-dimensional section $K \cap F$ of K is an isomorphic Euclidean ball.

2 Notation and background material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ_n for the rotationally invariant probability measure on S^{n-1}.

We also denote the Haar measure on $O(n)$ by ν_n. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $1 \leqslant k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters $c, c', \bar{c}, c_1, c_2$ etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Similarly, if $K, T \subseteq \mathbb{R}^n$ we will write $K \simeq T$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq T \subseteq c_2 K$.

A convex body is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $-x \in K$ whenever $x \in K$. The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by $h_K(x) = \max\{\langle x, y\rangle : y \in K\}$. The mean width $w(K)$ of K is defined by

$$w(K) = \int_{S^{n-1}} h_K(x) d\sigma_n(x). \tag{2.1}$$

The radius of K is defined as $R(K) = \max\{\|x\|_2 : x \in K\}$ and, if the origin is an interior point of K, the polar body K° of K is

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\}. \tag{2.2}$$

Finally, if K is a symmetric convex body in $\mathbb{R}^n$ and $\|\cdot\|_K$ is the norm induced to $\mathbb{R}^n$ by K, we set

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_n(x) \tag{2.3}$$

and write $b(K)$ for the smallest positive constant b with the property $\|x\|_K \leqslant b\|x\|_2$ for all $x \in \mathbb{R}^n$.

We set $k(K) = k(X_K)$, where $X_K = (\mathbb{R}^n, \|\cdot\|_K)$, and $k^*(K) := k(K^\circ)$. Note that $M(K^\circ) = w(K)$ and $b(K^\circ) = R(K)$, and hence

$$k^*(K) = c_0 n \left(\frac{w(K)}{R(K)}\right)^2. \tag{2.4}$$

From Theorem 1.1 we know that if $k \leqslant k(K)$ then for most $F \in G_{n,k}$ we have $K \cap F \simeq \frac{1}{M(K)} B_F$. By duality, if $k \leqslant k^*(K)$ then $P_F(K) \simeq w(K)\, B_F$.

If K and T are two convex bodies in $\mathbb{R}^n$ that contain the origin in their interior, their geometric distance $d_{\mathrm{G}}(K, T)$ is defined by

$$d_{\mathrm{G}}(K, T) = \inf\{ab : a, b > 0, K \subseteq bT \text{ and } T \subseteq aK\}. \tag{2.5}$$

The natural distance between two n-dimensional normed spaces X_K and X_T is the Banach-Mazur distance

$$d_{\mathrm{BM}}(X_K, X_T) = \inf_{A \in GL(n)} \|A : X_K \to X_T\| \, \|A^{-1} : X_T \to X_K\|. \tag{2.6}$$

From the definition of the geometric distance we see that

$$d_{\mathrm{BM}}(X_K, X_T) = \inf\{d_{\mathrm{G}}(K, A(T)) : A \in GL(n)\}. \tag{2.7}$$

In other words, the Banach-Mazur distance $d_{\mathrm{BM}}(X_K, X_T)$ is the smallest positive real λ for which we may find $A \in GL(n)$ such that $K \subseteq A(T) \subseteq \lambda K$. It is clear that $d_{\mathrm{BM}}(X_K, X_T) \geqslant 1$ with equality if and only if X_K are X_T isometrically isomorphic. Note that $d_{\mathrm{BM}}(X, Z) \leqslant d_{\mathrm{BM}}(X, Y) d_{\mathrm{BM}}(Y, Z)$ for any triple of n-dimensional normed spaces.

A symmetric convex body K in $\mathbb{R}^n$ is said to be in John's position if the Euclidean unit ball B_2^n is contained in K and for every $T \in GL(n)$ with $T(B_2^n) \subseteq K$ we have $|T(B_2^n)| \leqslant |B_2^n|$; in other words, if B_2^n is the ellipsoid of maximal volume inscribed in K. One can check that this position is uniquely determined up to orthogonal transformations. It is easily checked that K is in John's position if and only if B_2^n is the ellipsoid of minimal volume containing K°.

A classical theorem of John [13] (see also [2] for the reverse implication) states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist contact points $x_1, \dots, x_m \in \mathbb{R}^n$ of K and B_2^n and positive real numbers $c_1, \dots, c_m$ such that

$$x = \sum_{j=1}^{m} c_j \langle x, x_j \rangle x_j \tag{2.8}$$

for all $x \in \mathbb{R}^n$. This implies that $K \subseteq \sqrt{n} B_2^n$, and hence, $d_{\mathrm{G}}(K, B_2^n) \leqslant \sqrt{n}$. Starting from John's decomposition (2.8), Dvoretzky and Rogers [6] obtained a series of results on the distribution of the x_j's in the unit sphere. One of the consequences of the "Dvoretzky-Rogers lemma" is that if K is in John's position then

$$k(K) = c_0 n \left(\frac{M}{b}\right)^2 \geqslant c \log(n+1). \tag{2.9}$$

This estimate plays a key role in the proof of Dvoretzky's theorem.

We refer the reader to the books [21] and [1] for more information on the asymptotic theory of convex bodies; the proofs of all the results that are mentioned in the first two sections of this note can be found in [1].

3 Regularization in John's position and Dvoretzky-type theorems

Let K be a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subseteq K$. For every $t \geqslant 1$ we define

$$K_t = \mathrm{conv}(K \cup t B_2^n). \tag{3.1}$$

Note that $K_1 = K$. We set

$$M_t := \int_{S^{n-1}} \|x\|_{K_t} \, d\sigma_n(x) \quad \text{and} \quad b_t := \max\{\|x\|_{K_t} : x \in S^{n-1}\}. \tag{3.2}$$

Since $t \geqslant 1$ and $B_2^n \subseteq K$, we have

$$K \subseteq K_t \subseteq tK, \quad \text{or equivalently,} \quad \frac{1}{t}\|x\| \leqslant \|x\|_{K_t} \leqslant \|x\| \tag{3.3}$$

for all $x \in \mathbb{R}^n$. In other words,

$$d_{\mathrm{BM}}(K, K_t) \leqslant d_{\mathrm{G}}(K, K_t) \leqslant t. \tag{3.4}$$

From (3.3) we see that $M_1 \leqslant t M_t$ and $b_t \leqslant b_1$, therefore

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant \frac{1}{t^2} \left(\frac{M_1}{b_1}\right)^2 \quad \text{for all } t \geqslant 1. \tag{3.5}$$

Fresen's result (Theorem 1.3) provides lower bounds for M_t/b_t in the case where K is in John's position. The proof makes essential use of proportional Dvoretzky-Rogers factorization results (the main tool is in fact a theorem of Vershynin from [24]). It will be convenient to state Theorem 1.3 in a slightly different way.

Lemma 3.1 (Fresen). *Let K be a symmetric convex body in $\mathbb{R}^n$ such that B_2^n is the maximal volume ellipsoid of K. For every $1 \leqslant t \leqslant \sqrt{n}$ we have*

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant c \frac{t^2}{n} \log\left(1 + \frac{n}{t^2}\right), \tag{3.6}$$

where $c > 0$ is an absolute constant.

Proof. The exact statement of Fresen's lemma is that there exist absolute constants $\alpha > 1$ and $\beta < 1$ such that, for every $t \in [\alpha, \beta\sqrt{n}]$, the body K_t defined by (3.1) satisfies

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant c_1 \frac{t^2}{n} \log\left(\frac{c_2 n}{t^2}\right), \tag{3.7}$$

where $c_1, c_2 > 0$ are absolute constants. We observe that the function $f(y) = \log(c_2 y) / \log(1+y)$ tends to 1 as $y \to \infty$, so we can find $M > 0$ such that

$$\log(c_2 y) \geqslant \frac{1}{2} \log(1+y) \tag{3.8}$$

for all $y \geqslant M$. If $\alpha^2 \leqslant t^2 \leqslant \min\{\beta^2, M^{-1}\} n$ then, setting $y = n/t^2$ and using (3.7) and (3.8) we get

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant \frac{c_1}{2} \frac{t^2}{n} \log\left(1 + \frac{n}{t^2}\right). \tag{3.9}$$

Now, let $\min\{\beta^2, M^{-1}\} n \leqslant t^2 \leqslant n$. We observe that $tB_2^n \subseteq K_t \subseteq \sqrt{n} B_2^n$, and hence

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant \frac{t^2}{n} \geqslant \frac{1}{C(\beta, M)} \frac{t^2}{n} \log\left(1 + \frac{n}{t^2}\right), \tag{3.10}$$

where $C(\beta, M) = \log(1 + \max\{\beta^{-2}, M\})$. Finally, if $1 \leqslant t \leqslant \alpha$ then we use (2.9) and (3.5) to write

$$\left(\frac{M_t}{b_t}\right)^2 \geqslant \frac{1}{t^2} \left(\frac{M_1}{b_1}\right)^2 \geqslant \frac{c}{\alpha^2} \frac{\log(n+1)}{n} \geqslant \frac{c}{\alpha^4} \frac{t^2}{n} \log\left(1 + \frac{n}{t^2}\right). \tag{3.11}$$

The result follows if we choose $c = c(\alpha, \beta, M) > 0$ suitably, taking into accout (3.9), (3.10) and (3.11). $\square$

Remark 3.2. It is useful to note that, in fact, what Fresen proved is a lower bound for M_t: for every $1 \leqslant t \leqslant \sqrt{n}$ we have

$$M_t^2 \geqslant \frac{c \log\left(1 + \frac{n}{t^2}\right)}{n}, \tag{3.12}$$

where $c > 0$ is an absolute constant. Then, Lemma 3.1 (or Theorem 1.3) follows immediately from the fact that $K_t \supseteq tB_2^n$ and hence $b_t \leqslant \frac{1}{t}$.

We use Lemma 3.1 to "increase" the Dvoretzky dimension of K as much as we need using the regularization operation $K \mapsto K_t$ for a suitable value of t. The cost of this operation is measured by the Banach-Mazur distance $d_{\mathrm{BM}}(K, K_t)$ which is controlled by t. The idea will become clear in the argument that follows.

Proof of Theorem 1.2. We may assume that the unit ball K of X is in John's position. Consider the smallest value t_k for which

$$c_0 c t_k^2 \log\left(1 + \frac{n}{t_k^2}\right) \geqslant k, \tag{3.13}$$

where c_0 is the constant in Theorem 1.1. Then, we may apply Lemma 3.1 to get

$$k(K_{t_k}) = c_0 n \left(\frac{M_{t_k}}{b_{t_k}}\right)^2 \geqslant c_0 c t_k^2 \log\left(1 + \frac{n}{t_k^2}\right) \geqslant k. \tag{3.14}$$

Therefore, we can find a subset $A_{n,k} \subset G_{n,k}$ of measure $\nu_{n,k}(A_{n,k}) \geqslant 1 - \exp(-\bar{c}_0 k)$ such that for any subspace $F \in A_{n,k}$ we have

$$d_{\mathrm{G}}(K_{t_k} \cap F, B_2^n \cap F) \leqslant 9/4 \leqslant 3. \tag{3.15}$$

From (3.4) we conclude that

$$d_{\mathrm{G}}(K\cap F, B_2^n\cap F)\leqslant d_{\mathrm{G}}(K_{t_k}\cap F, B_2^n\cap F)d_{\mathrm{G}}(K, K_{t_k})\leqslant 3t_k \tag{3.16}$$

with probability greater than $1-e^{-\overline{c}_0k}$ on $G_{n,k}$.

It remains to estimate t_k: Let us assume that $k\leqslant \delta n$ for some small enough $\delta>0$ that will be specified shortly, and set $C=\sqrt{\frac{2}{c_0c}}$. The real function $g(y)=\log\left(1+\frac{y(\log 2)}{C^2}\right)/\log(1+y)$ tends to 1 as $y\to\infty$, so we can find $M>0$ such that

$$\log\left(1+\frac{y(\log 2)}{C^2}\right)\geqslant \frac{1}{2}\log(1+y).$$

for all $y\geqslant M$. Now choose any $\delta\in(0,M^{-1})$, so that $y:=n/k\geqslant \delta^{-1}\geqslant M$. Then, the choice $t_k = C\sqrt{k}/\sqrt{\log\left(1+\frac{n}{k}\right)}$ gives

$$c_0ct_k^2\log\left(1+\frac{n}{t_k^2}\right)\geqslant C^2c_0ck\frac{\log\left(1+\frac{n(\log 2)}{C^2k}\right)}{\log\left(1+\frac{n}{k}\right)}\geqslant k. \tag{3.17}$$

We are left with the case $k\geqslant \delta n$. Then, by John's theorem, we have

$$d(K\cap F, \ell_2^k)\leqslant \sqrt{n}\leqslant C(\delta)\frac{\sqrt{k}}{\sqrt{\log\left(1+\frac{n}{k}\right)}}, \tag{3.18}$$

where $C(\delta)=\sqrt{\log\left(1+\frac{1}{\delta}\right)}/\sqrt{\delta}$. This concludes the proof of the theorem. □

The proof of the isomorphic global Dvoretzky theorem follows the same lines.

Proof of Theorem 1.5. We assume that K is in John's position, and choose $t_k>1$ so that

$$k\geqslant \frac{16c_3}{c}\frac{n/t_k^2}{\log\left(1+\frac{n}{t_k^2}\right)}, \tag{3.19}$$

where $c, c_3>0$ are the constants in Lemma 3.1 and Theorem 1.4. Applying Lemma 3.1 we see that

$$k\geqslant 16c_3\left(\frac{b_{t_k}}{M_{t_k}}\right)^2. \tag{3.20}$$

Then, using Theorem 1.4 for the body K_{t_k} (with this value of k and $\varepsilon=1/4$) we see that a random k-tuple $(U_1,\dots,U_k)$ from $O(n)$ satisfies

$$\frac{4}{5}M_{t_k}\|x\|_2\leqslant \frac{1}{k}\sum_{i=1}^k\|U_i(x)\|_{K_{t_k}}\leqslant \frac{5}{4}M_{t_k}\|x\|_2. \tag{3.21}$$

with probability greater than $1-\exp(-c_4n)$. Taking into account (3.3) we get

$$\frac{4}{5}M_{t_k}\|x\|_2\leqslant \frac{1}{k}\sum_{i=1}^k\|U_i(x)\|_{K}\leqslant \frac{5t_k}{4}M_{t_k}\|x\|_2 \tag{3.22}$$

for all $x\in\mathbb{R}^n$, and hence,

$$d_{\mathrm{G}}\left(\frac{1}{k}\sum_{i=1}^k U_i^*(K^\circ), B_2^n\right)\leqslant 2t_k \tag{3.23}$$

with probability greater than $1-\exp(-c_4 n)$.

It remains to choose t_k. Observe that setting $t_k = 4\sqrt{\frac{c_3 n}{ck\log(k+1)}}$ we have $t_k > 1$ (because $k \leqslant \delta n/\log(n+1)$ and provided that δ is small enough) and

$$\frac{16c_3}{c}\frac{n/t_k^2}{\log(1+n/t_k^2)} = \frac{k\log(k+1)}{\log\left(1+\frac{ck\log(k+1)}{16c_3}\right)} \leqslant k, \tag{3.24}$$

provided that $\log(k+1) \geqslant \frac{16c_3}{c}$. Since, by John's theorem, the result is obviously true (with a slightly larger value of C) for small values of k, the proof is complete. □

Remark 3.3. Consider the case where $K = [-1,1]^n$ is the unit cube. Then, $K^\circ = B_1^n = \{x \in \mathbb{R}^n : \|x\|_1 \leqslant 1\}$. Let $U_1, \ldots, U_k$ be a k-tuple in $O(n)$ and set $V_j = U_j^*$, $1 \leqslant j \leqslant k$. Assume that

$$\frac{1}{k}\sum_{i=1}^{k} V_i(B_1^n) \supseteq \alpha B_2^n \tag{3.25}$$

for some $\alpha > 0$. Recall that the covering number $N(A,B)$ of a body A by a second body B is the least integer N for which there exist N translates of B whose union covers A. An elementary argument shows that

$$\begin{aligned} \left|\sum_{i=1}^{k} V_i(B_1^n)\right| &\leqslant N\left(\sum_{i=1}^{k} V_i(B_1^n), krB_2^n\right)|krB_2^n| = N\left(\sum_{i=1}^{k} V_i(B_1^n), \sum_{i=1}^{k} rB_2^n\right)|krB_2^n| \\ &= |krB_2^n|\prod_{i=1}^{k} N(V_i(B_1)^n, rB_2^n) = (kr)^n|B_2^n|\big(N(B_1^n, rB_2^n)\big)^k \end{aligned} \tag{3.26}$$

for all $r > 0$. Now, we use the following result of Schütt [23]: If $\log n \leqslant s \leqslant n$ then there exists

$$r_{n,s} \leqslant c_5\sqrt{\frac{\log(n/s+1)}{s}}, \tag{3.27}$$

where $c_5 > 0$ is an absolute constant, such that $N(B_1^n, r_{n,s}B_2^n) \leqslant 2^s$. From (3.25) and (3.26) it follows that

$$\alpha|B_2^n|^{1/n} \leqslant \frac{1}{k}\left|\sum_{i=1}^{k} V_i(B_1^n)\right|^{1/n} \leqslant r_{n,s}2^{\frac{ks}{n}}|B_2^n|^{1/n} \tag{3.28}$$

for all $\log n \leqslant s \leqslant n$. Assuming that $k \leqslant \delta\frac{n}{\log(n+1)}$ we may choose $s = \frac{n}{k}$ in (3.28). This gives

$$\alpha \leqslant 2r_{n,s} \leqslant \frac{c_6\sqrt{k\log(k+1)}}{\sqrt{n}}. \tag{3.29}$$

On the other hand, if $k \geqslant \frac{\delta n}{\log n}$ then $k\log(k+1) \geqslant c_7(\delta)n$ and it is clear that

$$\alpha \leqslant R\left(\frac{1}{k}\sum_{i=1}^{k} V_i(B_1^n)\right) \leqslant 1 \leqslant \frac{c_8(\delta)\sqrt{k\log(k+1)}}{\sqrt{n}}. \tag{3.30}$$

Next, assume that

$$\frac{1}{k}\sum_{i=1}^{k}\|U_i(x)\|_K \leqslant \beta\|x\|_2 \tag{3.31}$$

for some $\beta > 0$ and for all $x \in \mathbb{R}^n$. From a result of Litvak, V. Milman and Schectman [16] it follows that $\|x\|_K \leqslant \beta\sqrt{k}\|x\|_2$ for all $x \in \mathbb{R}^n$, and hence, $B_2^n \subseteq \beta\sqrt{k}K$. Since B_2^n and K have contact points, this implies that $\beta \geqslant 1/\sqrt{k}$. Combining the above we get:

Proposition 3.4. *Let B_1^n denote the unit ball of ℓ_1^n. For any $k \geqslant 1$ and any $V_1, \dots, V_k \in O(n)$ we have*

$$d_{\mathrm{G}}\left(\frac{1}{k}\sum_{i=1}^{k} V_i(B_1^n), B_2^n\right) \geqslant c\sqrt{\frac{n}{k^2 \log k}}, \tag{3.32}$$

where $c > 0$ is an absolute constant.

Note. The estimate of Proposition 3.4 is probably not optimal for large values of k; in fact, it provides some non-trivial information only if $k \leqslant c\sqrt{n/\log(n+1)}$.

4 Inradius and circumradius of random sections in John's position

In this section we prove Theorem 1.6 in its dual form:

Theorem 4.1. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that the Euclidean unit ball B_2^n is the minimal volume ellipsoid of K. For every $k^*(K) \leqslant k \leqslant n$ we have that a subspace $F \in G_{n,k}$ satisfies*

$$c_1\sqrt{\frac{\log\left(1+\frac{n}{k}\right)}{n}}\, B_2^n \cap F \subseteq P_F(K) \subseteq c_2\sqrt{\frac{k}{n}}\, B_2^n \cap F \tag{4.1}$$

with probability greater than $1 - \exp(-c_3 k)$.

Proof. For the right hand side inclusion we use the next general fact, which is a standard application of concentration of measure on the Euclidean sphere (see [1, Section 5.7] for the details). If K is a symmetric convex body in $\mathbb{R}^n$ then, for any $1 \leqslant k < n$ and any $s > 1$ there exists a subset $A_{n,k} \subset G_{n,k}$ with measure greater than $1 - e^{-c_1 s^2 k}$ such that the orthogonal projection of K onto any subspace $F \in A_{n,k}$ satisfies

$$R(P_F(K)) \leqslant w(K) + c_2 s\sqrt{k/n}R(K), \tag{4.2}$$

where $c_1 > 0, c_2 > 1$ are absolute constants. Note that if $k \geqslant k^*(K)$ then, by (2.4),

$$w(K) \leqslant c\sqrt{k/n}R(K).$$

Therefore, since $R(K) = R(B_2^n) = 1$, applying (4.2) we see that

$$P_F(K) \subseteq c(s)\sqrt{k/n}\, B_2^n \cap F \tag{4.3}$$

with probability greater than $1 - e^{-c_1 s^2 k}$ on $G_{n,k}$.

For the other inclusion, let $T = K^\circ$. Then, T is in John's position and, as in the proof of Theorem 1.2, we see that if $t_k = C\sqrt{k}/\sqrt{\log\left(1+\frac{n}{k}\right)}$, for an absolute constant $C > 0$, then

$$k(T_{t_k}) = c_0 n\left(\frac{M(T_{t_k})}{b(T_{t_k})}\right)^2 \geqslant k, \tag{4.4}$$

and hence, a random k-dimensional projection of $(T_{t_k})^\circ$ satisfies, with probability greater than $1 - e^{-ck}$,

$$P_F((T_{t_k})^\circ) \supseteq cw((T_{t_k})^\circ)\, B_2^n \cap F = cM(T_{t_k})\, B_2^n \cap F. \tag{4.5}$$

Since $T \subseteq T_{t_k}$, using also (3.12), we see that

$$P_F(K) = P_F(T^\circ) \supseteq P_F((T_{t_k})^\circ) \supseteq \frac{c'\sqrt{\log\left(1+\frac{n}{t_k^2}\right)}}{\sqrt{n}}\, B_2^n \cap F \supseteq \frac{c_1\sqrt{\log\left(1+\frac{n}{k}\right)}}{\sqrt{n}}\, B_2^n \cap F \tag{4.6}$$

as required. $\square$

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GIORGOS CHASAPIS: Department of Mathematics, University of Athens, Panepistimioupolis 157-84, Athens, Greece.
E-mail: `gchasapis@math.uoa.gr`

APOSTOLOS GIANNOPOULOS: Department of Mathematics, University of Athens, Panepistimioupolis 157-84, Athens, Greece.
E-mail: `apgiannop@math.uoa.gr`