REMARKS ON AN INEQUALITY OF ROGERS AND SHEPHARD

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Abstract. A classical inequality of Rogers and Shephard states that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$1 \leqslant g(K,k;F) := \big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{1/k} \leqslant \binom{n}{k}^{1/k} \leqslant \frac{cn}{k}$$

for every $F \in G_{n,k}$, where $c > 0$ is an absolute constant. We show that if K is origin symmetric and isotropic then, for every $1 \leqslant k \leqslant n-1$, a random $F \in G_{n,k}$ satisfies

$$c_1 L_K^{-1}\sqrt{n/k} \leqslant g(K,k;F) \leqslant c_2\sqrt{n/k}\,(\log n)^2 L_K$$

with probability greater than $1-e^{-k}$, where L_K is the isotropic constant of K and $c_1, c_2 > 0$ are absolute constants.

1. Introduction

Let K be a convex body of volume 1 in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. For every $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we define

$$g(K,k;F) := \big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{1/k}, \tag{1.1}$$

where $F^{\perp}$ denotes the orthogonal subspace of F in $\mathbb{R}^n$. A classical inequality of Rogers and Shephard [13] (see also Chakerian [5]) states that if K is origin symmetric then

$$1 \leqslant g(K,k;F) \leqslant \binom{n}{k}^{1/k} \leqslant \frac{c_0 n}{k}, \tag{1.2}$$

where $c_0 > 0$ is an absolute constant. The right-hand side inequality holds true under the more general assumption that $0 \in \mathrm{int}(K)$. On the other hand, Spingarn [15] showed that the lower bound remains valid if we assume that K is centered, i.e. that the barycenter of K is at the origin.

Both estimates are sharp: let $\{e_1,\dots,e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and set $F = \mathrm{span}\{e_1,\dots,e_k\}$. Consider a convex body $A \subset F$ and a convex body $B \subset F^{\perp}$ with $0 \in \mathrm{int}(A)\cap\mathrm{int}(B)$. One can check that if $K = A\times B = \{a+b : a\in A, b\in B\}$ then $P_F(K) = A$, $K\cap F^{\perp} = B$ and $\mathrm{vol}_n(K) = \mathrm{vol}_k(A)\mathrm{vol}_{n-k}(B)$. On the other hand, if we consider the convex body $K' = \mathrm{conv}(A\cup B) = \{(1-t)a+tb :$

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$a \in A, b \in B, 0 \leqslant t \leqslant 1\}$ then $P_F(K') = A$, $K' \cap F^\perp = B$ and $\mathrm{vol}_n(K') = \binom{n}{k}\mathrm{vol}_k(A)\mathrm{vol}_{n-k}(B)$.

Our starting point is the observation that the behavior of $g(\mathcal{E}, k; F)$ lies "in the middle" when $\mathcal{E}$ is an ellipsoid.

Proposition 1.1. *For every ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ and for all $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$ the product $\mathrm{vol}_k\big(P_F(\mathcal{E})\big)\mathrm{vol}_{n-k}(\mathcal{E} \cap F^\perp)$ is independent of the subspace F. More precisely, we have*

$$\mathrm{vol}_k\big(P_F(\mathcal{E})\big)\mathrm{vol}_{n-k}(\mathcal{E} \cap F^\perp) = \frac{\mathrm{vol}_k(B_2^k)\mathrm{vol}_{n-k}(B_2^{n-k})}{\mathrm{vol}_n(B_2^n)}\mathrm{vol}_n(\mathcal{E}). \tag{1.3}$$

Therefore,

$$\left(\frac{c_1 n}{k}\right)^{k/2}\mathrm{vol}_n(\mathcal{E}) \leqslant \mathrm{vol}_k\big(P_F(\mathcal{E})\big)\mathrm{vol}_{n-k}(\mathcal{E} \cap F^\perp) \leqslant \left(\frac{c_2 n}{k}\right)^{k/2}\mathrm{vol}_n(\mathcal{E}), \tag{1.4}$$

where $c_1, c_2 > 0$ are absolute constants.

For the reader's convenience we include a proof of this observation in Section 3. Assuming that $\mathrm{vol}_n(\mathcal{E}) = 1$, from Proposition 1.1 we see that

$$g(\mathcal{E}, k; F) \simeq \sqrt{n/k} \tag{1.5}$$

for all $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$. The question that we discuss in this note is if this is the typical (with respect to $F \in G_{n,k}$) behavior of $g(K, k; F)$ for any symmetric (or, more generally, centered) convex body K of volume 1 in $\mathbb{R}^n$. Our main result provides an (almost sharp) affirmative answer if we assume that K is in isotropic position.

Theorem 1.2. *Let K be an origin symmetric isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ a random $F \in G_{n,k}$ satisfies*

$$c_1 L_K^{-1}\sqrt{n/k} \leqslant g(K, k; F) \leqslant c_2\sqrt{n/k}(\log n)^2 L_K \tag{1.6}$$

with probability greater than $1 - e^{-k}$, where $c_1, c_2 > 0$ are absolute constants.

Our approach is presented in Section 4 and leads to some general lower and upper bounds that might be useful for other classical positions of K, such as the minimal surface area position or minimal mean width position or John position. In Section 5 we use the additional information that one has when K is isotropic, and obtain the bounds of Theorem 1.2. The left hand side inequality in (1.6) remains valid for any isotropic convex body K in $\mathbb{R}^n$. For the right hand side inequality we employ a recent result of E. Milman on the mean width of origin symmetric isotropic convex bodies, see [8]; this forces the assumption of symmetry in Theorem 1.2. Background information is provided in Section 2 and in the beginning of Section 5.

2. Notation and background information

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. The volume of an s-dimensional set A is denoted by $\mathrm{vol}_s(A)$. We write ω_n for the volume of B_2^n and σ_n for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$. We write $F^\perp$ for the orthogonal subspace of F in $\mathbb{R}^n$.

We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Similarly, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$. We also write $\overline{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\overline{A} := \mathrm{vol}_n(A)^{-1/n} A$.

A convex body is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is origin symmetric if $-x \in K$ whenever $x \in K$. We say that K is centered if it has barycenter at the origin, i.e. $\int_K \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by $h_K(x) = \max\{\langle x, y\rangle : y \in K\}$. The radius of K is defined as $R(K) = \max\{\|x\|_2 : x \in K\}$ and, if the origin is an interior point of K, the polar body K° of K is

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } x \in K\}. \tag{2.1}$$

We will use the fact that

$$c^n \mathrm{vol}_n(B_2^n)^2 \leqslant \mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ) \leqslant \mathrm{vol}_n(B_2^n)^2 \tag{2.2}$$

for every centered convex body K in $\mathbb{R}^n$. The right-hand side inequality is the Blaschke-Santaló inequality, while the left-hand side inequality is due to Bourgain and V. Milman [3] and holds true if we just assume that $0 \in \mathrm{int}(K)$.

For each $p > -n$, $p \neq 0$, we set

$$I_p(K) := \left(\int_K \|x\|_2^p dx \right)^{1/p} \tag{2.3}$$

and for each $-\infty < p < \infty$, $p \neq 0$, we define the p-mean width of K by

$$w_p(K) := \left(\int_{S^{n-1}} h_K^p(\theta) d\sigma_n(\theta) \right)^{1/p}. \tag{2.4}$$

From Hölder's inequality, both are increasing functions of p. The mean width of K is the quantity $w(K) = w_1(K)$. Note that

$$w_{-n}(K) = \left(\frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_n(K^\circ)} \right)^{\frac{1}{n}}. \tag{2.5}$$

This is immediate if we express $\mathrm{vol}_n(K^\circ)$ in polar coordinates. If K is an origin symmetric convex body in $\mathbb{R}^n$ and $\|\cdot\|_K$ is the norm induced to $\mathbb{R}^n$ by K, we set

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_n(x)$$

and write $b(K)$ for the smallest positive constant b with the property $\|x\|_K \leqslant b\|x\|_2$ for all $x \in \mathbb{R}^n$. From V. Milman's proof of Dvoretzky's theorem (see [10]) we know that if $k \leqslant cn(M(K)/b(K))^2$ then for most $F \in G_{n,k}$ we have $K \cap F \simeq \frac{1}{M(K)} B_F$.

For every convex body K in $\mathbb{R}^n$ and for every $1 \leqslant k \leqslant n-1$ we define the normalized k-th quermassintegral of K by

$$Q_k(K) = \left(\frac{1}{\omega_k} \int_{G_{n,k}} \mathrm{vol}_k(P_F(K))\, d\nu_{n,k}(F) \right)^{1/k}. \tag{2.6}$$

Note that $Q_1(K) = w(K)$. From the Aleksandrov-Fenchel inequality (see [14]) it follows that $Q_k(K)$ is a decreasing function of k. In particular,

$$\left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))\, d\nu_{n,k}(F)\right)^{1/k} \leqslant \frac{c_1 w(K)}{\sqrt{k}}. \tag{2.7}$$

where $c_1 > 0$ is an absolute constant. We refer to the books [14] and [10] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

The next two functionals will play an essential role in our argument.

(i) *p-mean projection function.* For every $1 \leqslant k \leqslant n-1$ and for every $p \neq 0$ we define the p-mean projection function $W_{[k,p]}(K)$ by

$$W_{[k,p]}(K) := \left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}}.$$

We also set $W_{[n]}(K) := \mathrm{vol}_n(K)^{1/n}$.

(ii) *p-mean section function.* For every $1 \leqslant k \leqslant n-1$ and for every $p \neq 0$ we define the p-mean section function $\tilde{W}_{[k,p]}(K)$ by

$$\tilde{W}_{[k,p]}(K) = \left(\int_{G_{n,k}} \mathrm{vol}_{n-k}(K \cap F^{\perp})^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}}.$$

The normalized dual k-th quermassintegral of K is the quantity $\tilde{W}_{[k]}(K) := \tilde{W}_{[k,1]}(K)$.

3. Ellipsoids

We start with the proof of Proposition 1.1. We will use the classical fact that Steiner symmetrization transforms an ellipsoid to an ellipsoid (see for example [2]). Here we state it as a lemma and include its proof for the sake of completeness.

Lemma 3.1. *For every $u \in S^{n-1}$ and for every ellipsoid $\mathcal{E}$ the Steiner symmetral $S_u(\mathcal{E})$ of $\mathcal{E}$ with respect to u is an ellipsoid.*

Proof. Assume without loss of generality that the ellipsoid is centered at the origin. Consider a positive definite map $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\mathcal{E} = \{x \in \mathbb{R}^n : \langle Tx, x\rangle \leqslant 1\}.$$

By the definition of Steiner symmetrization, a point $y \in \mathbb{R}^n$ belongs to $S_u(\mathcal{E})$ if the line $L = \{y + \lambda u : \lambda \in \mathbb{R}\}$ intersects $\mathcal{E}$ and

$$|\langle y, u\rangle| \leqslant \frac{1}{2}\mathrm{length}(\mathcal{E} \cap L). \tag{3.1}$$

The assumption that L intersects $\mathcal{E}$ means that there exists $\lambda \in \mathbb{R}$ so that $\langle T(y + \lambda u), (y+\lambda u)\rangle \leqslant 1$. The left-hand side is a quadratic function of λ, so its discriminant is non-negative, that is

$$\langle Ty, u\rangle^2 + \langle Tu, u\rangle - \langle Tu, u\rangle\langle Ty, y\rangle \geqslant 0.$$

In this case the length in (3.1) equals

$$\frac{2\sqrt{\langle Ty, u\rangle^2 - \langle Tu, u\rangle\big(\langle Ty, y\rangle - 1\big)}}{\langle Tu, u\rangle}.$$

Substituting in (3.1) we get that

$$S_u(\mathcal{E}) = \Big\{ y \in \mathbb{R}^n : \langle Tu, u\rangle^2 \langle y, u\rangle^2 \leqslant \langle Ty, u\rangle^2 - \langle Tu, u\rangle\big(\langle Ty, y\rangle - 1\big)\Big\}.$$

This set is clearly an ellipsoid (it is defined by a quadratic form). □

Note. In fact, it is known that Lemma 3.1 characterizes ellipsoids in the following sense: if K is a convex body with the property that all its Steiner symmetrals $S_u(K)$ are affine images of K, then K is an ellipsoid (see e.g. [7]).

Proof of Proposition 1.1. Assume without loss of generality that $\mathcal{E}$ is centered at the origin. We first prove (1.3). We distinguish two cases.

Case 1: F is generated by the unit vectors of k semiaxes of $\mathcal{E}$. In this case if $\lambda_1, \ldots, \lambda_n$ are the positive lengths of the ellipsoid's semiaxes then obviously

$$\begin{aligned}\mathrm{vol}_k\big(P_F(\mathcal{E})\big)\mathrm{vol}_{n-k}(\mathcal{E} \cap F^\perp) &= \left(\prod_{j=1}^n \lambda_j\right)\mathrm{vol}_k(B_2^k)\mathrm{vol}_{n-k}(B_2^{n-k}) \\ &= \frac{\mathrm{vol}_k(B_2^k)\mathrm{vol}_{n-k}(B_2^{n-k})}{\mathrm{vol}_n(B_2^n)}\mathrm{vol}_n(\mathcal{E}).\end{aligned}$$

Case 2: F is any element of $G_{n,k}$. Let $u_1, \ldots, u_k$ be any orthonormal basis of F. We write $\mathcal{E}' = S_{u_1}(\ldots(S_{u_k}(\mathcal{E})\ldots)$ for the ellipsoid obtained by successive Steiner symmetrizations of $\mathcal{E}$ in the directions $u_1, \ldots, u_k$. By the properties of Steiner symmetrization we have that

$$\mathrm{vol}_k(P_F(\mathcal{E})) = \mathrm{vol}_k(P_F(\mathcal{E}')) \quad \text{and} \quad \mathrm{vol}_{n-k}(\mathcal{E} \cap F^\perp) = \mathrm{vol}_{n-k}(\mathcal{E}' \cap F^\perp).$$

From Lemma 3.1 it follows that $\mathcal{E}'$ is an ellipsoid which in addition has the same volume as $\mathcal{E}$. Moreover, observe that Case 1 applies now to the ellipsoid $\mathcal{E}'$ and the subspace F. Thus, we get

$$\begin{aligned}\mathrm{vol}_k\big(P_F(\mathcal{E})\big)\mathrm{vol}_{n-k}(\mathcal{E} \cap F^\perp) &= \mathrm{vol}_k\big(P_F(\mathcal{E}')\big)\mathrm{vol}_{n-k}(\mathcal{E}' \cap F^\perp) \\ &= \frac{\mathrm{vol}_k(B_2^k)\mathrm{vol}_{n-k}(B_2^{n-k})}{\mathrm{vol}_n(B_2^n)}\mathrm{vol}_n(\mathcal{E}') \\ &= \frac{\mathrm{vol}_k(B_2^k)\mathrm{vol}_{n-k}(B_2^{n-k})}{\mathrm{vol}_n(B_2^n)}\mathrm{vol}_n(\mathcal{E}),\end{aligned}$$

completing the proof of (1.3).

Since $\mathrm{vol}_n(B_2^n) = \pi^{n/2}/\Gamma(1+n/2)$ it is elementary to check that (1.4) holds true as well. □

4. General bounds

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. In order to obtain a lower bound for $g(K,k;F)$ we will estimate the expectation $\mathbb{E}_{\nu_{n,k}}\Big[\big(g(K,k;F)\big)^{-a}\Big]$ for some $a > 0$. For any pair (p,q) of conjugate exponents, using Hölder's inequality

we write
(4.1)

$$\begin{aligned}&\int_{G_{n,k}}\frac{1}{\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})}d\nu_{n,k}(F)\\ &\leqslant\left(\int_{G_{n,k}}\frac{1}{\mathrm{vol}_k(P_F(K))^p}d\nu_{n,k}(F)\right)^{1/p}\left(\int_{G_{n,k}}\frac{1}{\mathrm{vol}_{n-k}(K\cap F^{\perp})^q}d\nu_{n,k}(F)\right)^{1/q}.\end{aligned}$$

For the first integral in the right-hand side of (4.1) one may use the next lemma (from [6]) which relates it to the mixed widths of K.

Lemma 4.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1\leqslant k\leqslant n-1$ and $p\geqslant 1$,*

$$W_{[k,-p]}(K)=\left(\int_{G_{n,k}}\mathrm{vol}_k(P_F(K))^{-p}d\nu_{n,k}(F)\right)^{-\frac{1}{kp}}\geqslant c_1\frac{w_{-kp}(K)}{\sqrt{k}},\tag{4.2}$$

where $c_1>0$ is an absolute constant.

Proof. Using Hölder's inequality, the Blaschke-Santaló and the reverse Santaló inequality, for every $p\geqslant 1$ we can write

$$\begin{aligned}\left(\int_{G_{n,k}}\mathrm{vol}_k(P_F(K))^{-p}d\nu_{n,k}(F)\right)^{\frac{1}{kp}}&\simeq\left(\int_{G_{n,k}}\frac{\mathrm{vol}_k((P_F(K))^{\circ})^p}{\omega_k^{2p}}d\nu_{n,k}(F)\right)^{\frac{1}{kp}}\\ &\simeq\sqrt{k}\left(\int_{G_{n,k}}\left(\int_{S_F}\frac{1}{h^k_{P_F(K)}(\theta)}d\sigma_F(\theta)\right)^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}}\\ &\simeq\sqrt{k}\left(\int_{G_{n,k}}\left(\int_{S_F}\frac{1}{h^k_K(\theta)}d\sigma_F(\theta)\right)^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}}\\ &\leqslant c\sqrt{k}\left(\int_{G_{n,k}}\int_{S_F}\frac{1}{h^{kp}_K(\theta)}d\sigma_F(\theta)\,d\nu_{n,k}(F)\right)^{\frac{1}{kp}}\\ &=c\sqrt{k}\left(\int_{S^{n-1}}\frac{1}{h^{kp}_K(\theta)}d\sigma(\theta)\right)^{\frac{1}{kp}}\\ &=c\sqrt{k}w^{-1}_{-kp}(K).\end{aligned}$$

The lemma follows. □

We set $p:=n/k>1$. Then, from Lemma 4.1, (2.5) and (2.2) we get

$$W_{[k,-n/k]}(K)\geqslant\frac{w_{-n}(K)}{c_1\sqrt{k}}\simeq\frac{1}{c_1\sqrt{k}}\left(\frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_n(K^{\circ})}\right)^{1/n}\simeq\sqrt{n/k}.\tag{4.3}$$

This gives:

Lemma 4.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1\leqslant k\leqslant n-1$,*

$$W^{-1}_{[k,-n/k]}(K)=\left(\int_{G_{n,k}}\mathrm{vol}_k(P_F(K))^{-n/k}d\nu_{n,k}(F)\right)^{1/n}\leqslant c_2\sqrt{k/n}\tag{4.4}$$

where $c_2>0$ is an absolute constant.

Taking into account (4.1) we get the next general estimate.

Proposition 4.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $1\leqslant k\leqslant n-1$ we have*

$$\int_{G_{n,k}}\frac{1}{\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})}d\nu_{n,k}(F) \leqslant \left(c_1\sqrt{k/n}\right)^k\left(\int_{G_{n,k}}\frac{1}{\mathrm{vol}_{n-k}(K\cap F^{\perp})^{\frac{n}{n-k}}}d\nu_{n,k}(F)\right)^{\frac{n-k}{n}}, \tag{4.5}$$

where $c_1>0$ is an absolute constant.

We turn to the upper bound. The next proposition shows that the normalized dual quermassintegrals $\tilde{W}_{[k]}(K)$ are strongly related to the quantities $I_p(K)$.

Lemma 4.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1\leqslant k\leqslant n-1$. Then,*

$$\tilde{W}_{[k]}(K)I_{-k}(K)=\left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k}=\tilde{W}_{[k]}(\overline{B}_2^n)I_{-k}(\overline{B}_2^n). \tag{4.6}$$

Direct computation shows that $\left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k}\simeq\sqrt{n}$.

Proof. We integrate in polar coordinates:

$$\begin{aligned}I_{-k}^{-k}(K)&=\frac{n\omega_n}{n-k}\int_{S^{n-1}}\frac{1}{\|x\|_K^{n-k}}d\sigma(x)\\&=\frac{n\omega_n}{(n-k)\omega_{n-k}}\int_{G_{n,n-k}}\omega_{n-k}\int_{S_F}\frac{1}{\|\theta\|_{K\cap F}^{n-k}}d\sigma(\theta)d\nu_{n,n-k}(F)\\&=\frac{n\omega_n}{(n-k)\omega_{n-k}}\int_{G_{n,n-k}}\mathrm{vol}_{n-k}(K\cap F)d\nu_{n,n-k}(F)\\&=\frac{n\omega_n}{(n-k)\omega_{n-k}}\int_{G_{n,k}}\mathrm{vol}_{n-k}(K\cap F^{\perp})d\nu_{n,k}(F),\end{aligned}$$

and the result follows from the definition of $\tilde{W}_{[k]}(K)$. □

It was proved in [12] that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then for any $p>-n$ we have

$$I_p(K)\geqslant I_p(\overline{B}_2^n).$$

One can also check that $\tilde{W}_{[k]}(\overline{B}_2^n)\simeq 1$ for all $1\leqslant k\leqslant n-1$. Then, Lemma 4.4 immediately gives:

Lemma 4.5. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1\leqslant k\leqslant n-1$,*

$$\tilde{W}_{[k]}(K)\leqslant\tilde{W}_{[k]}(\overline{B}_2^n)\simeq 1.$$

Now we write

$$\begin{aligned}(4.7)\quad &\int_{G_{n,k}} \big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{1/2} d\nu_{n,k}(F)\\ &\leqslant \left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))d\nu_{n,k}(F)\right)^{1/2}\left(\int_{G_{n,k}} \mathrm{vol}_{n-k}(K\cap F^{\perp})d\nu_{n,k}(F)\right)^{1/2},\end{aligned}$$

and taking into account Lemma 4.5 we get the next general estimate.

Proposition 4.6. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $1\leqslant k\leqslant n-1$ we have*

$$\begin{aligned}(4.8)\quad &\int_{G_{n,k}} \big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{1/2} d\nu_{n,k}(F)\\ &\qquad\leqslant c_2^k\left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))d\nu_{n,k}(F)\right)^{1/2},\end{aligned}$$

where $c_2>0$ is an absolute constant.

Taking into account (2.7) we see that

$$(4.9)\quad \int_{G_{n,k}} \big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{1/2} d\nu_{n,k}(F)\leqslant \left(\frac{c_3w(K)}{\sqrt{k}}\right)^{k/2}.$$

where $c_3>0$ is an absolute constant. Then, Markov's inequality implies the following.

Proposition 4.7. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $1\leqslant k\leqslant n-1$ we have that a random $F\in G_{n,k}$ satisfies*

$$(4.10)\quad g(K,k;F)=\big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{1/k}\leqslant \frac{c_4w(K)}{\sqrt{k}}$$

with probability greater than $1-e^{-k}$, where $c_4>0$ is an absolute constant.

5. The isotropic case

Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and its inertia matrix is a multiple of the identity: there exists a constant $L_K>0$ such that

$$(5.1)\quad \int_K \langle x,\theta\rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. More generally, a log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if its barycenter is at the origin and its inertia matrix is the identity; in this case, the isotropic constant of μ is defined as

$$(5.2)\quad L_\mu := \sup_{x\in\mathbb{R}^n}\big(f_\mu(x)\big)^{1/n},$$

where f_μ is the density of μ with respect to the Lebesgue measure. Note that a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic if and only if the log-concave probability measure μ_K with density $x\mapsto L_K^n\mathbf{1}_{K/L_K}(x)$ is isotropic. The reader may find a detailed and updated exposition of the theory of isotropic log-concave measures in the book [4].

Let μ be a probability measure on $\mathbb{R}^n$ with density f_μ with respect to the Lebesgue measure. For every $1 \leqslant k \leqslant n-1$ and every $E \in G_{n,k}$, the marginal of μ with respect to E is the probability measure with density

$$f_{\pi_E \mu}(x) = \int_{x+E^\perp} f_\mu(y)dy. \tag{5.3}$$

It is easily checked that if μ is centered, isotropic or log-concave, then $\pi_E\mu$ is also centered, isotropic or log-concave, respectively. For every log-concave probability measure μ on $\mathbb{R}^n$ and any $p > 0$ we define the set $K_p(\mu)$ as follows:

$$K_p(\mu) = \left\{ x \in \mathbb{R}^n : \int_0^\infty f_\mu(rx) r^{p-1}\, dr \geqslant \frac{f_\mu(0)}{p} \right\}.$$

The bodies $K_p(\mu)$ were introduced by K. Ball [1] who showed that they are convex. The next proposition is a generalization of a result of Ball from the same work (see also [9], and [4] for the precise statement below); it gives a very useful expression for the volume of central sections of an isotropic convex body.

Proposition 5.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. We denote by μ_K the isotropic log-concave measure with density $L_K^n \mathbf{1}_{L_K^{-1}K}$. Then, for every $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$, the body $\overline{K_{k+1}}(\pi_F(\mu_K))$ satisfies*

$$\mathrm{vol}_{n-k}(K \cap F^\perp)^{1/k} \simeq \frac{L_{\overline{K_{k+1}}(\pi_F(\mu_K))}}{L_K}. \tag{5.4}$$

Assume that K is an isotropic convex body in $\mathbb{R}^n$. From Proposition 5.1 we know that, for every $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$,

$$\mathrm{vol}_{n-k}(K \cap F^\perp)^{-1/k} \simeq \frac{L_K}{L_{\overline{K_{k+1}}(\pi_F(\mu_K))}} \leqslant c_2 L_K, \tag{5.5}$$

because $L_C \geqslant c$ for every convex body C, where $c > 0$ is an absolute constant (see for example Proposition 2.3.12 in [4]). Therefore, Proposition 4.3 gives

$$\begin{aligned}\int_{G_{n,k}} \frac{1}{\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^\perp)} d\nu_{n,k}(F) &\leqslant \left(c_1\sqrt{k/n}\right)^k (c_2 L_K)^k \\ &\leqslant (c_3\sqrt{k/n}L_K)^k.\end{aligned}$$

From Markov's inequality we get:

Proposition 5.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$, a random $F \in G_{n,k}$ satisfies*

$$g(K,k;F) := \left(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^\perp)\right)^{\frac{1}{k}} \geqslant \frac{c_4\sqrt{n/k}}{L_K} \tag{5.6}$$

with probability greater than $1 - e^{-k}$, where $c_4 > 0$ is an absolute constant.

For the upper bound we use (2.7) and a recent result of E. Milman [8]: if K is isotropic, and if we make the additional assumption that K is origin symmetric, then

$$w(K) \leqslant c_5\sqrt{n}(\log n)^2 L_K.$$

Thus, applying directly Proposition 4.7 we get:

Proposition 5.3. *Let K be an origin symmetric isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ a random $F \in G_{n,k}$ satisfies*

$$g(K,k;F) := \big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K\cap F^{\perp})\big)^{\frac{1}{k}} \leqslant c_6\sqrt{n/k}(\log n)^2 L_K \tag{5.7}$$

with probability greater than $1-e^{-k}$.

Combining Proposition 5.2 and Proposition 5.3 we obtain Theorem 1.2.

Remark 5.4. (i) It is known that for every isotropic convex body K in $\mathbb{R}^n$ we can find an origin-symmetric convex body T with the property that $L_T \simeq L_K$ (see [4, Proposition 2.5.10]): if we define a function f supported on $K-K$ by

$$f(x) = (\mathbf{1}_K * \mathbf{1}_{-K})(x) = \int_{\mathbb{R}^n} \mathbf{1}_K(y)\mathbf{1}_{-K}(x-y)\,dy = \mathrm{vol}_n(K\cap(x+K))$$

then f is an even isotropic log-concave density and one can check that $L_f = \sqrt{2}\,L_K$. It follows that the convex body $T = \overline{K_{n+2}(f)}$ has the desired properties. From Proposition 4.6 we see that the upper bound in Theorem 1.2 remains valid for a not necessarily symmetric isotropic convex body K and some $1 \leqslant k \leqslant n-1$, provided that

$$\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))d\nu_{n,k}(F) \leqslant C^k \int_{G_{n,k}} \mathrm{vol}_k(P_F(T))d\nu_{n,k}(F).$$

(ii) The logarithmic terms in (5.7) cannot be completely eliminated as long as the proof passes through estimates of the mean width of K. This is evident from the case of $K = \overline{B}_1^n$, where $w(\overline{B}_1^n) \simeq \sqrt{n\log(1+n)}$. However, some of these terms may not be needed. For example, if the body is in the ℓ-position (see [4, Section 1.11]) then the reverse Urysohn inequality $w(K) \leqslant c\sqrt{n}\log n$ and Proposition 4.7 imply that $g(K,k;F) \leqslant c_6\sqrt{n/k}\log n$ for a random $F \in G_{n,k}$.

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