On a multi-integral norm defined by weighted sums of log-concave random vectors

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Abstract

Let C and K be centrally symmetric convex bodies in $\mathbb{R}^n$. We show that if C is isotropic then

$$\|\mathbf{t}\|_{C^s,K} = \int_C \cdots \int_C \Big\|\sum_{j=1}^s t_j x_j\Big\|_K dx_s \cdots dx_1 \leqslant c_1 L_C (\log n)^5 \sqrt{n} M(K) \|\mathbf{t}\|_2$$

for every $s \geqslant 1$ and $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$, where L_C is the isotropic constant of C and $M(K) := \int_{S^{n-1}} \|\xi\|_K d\sigma(\xi)$. This reduces a question of V. Milman to the problem of estimating from above the parameter $M(K)$ of an isotropic convex body. The proof is based on an observation that combines results of Eldan, Lehec and Klartag on the slicing problem: If μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ then, for any centrally symmetric convex body K in $\mathbb{R}^n$ we have that

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K \, d\mu(x) \leqslant c_2 \sqrt{n} (\log n)^5 \, M(K).$$

We illustrate the use of this inequality with further applications.

1 Introduction

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. For any s-tuple $\mathcal{C} = (C_1, \ldots, C_s)$ of centrally symmetric convex bodies C_j of volume 1 in $\mathbb{R}^n$, consider the norm on $\mathbb{R}^s$, defined by

$$\|\mathbf{t}\|_{\mathcal{C},K} = \int_{C_1} \cdots \int_{C_s} \Big\|\sum_{j=1}^s t_j x_j\Big\|_K dx_s \cdots dx_1$$

where $\mathbf{t} = (t_1, \ldots, t_s)$. If $\mathcal{C} = (C, \ldots, C)$ then we write $\|\mathbf{t}\|_{C^s,K}$ instead of $\|\mathbf{t}\|_{\mathcal{C},K}$. A question posed by V. Milman is to determine if, in the case $C = K$, one has that $\|\cdot\|_{K^s,K}$ is equivalent to the standard Euclidean norm up to a term which is logarithmic in the dimension, and in particular, if under some cotype condition on the norm induced by K to $\mathbb{R}^n$ one has equivalence between $\|\cdot\|_{K^s,K}$ and the Euclidean norm.

This question was studied by Bourgain, Meyer, V. Milman and Pajor. For simplicity let us assume that $\mathrm{vol}_n(K) = 1$ (this is only a matter of normalization). It was proved in [6] that

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant c\sqrt{s}\Big(\prod_{j=1}^s |t_j|\Big)^{1/s}$$

where $c > 0$ is an absolute constant. Later, Gluskin and V. Milman obtained a better lower bound in [15], in fact working in a more general context: Let $\mathcal{A} = (A_1, \ldots, A_s)$ be an s-tuple of measurable sets of volume 1 in $\mathbb{R}^n$ and K be a star body of volume 1 in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Then,

(1.1) $$\|\mathbf{t}\|_{\mathcal{A},K} \geqslant c\,\|\mathbf{t}\|_2$$

for all $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$. Their argument was based on the Brascamp-Lieb-Luttinger inequality (see also [3, Chapter 4]).

We are mainly interested in upper bounds for the quantity $\|\mathbf{t}\|_{C^s,K}$. Since $\|\mathbf{t}\|_{C^s,K}=\|\mathbf{t}\|_{(TC)^s,TK}$ for any $T\in SL(n)$, we may restrict our attention to the case where C is isotropic (see Section 2 for the definition and background information). In fact, we are particularly interested in the case where C is isotropic and $K=C$, which corresponds to V. Milman's original question.

For any centered log-concave probability measure μ on $\mathbb{R}^n$ and any centrally symmetric convex body K in $\mathbb{R}^n$, consider the parameter

$$I_1(\mu,K):=\int_{\mathbb{R}^n}\|x\|_K d\mu(x). \tag{1.2}$$

Assume that C is isotropic. Chasapis, Giannopoulos and the author observed in [8] that one may write

$$\|\mathbf{t}\|_{C^s,K}=\|\mathbf{t}\|_2 L_C I_1(\mu_{\mathbf{t}},K), \tag{1.3}$$

where $\mu_{\mathbf{t}}$ is an isotropic, compactly supported log-concave probability measure depending on $\mathbf{t}$. One may easily check that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ and K is a centrally symmetric convex body in $\mathbb{R}^n$ then

$$\begin{aligned}\int_{O(n)} I_1(\mu,U(K))\,d\nu(U)&=\int_{\mathbb{R}^n}\int_{O(n)}\|x\|_{U(K)}d\nu(U)\,d\mu(x)\\ &=M(K)\int_{\mathbb{R}^n}\|x\|_2 d\mu(x)\approx\sqrt{n}M(K)\end{aligned}$$

where $M(K):=\int_{S^{n-1}}\|\xi\|_K d\sigma(\xi)$ and ν,σ denote the Haar probability measures on $O(n)$ and S^{n-1} respectively. It follows that

$$\int_{O(n)}\|\mathbf{t}\|_{C^s,U(K)}\,d\nu(U)\approx\left(L_C\sqrt{n}M(K)\right)\|\mathbf{t}\|_2. \tag{1.4}$$

Therefore, one might hope to obtain a quantity of the order of $L_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2$ as an upper estimate for $\|\mathbf{t}\|_{C^s,K}$. Using an argument which goes back to Bourgain (also, employing Paouris' inequality and Talagrand's comparison theorem) Chasapis, Giannopoulos and the author proved in [8] that if C is an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and K is a centrally symmetric convex body in $\mathbb{R}^n$ then

$$\|\mathbf{t}\|_{C^s,K}\leqslant c\left(L_C\max\left\{\sqrt[4]{n},\sqrt{\log(1+s)}\right\}\right)\sqrt{n}M(K)\|\mathbf{t}\|_2 \tag{1.5}$$

for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c>0$ is an absolute constant. To the best of our knowledge, even in the case $C=K$, general estimates such as (1.5) were not available before [8]. In the same work, starting from the basic identity (1.3), the authors also obtained an alternative proof of (1.1).

In this note we provide a poly-logarithmic in n, and independent from s, upper bound for $\|\mathbf{t}\|_{C^s,K}$.

Theorem 1.1. *Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then,*

$$\|\mathbf{t}\|_{C^s,K}\leqslant c_1L_C(\log n)^5\sqrt{n}M(K)\|\mathbf{t}\|_2$$

for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c_1>0$ is an absolute constant.

In the case $C = K$, Theorem 1.1 and (1.1) show that, for any centrally symmetric convex body K of volume 1 in $\mathbb{R}^n$,

$$c_1\|\mathbf{t}\|_2 \leqslant \|\mathbf{t}\|_{K^s,K} \leqslant c_2(\log n)^5\sqrt{n}M(K^*)L_{K^*}\|\mathbf{t}\|_2$$

for every $s \geqslant 1$ and every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$, where $c_i > 0$ are absolute constants and K^* is an isotropic linear image of K (note that $L_{K^*} = L_K$). Thus, we have a reduction of V. Milman's question to the problem of estimating the parameter $M(K^*)$ for an isotropic centrally symmetric convex body K^* in $\mathbb{R}^n$. One may hope that $\sqrt{n}M(K^*)L_{K^*} \leqslant c_3(\log n)^b$ for some absolute constant $b > 0$, which would completely settle the problem, modulo the best possible exponent of $\log n$. However, the latter problem remains open; see Remark 4.2 for a brief review of what is known.

The source of our improved estimate in Theorem 1.1 is the next result which we believe is of independent interest.

Theorem 1.2. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any centrally symmetric convex body K in $\mathbb{R}^n$ we have that*

$$I_1(\mu, K) \leqslant c_2\sqrt{n}(\log n)^5\, M(K)$$

where $c_2 > 0$ is an absolute constant.

Theorem 1.2 follows from the recent developments on the Kannan-Lovász-Simonovits conjecture and the slicing problem. We simply combine the recent estimate of Klartag and Lehec [18] with previous results of Eldan [10] and, in particular, with a theorem of Eldan and Lehec from [11] which now leads to this strong and general estimate for $I_1(\mu, K)$. We explain the details in Section 3. In Section 4 we combine Theorem 1.2 with the approach and method that was introduced in [8] to obtain Theorem 1.1.

In Section 5 we make some observations on the geometry of the unit ball $\mathcal{B}_s := \mathcal{B}_s(C^s, K)$ of the norm $\|\cdot\|_{C^s,K}$. By the symmetry of C we easily check that $\mathcal{B}_s$ is an unconditional 1-symmetric convex body in $\mathbb{R}^s$. Therefore, one can determine the volume radius of $\mathcal{B}_s$ up to a $\sqrt{\log s}$-factor: we have

$$\frac{c}{\sqrt{\log s}}M(\mathcal{B}_s) \leqslant \frac{1}{\mathrm{vrad}(\mathcal{B}_s)} \leqslant M(\mathcal{B}_s)$$

for any $s \geqslant 1$, where $c > 0$ is an absolute constant. In the case $s \geqslant c_1 n$, where $c_1 > 1$ is an absolute constant, a result of Bourgain, Meyer, V. Milman and Pajor from [6] is equivalent to the assertion

$$M(\mathcal{B}_s) \geqslant c_2 L_C\sqrt{n}M(K)$$

for some absolute constant $c_2 > 0$. Under the same assumption, that $s \geqslant c_1 n$, we obtain an upper bound of the same order, and hence determine $M(\mathcal{B}_s)$.

Theorem 1.3. *There exist absolute constants $c_i > 0$ such that: If C is an isotropic centrally symmetric convex body in $\mathbb{R}^n$ then, for any centrally symmetric convex body K in $\mathbb{R}^n$ and any $s \geqslant c_1 n$,*

$$c_2 L_C\sqrt{n}M(K) \leqslant M(\mathcal{B}_s) \leqslant c_3 L_C\sqrt{n}M(K)$$

where $\mathcal{B}_s$ is the unit ball of the norm $\|\cdot\|_{C^s,K}$ on $\mathbb{R}^s$.

Combining Theorem 1.3 and Theorem 1.1 with a well-known deviation inequality for Lipschitz functions on the Euclidean sphere of $\mathbb{R}^s$ we get:

Corollary 1.4. *There exist absolute constants $c_i > 0$ such that: If C is an isotropic centrally symmetric convex body in $\mathbb{R}^n$ then for any centrally symmetric convex body K in $\mathbb{R}^n$ and any $s \geqslant c_1 n$, an s-tuple $\mathbf{t} = (t_1, \dots, t_s)$ with $\|\mathbf{t}\|_2 = 1$ satisfies*

$$c_2 L_C \sqrt{n} M(K) \|\mathbf{t}\|_2 \leqslant \|\mathbf{t}\|_{\mathcal{B}_s} \leqslant c_3 L_C \sqrt{n} M(K) \|\mathbf{t}\|_2$$

with probability greater than $1 - \exp(-c_4 s/(\log n)^{10})$.

We believe that Theorem 1.2 is a useful result and in Section 6 we illustrate its use with some additional applications. In the language of convex bodies, Theorem 1.2 has the following reformulation.

Theorem 1.5. *Let C and K be two centrally symmetric convex bodies in $\mathbb{R}^n$. There exists $T \in SL(n)$ such that*

$$\frac{1}{\mathrm{vol}_n(C)} \int_C \|x\|_{TK} dx \leqslant c(\log n)^{10} \left(\frac{\mathrm{vol}_n(C)}{\mathrm{vol}_n(K)}\right)^{1/n}$$

where $c > 0$ is an absolute constant.

The inequality of Theorem 1.5 is a reverse form of the well-known fact (see [20, Section 2.2]) that for any pair of centrally symmetric convex bodies K and C in $\mathbb{R}^n$ one has

$$\frac{1}{\mathrm{vol}_n(C)} \int_C \|x\|_K dx \geqslant \frac{n}{n+1} \left(\frac{\mathrm{vol}_n(C)}{\mathrm{vol}_n(K)}\right)^{1/n}.$$

Our second application is an affirmative answer to a question from [13] where Giannopoulos, Paouris and Vritsiou presented a reduction of the slicing problem to the study of the parameter

$$I_1(K, Z_q^\circ(K)) = \int_K \|\langle \cdot, x\rangle\|_{L_q(K)} dx$$

(where $\{Z_q(K)\}_{q\geqslant 1}$ is the family of L_q-centroid bodies of K). It was shown in [13] that if (for some $1/2 \leqslant s < 1$) one had an upper bound of the form

$$I_1(K, Z_q^\circ(K)) \leqslant c_1 q^s \sqrt{n} L_K^2 \tag{1.6}$$

for every isotropic convex body K in $\mathbb{R}^n$ and all $1 \leqslant q \leqslant n$, then it would follow that

$$L_n \leqslant c_2 \sqrt[4]{n} \log n / q^{\frac{1-s}{2}}$$

where $L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\}$. However, to the best of our knowledge, it is only known that (1.6) holds true with $s = 1$, which via the reduction of [13] resulted in an estimate of the form $L_n = O(\sqrt[4]{n}(\log n)^b)$. Using Theorem 1.2 we confirm in Section 6 that (1.6) holds true with $s = 1/2$ up to some factors which are logarithmic in n.

Theorem 1.6. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ with center of mass at the origin. Then, for every $1 \leqslant q \leqslant n$ we have that*

$$I_1(K, Z_q^\circ(K)) \leqslant c\sqrt{qn} (\log n)^7 L_K^2$$

where $c > 0$ is an absolute constant.

Note that Theorem 1.6 establishes (1.6) with $s = \frac{1}{2}$ (up to a factor which is logarithmic in n) in accordance with the fact that the reduction of the slicing problem in [13] with $q \approx n$ then leads to a poly-logarithmic in n upper bound for L_n. Of course, the order has been now reversed since in the proof of Theorem 1.6 we use the results of Eldan, Lehec and Klartag as well as E. Milman's bounds for the mean width of the L_q-centroid bodies $Z_q(K)$. However, we believe that this application illustrates, once again, the strength and possible uses of Theorem 1.2.

2 Notation and background information

In this section we introduce notation and terminology that we use throughout this work, and provide background information on isotropic convex bodies. We write $\langle \cdot, \cdot \rangle$ for the standard inner product in $\mathbb{R}^n$ and denote the Euclidean norm by $\|\cdot\|_2$. In what follows, B_2^n is the Euclidean unit ball, S^{n-1} is the unit sphere, and σ is the rotationally invariant probability measure on S^{n-1}. Lebesgue measure in $\mathbb{R}^n$ is denoted by vol_n. The Grassmann manifold $G_{n,k}$ of all k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. For every $1 \leqslant k \leqslant n-1$ and $E \in G_{n,k}$ we write P_E for the orthogonal projection from $\mathbb{R}^n$ onto E. The letters c, c', c_j, c_j' etc. denote absolute positive constants whose value may change from line to line. Sometimes we might even relax our notation: $a \lesssim b$ will then mean "$a \leqslant cb$ for some (suitable) absolute constant $c > 0$", and $a \approx b$ will stand for "$a \lesssim b \wedge a \gtrsim b$".

A convex body in $\mathbb{R}^n$ is a compact convex set $C \subset \mathbb{R}^n$ with non-empty interior. We say that C is centrally symmetric if $-C = C$, and that C is centered if its barycenter $\frac{1}{\mathrm{vol}_n(C)} \int_C x\,dx$ is at the origin. We say that C is unconditional with respect to the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ if $x = (x_1, \ldots, x_n) \in C$ implies that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in C$ for any choice of signs $\epsilon_i \in \{-1, 1\}$. This is equivalent to the fact that the norm $\|\cdot\|_C$ on $\mathbb{R}^n$ with unit ball C satisfies

$$\Big\| \sum_{i=1}^n \epsilon_i t_i e_i \Big\|_C = \Big\| \sum_{i=1}^n t_i e_i \Big\|_C$$

for any choice of signs $\{\epsilon_i\}_{i=1}^n$ and scalars t_i. A centrally symmetric convex body C in $\mathbb{R}^n$ is called 1-symmetric if the norm induced by C to $\mathbb{R}^n$ is unconditional and permutation invariant, i.e.

$$\Big\| \sum_{i=1}^n \epsilon_i t_i e_{\sigma(i)} \Big\|_C = \Big\| \sum_{i=1}^n t_i e_i \Big\|_C$$

for any choice of scalars t_i, any choice of signs $\{\epsilon_i\}_{i=1}^n$ and any permutation σ of $\{1, \ldots, n\}$.

The radius $R(C)$ of a convex body C is the smallest $R > 0$ such that $C \subseteq RB_2^n$. The volume radius of C is the quantity $\mathrm{vrad}(C) = (\mathrm{vol}_n(C)/\mathrm{vol}_n(B_2^n))^{1/n}$. Integration in polar coordinates shows that if the origin is an interior point of C then the volume radius of C can be expressed as

$$\mathrm{vrad}(C) = \Big(\int_{S^{n-1}} \|\xi\|_C^{-n}\, d\sigma(\xi) \Big)^{1/n}$$

where $\|\xi\|_C = \inf\{t > 0 : \xi \in tC\}$. We also consider the parameter

$$M(C) = \int_{S^{n-1}} \|\xi\|_C d\sigma(\xi).$$

The support function of C is defined by $h_C(y) := \max\{\langle x, y\rangle : x \in C\}$, and the mean width of C is twice the average of h_C on S^{n-1}:

$$w(C) := 2\int_{S^{n-1}} h_C(\xi)\, d\sigma(\xi)$$

For notational convenience we write $\overline{C}$ for the homothetic image of volume 1 of a convex body $C \subseteq \mathbb{R}^n$, i.e. $\overline{C} := \mathrm{vol}_n(C)^{-1/n}C$.

The polar body C° of a centrally symmetric convex body C in $\mathbb{R}^n$ is defined by

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}.$$

The Blaschke-Santaló inequality states that $\mathrm{vol}_n(C)\mathrm{vol}_n(C^\circ) \leqslant \mathrm{vol}_n(B_2^n)^2$, with equality if and only if C is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman asserts that there exists an absolute constant $c > 0$ such that, conversely,

$$\left(\mathrm{vol}_n(C)\mathrm{vol}_n(C^\circ)\right)^{1/n} \geqslant c\,\mathrm{vol}_n(B_2^n)^{2/n} \approx 1/n.$$

A convex body C in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_C > 0$ such that

$$\|\langle\cdot, \xi\rangle\|_{L_2(C)}^2 := \int_C \langle x, \xi\rangle^2 dx = L_C^2 \tag{2.1}$$

for all $\xi \in S^{n-1}$. The hyperplane conjecture asks if there exists an absolute constant $A > 0$ such that

$$L_n := \max\{L_C : C \text{ is isotropic in } \mathbb{R}^n\} \leqslant A \tag{2.2}$$

for all $n \geqslant 1$. Bourgain proved in [5] that $L_n \leqslant c\sqrt[4]{n}\log n$; later, Klartag [17] improved this bound to $L_n \leqslant c\sqrt[4]{n}$. In a breakthrough work, Chen [9] proved that for any $\epsilon > 0$ there exists $n_0(\epsilon) \in \mathbb{N}$ such that $L_n \leqslant n^\epsilon$ for every $n \geqslant n_0(\epsilon)$. Very recently, Klartag and Lehec [18] showed that the hyperplane conjecture and the stronger Kannan-Lovász-Simonovits isoperimetric conjecture hold true up to a factor that is poly-logarithmic in the dimension; more precisely, they achieved the bound $L_n \leqslant c(\log n)^4$, where $c > 0$ is an absolute constant. Even more recently, Jambulapati, Lee and Vempala announced in [16] a slight improvement of this estimate to $L_n \leqslant c(\log n)^{2.2226}$. These developments will be discussed in Section 3 and, together with previous work of Eldan and Lehec, is the key for our main result.

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set and the restriction of $\log f$ to it is concave. A theorem of Borell [4] shows that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then μ has a log-concave density f_μ. Note that if C is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_C$ is the density of a log-concave measure.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \text{Cov}(\mu)]^{\frac{1}{2n}} \tag{2.3}$$

where $\text{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\text{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}. \tag{2.4}$$

We say that a $\log$-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered, i.e. if

$$\int_{\mathbb{R}^n} \langle x,\xi\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x,\xi\rangle f_\mu(x)dx = 0 \tag{2.5}$$

for all $\xi \in S^{n-1}$, and $\text{Cov}(\mu)$ is the identity matrix.

Let C be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$, the L_q-centroid body $Z_q(C)$ of C is the centrally symmetric convex body with support function

$$h_{Z_q(C)}(y) = \left(\int_C |\langle x,y\rangle|^q dx\right)^{1/q}. \tag{2.6}$$

From Hölder's inequality it follows that $Z_1(C) \subseteq Z_p(C) \subseteq Z_q(C) \subseteq Z_\infty(C)$ for all $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(C) = \text{conv}\{C, -C\}$. Using Borell's lemma (see e.g. [7, Lemma 2.4.5 and Theorem 2.4.6]) one can check that

$$Z_q(C) \subseteq c_1 \frac{q}{p} Z_p(C) \tag{2.7}$$

for all $1 \leqslant p < q$. One can also check that $Z_q(C) \supseteq c_2 Z_\infty(C)$ for all $q \geqslant n$.

We refer to the books [2] and [3] for basic facts from asymptotic convex geometry. We also refer to [20] and [7] for more information on isotropic convex bodies and log-concave probability measures.

3 Proof of Theorem 1.2

Definition 3.1. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any convex body K in $\mathbb{R}^n$ with $0 \in \text{int}(K)$ we define

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K d\mu(x).$$

In this section we provide an estimate for $I_1(\mu, K)$ in the case where μ is isotropic. It follows from an upper bound for the same quantity, due to Eldan and Lehec [11], which involves the constant

$$\tau_n^2 = \sup_\mu \sup_{\xi\in S^{n-1}} \sum_{i,j=1}^n \mathbb{E}_\mu(x_i x_j \langle x,\xi\rangle)^2$$

where the first supremum is over all isotropic log-concave probability measures μ on $\mathbb{R}^n$.

Theorem 3.2 (Eldan-Lehec)**.** *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any centrally symmetric convex body K in $\mathbb{R}^n$ we have that*

$$\int_{\mathbb{R}^n} \|x\|_K \, d\mu(x) \leqslant c_1 \sqrt{\log n}\, \tau_n \int_{\mathbb{R}^n} \|x\|_K \, d\gamma_n(x)$$

where γ_n is the standard Gaussian measure on $\mathbb{R}^n$ and $c_1 > 0$ is an absolute constant.

Proof of Theorem 1.2. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. A result of Eldan [10] relates the constant τ_n with the thin-shell constant

$$\sigma_n = \sup_{\mu} \sqrt{\mathrm{Var}_\mu(|x|)}$$

where the supremum is over all isotropic log-concave probability measures μ on $\mathbb{R}^n$. Eldan proved that

$$\tau_n^2 \leqslant c_2 \sum_{k=1}^{n} \frac{\sigma_k^2}{k}$$

where $c_2 > 0$ is an absolute constant. On the other hand, Klartag and Lehec [18] have obtained a poly-logarithmic in n upper bound for σ_n: their main result states that $\sigma_n \leqslant c_3(\log n)^4$ where $c_3 > 0$ is an absolute constant. In fact, an improved upper bound $\sigma_n \leqslant c_3(\log n)^{2.2226}$ has been announced in [16]. In what follows, we write these estimates as

$$\sigma_n \leqslant c_3(\log n)^{\gamma} \tag{3.1}$$

in order to show the dependence of the next results on γ; it is already known that this inequality holds true for some absolute constant $0 \leqslant \gamma \leqslant 2.3$.

Combining the above estimates, one gets

$$\tau_n^2 \leqslant c_2 \sum_{k=1}^{n} \frac{\sigma_k^2}{k} \leqslant c_4(\log n)^{2\gamma+1}.$$

Therefore, the estimate of Eldan and Lehec immediately implies that

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K \, d\mu(x) \leqslant c_5(\log n)^{\gamma+1} \int_{\mathbb{R}^n} \|x\|_K \, d\gamma_n(x)$$

where $c_5 > 0$ is an absolute constant. Finally, integration in polar coordinates shows that

$$\int_{\mathbb{R}^n} \|x\|_K \, d\gamma_n(x) \approx \sqrt{n} \int_{S^{n-1}} \|\xi\|_K \, d\sigma(\xi) \approx \sqrt{n} M(K)$$

and hence the proof of Theorem 1.2 is complete. □

4 Proof of Theorem 1.1

In this section we provide the proof of Theorem 1.1. It combines the approach of [8] with Theorem 1.2.

Proof of Theorem 1.1. Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and $X_1,\ldots,X_s$ be independent random vectors, uniformly distributed in C. For any $\mathbf{t}=(t_1\ldots,t_s)\in\mathbb{R}^s$ we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1+\cdots+t_sX_s$. Since $\|\mathbf{t}\|_{C^s,K}$ is a norm, we may always assume that $\|\mathbf{t}\|_2=1$. Note that $\nu_{\mathbf{t}}$ is an even log-concave probability measure on $\mathbb{R}^n$ (this is a consequence of the Prékopa-Leindler inequality; see [2]). We write $g_{\mathbf{t}}$ for the density of $\nu_{\mathbf{t}}$. Our starting point is the next observation from [8].

Lemma 4.1. *For any $\mathbf{t}=(t_1\ldots,t_s)\in\mathbb{R}^s$, we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1+\cdots+t_sX_s$. Then,*

$$\|\mathbf{t}\|_{C^s,K}=\int_{\mathbb{R}^n}\|x\|_K d\nu_{\mathbf{t}}(x).$$

It is easily verified that the covariance matrix $\mathrm{Cov}(\nu_{\mathbf{t}})$ of $\nu_{\mathbf{t}}$ is a multiple of the identity: more precisely,

$$\mathrm{Cov}(\nu_{\mathbf{t}})=L_C^2\,I_n.$$

It follows that the function $f_{\mathbf{t}}(x)=L_C^n g_{\mathbf{t}}(L_Cx)$ is the density of an isotropic log-concave probability measure $\mu_{\mathbf{t}}$ on $\mathbb{R}^n$. Indeed, we have

$$\int_{\mathbb{R}^n}f_{\mathbf{t}}(x)x_ix_j\,dx=L_C^n\int_{\mathbb{R}^n}g_{\mathbf{t}}(L_Cx)x_ix_j\,dx=L_C^{-2}\int_{\mathbb{R}^n}g_{\mathbf{t}}(y)y_iy_j\,dy=\delta_{ij}$$

for all $1\leqslant i,j\leqslant n$.

Note. It is proved in [8, Lemma 3.2] that if $\|\mathbf{t}\|_2=1$ then $\|g_{\mathbf{t}}\|_\infty\leqslant e^n$. From this inequality we see that

$$L_{\mu_{\mathbf{t}}}=\|f_{\mathbf{t}}\|_\infty^{\frac{1}{n}}=L_C\|g_{\mathbf{t}}\|_\infty^{\frac{1}{n}}\leqslant eL_C$$

for all $\mathbf{t}\in\mathbb{R}^s$ with $\|\mathbf{t}\|_2=1$.

We compute

$$\|\mathbf{t}\|_{C^s,K}=\int_{\mathbb{R}^n}\|x\|_K\,d\nu_{\mathbf{t}}(x)=L_C^{-n}\int_{\mathbb{R}^n}\|x\|_Kf_{\mathbf{t}}(x/L_C)\,dx=L_C\int_{\mathbb{R}^n}\|y\|_Kd\mu_{\mathbf{t}}(y)$$

and hence we get

$$\|\mathbf{t}\|_{C^s,K}=L_CI_1(\mu_{\mathbf{t}},K)\tag{4.1}$$

for all $\mathbf{t}\in\mathbb{R}^s$ with $\|\mathbf{t}\|_2=1$. Now, we use Theorem 1.2 to estimate $I_1(\mu_{\mathbf{t}},K)$. As a result, we obtain the upper bound

$$\|\mathbf{t}\|_{C^s,K}\leqslant c_1L_C(\log n)^{\gamma+1}\sqrt{n}M(K)$$

which is the assertion of Theorem 1.1. □

Remark 4.2. We mentioned in the introduction that the question to estimate the parameter $M(K)$ for an isotropic centrally symmetric convex body K in $\mathbb{R}^n$ remains open. The currently best known estimates are due to Giannopoulos and E. Milman (see [12]; also, [14] for previous work on this question). In view of the recent developments on the slicing problem, we briefly recall the main computations in their argument and make some slight modifications to get the next result.

Theorem 4.3 (Giannopoulos-E. Milman). *Let K be an isotropic centrally symmetric convex body in $\mathbb{R}^n$. Then,*

$$M(K) \leqslant \frac{cn^{1/3}(\log n)^{\frac{\gamma+1}{3}}}{\sqrt{n}L_K} \tag{4.2}$$

where $c > 0$ is an absolute constant and $0 \leqslant \gamma \leqslant 4$ is the constant in (3.1).

Proof. It is proved in [12, Corollary 4.4] that for every centrally symmetric convex body K in $\mathbb{R}^n$ with $K \supseteq rB_2^n$, one has

$$\sqrt{n}M(K) \leqslant c_1 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left\{\frac{1}{r}, \frac{n}{k}\log\left(e + \frac{n}{k}\right)\frac{1}{v_k^-(K)}\right\} \tag{4.3}$$

where $v_k^-(K) := \inf\{\mathrm{vrad}(P_E(K)) : E \in G_{n,k}\}$. Now, since K is isotropic and centrally symmetric, we know that

$$h_K(\xi) = \|\langle \cdot, \xi\rangle\|_{L_\infty(K)} \geqslant \|\langle \cdot, \xi\rangle\|_{L_2(K)} = L_K,$$

and hence $K \supseteq L_K B_2^n$, therefore we may use (4.3) with $r = L_K$. We also know (see [7, Proposition 5.1.15]) that

$$\mathrm{vol}(K \cap E^\perp)^{1/k} \leqslant c_2 \frac{L_k}{L_K}$$

for every $E \in G_{n,k}$. Applying the Rogers-Shephard inequality

$$\mathrm{vol}(P_E(K))^{1/k}\mathrm{vol}(K \cap E^\perp)^{1/k} \geqslant 1$$

(see e.g. [2, Lemma 1.5.6]) we see that $\mathrm{vol}(P_E(K))^{1/k} \geqslant c_3 L_K/L_k$ for every $E \in G_{n,k}$. Therefore, $\mathrm{vrad}(P_E(K)) \geqslant c_4\sqrt{k}L_K/L_k$, which gives

$$v_k^-(K) \geqslant c_4\sqrt{k}L_K/L_k.$$

Set $k_n = n^{2/3}(\log n)^{\frac{2(\gamma+1)}{3}}$. Inserting the above estimates into (4.3) and using the fact that $L_k = O((\log n)^{\gamma+1})$, we get

$$\begin{aligned}\sqrt{n}M(K) &\leqslant \frac{c_5}{L_K}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\min\left\{1, \frac{n(\log n)^{\gamma+1}}{k^{3/2}}\right\} \\ &\leqslant \frac{c_6}{L_K}\left(\sum_{k=1}^{k_n}\frac{1}{\sqrt{k}} + \sum_{k=k_n}^{n}\frac{n(\log n)^{\gamma+1}}{k^2}\right) \approx \frac{n^{1/3}(\log n)^{\frac{\gamma+1}{3}}}{L_K}.\end{aligned}$$

This proves the theorem. □

5 Geometry of the unit ball

Let C and K be centrally symmetric convex bodies in $\mathbb{R}^n$ and assume that C is isotropic. In this section we fix C and K, and write $\mathcal{B}_s := \mathcal{B}_s(C^s, K) \subset \mathbb{R}^s$ for the unit ball of the norm

$$\|\mathbf{t}\|_{C^s,K} = \int_C \cdots \int_C \Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K \, dx_s \cdots dx_1.$$

By the symmetry of C we easily check that $\|\cdot\|_{C^s,K}$ is an unconditional norm. Moreover, it is 1-symmetric. Therefore, $\mathcal{B}_s$ is a 1-symmetric convex body in $\mathbb{R}^s$ and hence it is known that it satisfies the MM^*-estimate

$$M(\mathcal{B}_s)w(\mathcal{B}_s) \leqslant c\sqrt{\log s} \tag{5.1}$$

where $c > 0$ is an absolute constant (see [22]). We also recall the general bounds

$$\frac{1}{M(\mathcal{B}_s)} \leqslant \mathrm{vrad}(\mathcal{B}_s) \leqslant w(\mathcal{B}_s) \tag{5.2}$$

that hold for any convex body which contains the origin as an interior point (see e.g. [21, Page xi]).

Lemma 5.1. *For any $s \geqslant 1$ we have that*

$$\frac{c_1}{\sqrt{\log s}}M(\mathcal{B}_s) \leqslant \frac{c_2}{w(\mathcal{B}_s)} \leqslant \frac{1}{\sqrt{s}}\Big\|\sum_{i=1}^s e_i\Big\|_{\mathcal{B}_s} \approx \frac{1}{\mathrm{vrad}(\mathcal{B}_s)} \leqslant M(\mathcal{B}_s)$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Since $\mathcal{B}_s$ is 1-symmetric, we know that $\overline{\mathcal{B}_s} := |\mathcal{B}_s|^{-1/s}\mathcal{B}_s$ is isotropic. A well-known result of Bobkov and Nazarov (see [7, Chapter 4]) shows that

$$c_1|\mathcal{B}_s|^{1/s}B_\infty^s \subseteq \mathcal{B}_s \subseteq c_2 s|\mathcal{B}_s|^{1/s}B_1^s.$$

Equivalently, for all $t_1,\dots,t_s \in \mathbb{R}$ we have that

$$c_2'|\mathcal{B}_s|^{-1/s}\cdot\frac{1}{s}\sum_{i=1}^s|t_i| \leqslant \Big\|\sum_{i=1}^s t_ie_i\Big\|_{\mathcal{B}_s} \leqslant c_1'|\mathcal{B}_s|^{-1/s}\cdot\max_{1\leqslant i\leqslant s}|t_i|.$$

Choosing $t_1 = \cdots = t_s = 1$ we get the equivalence

$$\frac{1}{\sqrt{s}}\Big\|\sum_{i=1}^s e_i\Big\|_{\mathcal{B}_s} \approx \frac{1}{\mathrm{vrad}(\mathcal{B}_s)}.$$

The remaining assertions of the lemma follow from (5.1) and (5.2). □

The following theorem is proved in [6].

Theorem 5.2 (Bourgain-Meyer-V. Milman-Pajor)**.** *There exist absolute constants $c_i > 0$ such that, if C is an isotropic centrally symmetric convex body in $\mathbb{R}^n$ then for any centrally symmetric convex body K in $\mathbb{R}^n$ and any $s \geqslant c_1 n$,*

$$\int_C\cdots\int_C\int_\Omega\Big\|\sum_{j=1}^s g_j(\omega)x_j\Big\|_K\,dP(\omega)\,dx_s\cdots dx_1 \geqslant c_2\sqrt{s}L_C\sqrt{n}M(K),$$

where g_j are independent standard Gaussian random variables on some probability space $(\Omega,\mathcal{F},P)$.

Note. Let γ_s denote the standard Gaussian measure on $\mathbb{R}^s$. With the notation of Theorem 5.2 we easily check that

$$\int_C \cdots \int_C \int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) x_j \Big\|_K d\omega\, dx_s \cdots dx_1 = \int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) e_j \Big\|_{C^s,K} d\omega$$

$$= \int_{\mathbb{R}^s} \|y\|_{C^s,K} d\gamma_s(y) \approx \sqrt{s} M(\mathcal{B}_s).$$

Therefore, Theorem 5.2 states that if $s \geqslant c_1 n$ then

$$M(\mathcal{B}_s) \geqslant c_2 L_C \sqrt{n} M(K) \tag{5.3}$$

for some absolute constant $c_2 > 0$. Theorem 1.3 asserts that (for the same values of s) a reverse inequality is also true.

Proof of Theorem 1.3. We shall use a well-known result of Adamczak, Litvak, Pajor and Tomczak-Jaegermann from [1]: For every $\epsilon > 0$ and $t \geqslant 1$ there exists a constant $C = C(t, \epsilon) > 0$ with the following property: if $s \geqslant C(t, \epsilon) n$ and $X_1, \ldots, X_s$ are independent random vectors in $\mathbb{R}^n$ with the same isotropic log-concave distribution, then we have

$$\Big\| \frac{1}{s} \sum_{j=1}^s X_j \otimes X_j - I_n \Big\| = \sup_{\xi \in S^{n-1}} \Big| \frac{1}{s} \sum_{j=1}^s \langle X_j, \xi \rangle^2 - 1 \Big| \leqslant \epsilon \tag{5.4}$$

with probability greater than $1 - \exp(-ct\sqrt{n})$. Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and consider independent random vectors $X_1, \ldots, X_s$ distributed according to the isotropic log-concave probability measure ν_C with density $L_C^n \mathbf{1}_{C/L_C}$. Choosing $\epsilon = \frac{1}{2}$, $t = 1$ and applying the above result we get that there exists an absolute constant $c_1 > 0$ such that if $s \geqslant c_1 n$ then

$$\frac{1}{2} s \|z\|_2^2 \leqslant \sum_{j=1}^s \langle X_j, z \rangle^2 \leqslant \frac{3}{2} s \|z\|_2^2$$

for all $z \in \mathbb{R}^n$, with probability greater than $1 - \exp(-c_2\sqrt{n})$. It follows that if $s \geqslant c_1 n$ then there exists $\mathcal{A} \subset C^s$ of measure $|\mathcal{A}| > 1 - e^{-c_2\sqrt{n}}$ such that for any s-tuple $(x_1, \ldots, x_s) \in \mathcal{A}$ we have that

$$\frac{1}{2} s L_C^2 \|z\|_2^2 \leqslant \sum_{j=1}^s \langle x_j, z \rangle^2 \leqslant \frac{3}{2} s L_C^2 \|z\|_2^2 \tag{5.5}$$

for every $z \in \mathbb{R}^n$. Fix $(x_1, \ldots, x_s) \in \mathcal{A}$. We have

$$\frac{1}{2} s L_C^2 \sum_{j=1}^n \langle z, e_j \rangle^2 \leqslant \sum_{j=1}^s \langle z, x_j \rangle^2 \leqslant \frac{3}{2} s L_C^2 \sum_{j=1}^n \langle z, e_j \rangle^2$$

for all $z \in \mathbb{R}^n$. Consider the Gaussian processes

$$X_z := \Big\langle \sum_{j=1}^s g_j x_j, z \Big\rangle \quad \text{and} \quad Y_z := \sqrt{3s/2} L_C \sum_{j=1}^s z_j g_j' \qquad (z \in K^\circ)$$

where g_j, g'_j are independent standard Gaussian random variables on some probability space $(\Omega, \mathcal{F}, P)$. Taking into account the right hand side inequality in (5.5) we check that $\|X_z - X_{z'}\|_2 \leqslant \|Y_z - Y_{z'}\|_2$ for all $z, z' \in K^\circ$. Applying Slepian's comparison principle (see [21, Lemma 5.7] or [2, Theorem 9.1.7]) we get

$$\int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) x_j \Big\|_K d\omega \leqslant c_3 \sqrt{s} L_C \int_\Omega \Big\| \sum_{j=1}^n g'_j(\omega) e_j \Big\|_K d\omega \leqslant c_4 \sqrt{s} L_C \sqrt{n} M(K) \tag{5.6}$$

for every $(x_1, \dots, x_s) \in \mathcal{A}$. On the other hand, if $(x_1, \dots, x_s) \notin \mathcal{A}$ then we may write

$$\sum_{j=1}^s \langle x_j, z\rangle^2 \leqslant \sum_{j=1}^s \|x_j\|_2^2 \|z\|_2^2 \leqslant sR(C)^2 \|z\|_2^2 \leqslant c_5 s n^2 L_C^2 \sum_{j=1}^n \langle z, e_j\rangle^2,$$

recalling that $R(C) \leqslant c_6 n L_C$ because C is isotropic (see [7, Theorem 3.2.1]). Applying Slepian's comparison principle in a similar way as before, we obtain the bound

$$\int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) x_j \Big\|_K d\omega \leqslant c_7 \sqrt{s} n L_C \sqrt{n} M(K) \tag{5.7}$$

for every $(x_1, \dots, x_s) \notin \mathcal{A}$. Integrating on $\mathcal{A}$ and using (5.6) and (5.7) we finally get

$$\begin{aligned}
\sqrt{s} M(\mathcal{B}_s) &\approx \int_C \cdots \int_C \int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) x_j \Big\|_K d\omega \, dx_s \cdots dx_1 \\
&= \int_{\mathcal{A}} \int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) x_j \Big\|_K d\omega + \int_{C^s \setminus \mathcal{A}} \int_\Omega \Big\| \sum_{j=1}^s g_j(\omega) x_j \Big\|_K d\omega \\
&\leqslant c_8 \sqrt{s} \big(1 + n \cdot \exp(-c\sqrt{n})\big) L_C \sqrt{n} M(K) \leqslant c_9 \sqrt{s} L_C \sqrt{n} M(K),
\end{aligned}$$

which shows that

$$M(\mathcal{B}_s) \leqslant c_{10} L_C \sqrt{n} M(K).$$

The proof of (5.3), which was already obtained in [6], follows by a similar (and simpler) argument if we start from the left-hand side inequality of (5.5) . □

Proof of Corollary 1.4. Let $s \geqslant c_1 n$. Consider the norm $\|\mathbf{t}\|_{C^s,K}$ as a function on the unit sphere S^{s-1}. We have

$$\int_{S^{s-1}} \|\mathbf{t}\|_{C^s,K} \, d\sigma(\mathbf{t}) = M(\mathcal{B}_s) \gtrsim L_C \sqrt{n} M(K)$$

by (5.3), and if b is the best constant for which $\|\mathbf{t}\|_{C^s,K} \leqslant b\|\mathbf{t}\|_2$ for all $\mathbf{t} \in \mathbb{R}^s$ then from Theorem 1.1 we know that $b \lesssim c_1 L_C (\log n)^{\gamma+1} \sqrt{n} M(K)$. A standard application of the deviation inequality for Lipschitz functions on the Euclidean sphere of $\mathbb{R}^s$ (see [2, Proposition 5.2.4]) shows that an s-tuple $\mathbf{t} = (t_1, \dots, t_s)$ with $\|\mathbf{t}\|_2 = 1$ satisfies

$$\frac{1}{2} M(\mathcal{B}_s) \|\mathbf{t}\|_2 \leqslant \|\mathbf{t}\|_{\mathcal{B}_s} \leqslant \frac{3}{2} M(\mathcal{B}_s) \|\mathbf{t}\|_2$$

with probability greater than $1 - \exp(-c_3 s/(\log n)^{2(\gamma+1)})$ and hence, by Theorem 1.3,

$$\|\mathbf{t}\|_{\mathcal{B}_s} \approx L_C \sqrt{n} M(K) \|\mathbf{t}\|_2$$

with probability greater than $1 - \exp(-c_3 s/(\log n)^{2(\gamma+1)})$ on S^{s-1}. □

6 Further remarks and applications

We believe that Theorem 1.2 is a useful result that should find several applications. As a first example, we prove Theorem 1.5: If C and K are two centrally symmetric convex bodies in $\mathbb{R}^n$ then there exists $T \in SL(n)$ such that

$$\frac{1}{\mathrm{vol}_n(C)} \int_C \|x\|_{TK} dx \leqslant c(\log n)^9 \left(\frac{\mathrm{vol}_n(C)}{\mathrm{vol}_n(K)}\right)^{1/n}$$

where $c > 0$ is an absolute constant.

Proof of Theorem 1.5. By homogeneity we may assume that $\mathrm{vol}_n(C) = 1$. We may find $T_1 \in SL(n)$ such that $C_1 := T_1(C)$ is isotropic. Consider the isotropic log-concave probability measure ν_C with density $L_C^n \mathbf{1}_{C/L_C}$. Direct computation shows that

$$\int_{C_1} \|x\|_{T_1 T(K)} dx = I_1(\nu_{C_1}, T_1 T(K)) L_C$$

for every $T \in SL(n)$. Using Theorem 1.2 we get

$$\int_C \|x\|_{TK} dx = \int_{C_1} \|x\|_{T_1 T(K)} dx \leqslant c_1 \sqrt{n} (\log n)^{\gamma+1} L_C M(T_1 T(K))$$

for every $T \in SL(n)$, where $c_1 > 0$ is an absolute constant. From Pisier's inequality (see [2, Corollary 6.5.3]) and the Blaschke-Santaló inequality we may choose T so that

$$\begin{aligned} M(T_1 T(K)) &= w((T_1 T(K))^\circ) = w((T_1 T)^{-*}(K^\circ)) \leqslant c_2 \sqrt{n} (\log n) \mathrm{vol}_n(K^\circ)^{1/n} \\ &\leqslant c_3 \sqrt{n} (\log n) \frac{\mathrm{vol}_n(B_2^n)^{2/n}}{\mathrm{vol}_n(K)^{1/n}} \approx \frac{\log n}{\sqrt{n}} \mathrm{vol}_n(K)^{-1/n}, \end{aligned}$$

since $\mathrm{vol}_n(B_2^n)^{2/n} \approx 1/n$. Combining the above we see that there exists $T \in SL(n)$ such that

$$\int_C \|x\|_{TK} dx \leqslant c(\log n)^{\gamma+2} L_C \mathrm{vol}_n(K)^{-1/n}$$

and the result follows from the bound $L_C \leqslant L_n = O((\log n)^\gamma)$. □

As a second application we prove Theorem 1.6 which gives an answer to a question from [13] regarding the parameter

$$I_1(K, Z_q^\circ(K)) := \int_K \|\langle \cdot, x \rangle\|_{L_q(K)} dx.$$

We shall use E. Milman's estimates [19] on the mean width $w(Z_q(K))$ of the L_q-centroid bodies $Z_q(K)$ of an isotropic convex body K in $\mathbb{R}^n$.

Theorem 6.1 (E. Milman)**.** *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for all $1 \leqslant q \leqslant n$ one has*

(6.1) $$w(Z_q(K)) \leqslant c_1 \sqrt{q} \log(1+q) \max\left\{\frac{\sqrt{q}\log(1+q)}{\sqrt{n}}, 1\right\} L_K$$

where $c_1 > 0$ is an absolute constant.

Proof of Theorem 1.6. Note that $I_1(K, Z_q^\circ(K))$ is invariant under invertible linear transformations of K and hence we may assume that K is isotropic. As in the proof of Theorem 1.5, consider the isotropic log-concave probability measure ν_K with density $L_K^n \mathbf{1}_{K/L_K}$. Direct computation shows that

$$I_1(K, Z_q^\circ(K)) = I_1(\nu_K, Z_q^\circ(K)) L_K.$$

Using Theorem 1.2 we immediately see that

$$I_1(\nu_K, Z_q^\circ(K)) \leqslant c_1(\log n)^5 \int_{\mathbb{R}^n} \|x\|_{Z_q(K)^\circ}\, d\gamma_n(x) \leqslant c_2\sqrt{n}(\log n)^{\gamma+1} w(Z_q(K)).$$

Combining the above we get

$$\begin{aligned} I_1(K, Z_q^\circ(K)) &= I_1(\nu_K, Z_q^\circ(K)) L_K \\ &\leqslant c_1\sqrt{qn}(\log n)^{\gamma+1}\log(1+q)\max\left\{\frac{\sqrt{q}\log(1+q)}{\sqrt{n}}, 1\right\} L_K^2 \end{aligned}$$

for all $1 \leqslant q \leqslant n$, which shows that $I_1(K, Z_q^\circ(K)) \leqslant c_2\sqrt{qn}(\log n)^{\gamma+3} L_K^2$. □

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