Random approximation and the vertex index of convex bodies

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Abstract

We prove that there exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body in $\mathbb{R}^n$ whose center of mass is at the origin, then a random subset $X \subset K$ of cardinality $\mathrm{card}(X) = \lceil \alpha n \rceil$ satisfies with probability greater than $1 - e^{-c_1 n}$

$$K \subseteq c_2 n \,\mathrm{conv}(X),$$

where $c_1, c_2 > 0$ are absolute constants. As an application we show that the vertex index of any convex body K in $\mathbb{R}^n$ is bounded by $c_3 n^2$, where $c_3 > 0$ is an absolute constant, thus extending an estimate of Bezdek and Litvak for the symmetric case.

1 Introduction

The starting point of this article is the following fact, which appears in [13] and [3]: If K is an origin symmetric convex body in $\mathbb{R}^n$ then for any $d > 1$ there exist $N \leqslant dn$ points $x_1, \ldots, x_N \in K$ such that

$$\mathrm{absconv}(\{x_1, \ldots, x_N\}) \subseteq K \subseteq \gamma_d \sqrt{n}\, \mathrm{absconv}(\{x_1, \ldots, x_N\}),$$

where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$. For the proof one assumes that the Euclidean unit ball B_2^n is the ellipsoid of minimal volume containing K, and makes essential use of a theorem of Batson, Spielman and Srivastava [4] on extracting an approximate John's decomposition with few vectors from a John's decomposition of the identity. An extension of this result to the non-symmetric case was recently obtained by the first named author in [8]: There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there exist $N \leqslant \alpha n$ points $x_1, \ldots, x_N \in K \cap S^{n-1}$ such that

$$K \subseteq B_2^n \subseteq cn^{3/2} \mathrm{conv}(\{x_1, \ldots, x_N\}), \tag{1.1}$$

where $c > 0$ is an absolute constant. The proof involves a more delicate theorem of Srivastava from [26]. Using (1.1) one can establish the following "quantitative diameter version" of Helly's theorem (see [8]): If $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$, then there exist $s \leqslant \alpha n$ and $i_1, \ldots i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{3/2},$$

where $c > 0$ is an absolute constant. For this application it is crucial to choose the points $x_1, \ldots, x_N$ in (1.1) among the contact points of K and its minimal volume ellipsoid.

Our first main result shows that if we choose $x_1, \ldots, x_N$ independently and uniformly from K then we can have a random version of (1.1) with an improved dependence on the dimension.

Theorem 1.1. *There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body in $\mathbb{R}^n$ whose center of mass is at the origin and $x_1, \ldots, x_N$, $N = \lceil \alpha n \rceil$, are independent random points uniformly distributed in K then, with probability greater than $1 - e^{-c_1 n}$ we have*

$$K \subseteq c_2 n \,\mathrm{conv}(\{x_1, \ldots, x_N\}),$$

where $c_1, c_2 > 0$ are absolute constants.

For the proof we may assume that K is an isotropic convex body (see Section 2 for background information) and we use the so-called one-sided L_q-centroid bodies of K; these are the convex bodies $Z_q^+(K)$, $q \geqslant 1$, with support functions

$$h_{Z_q^+(K)}(y) = \left(2\int_K \langle x, y\rangle_+^q \, dx\right)^{1/q},$$

where $a_+ = \max\{a, 0\}$. We show that if $N \geqslant \alpha n$, where $\alpha > 1$ is an absolute constant, then N independent random points $x_1, \dots, x_N$ uniformly distributed in K satisfy

$$\mathrm{conv}(\{x_1, \dots, x_N\}) \supseteq c_1 Z_2^+(K) \supseteq c_2 L_K B_2^n$$

with probability greater than $1 - \exp(-c_3 n)$, where $c_1, c_2, c_3 > 0$ are absolute constants. Since K is contained in $(n+1)L_K B_2^n$, Theorem 1.1 follows.

A natural question, which is closely related to Theorem 1.1, is to fix $N \geqslant \alpha n$ and to ask for the largest value $t(N, n)$ for which N independent random points $x_1, \dots, x_N$ uniformly distributed in K satisfy

$$\mathrm{conv}(\{x_1, \dots, x_N\}) \supseteq t(N, n)\, K$$

with probability "exponentially close" to 1. A sharp answer to this question would unify Theorem 1.1 and the following result from [11] which deals with the case where N is exponential in n: For every $\delta \in (0,1)$ there exists $n_0 = n_0(\delta)$ such that for every $n \geqslant n_0$, if $C \log n/n \leqslant \gamma \leqslant 1$ and K is a centered convex body in $\mathbb{R}^n$, then $N = \exp(\gamma n)$ independent random points $x_1, \dots, x_N$ chosen uniformly from K satisfy with probability greater than $1 - \delta$

$$K \supseteq \mathrm{conv}(\{x_1, \dots, x_N\}) \supseteq c(\delta)\gamma K,$$

where $c(\delta)$ is a constant depending on δ. We prove the following.

Theorem 1.2. *Let $\beta \in (0,1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on β and an absolute constant $c_1 > 0$ with the following property: let K be a centered convex body in $\mathbb{R}^n$, $\alpha n \leqslant N \leqslant e^n$ and $x_1, \dots, x_N$ be independent random points uniformly distributed in K; then*

$$\mathrm{conv}(\{x_1, \dots, x_N\}) \supseteq \frac{c_1 \beta \log(N/n)}{n} K.$$

with probability greater than $1 - e^{-N^{1-\beta} n^\beta}$.

In fact, Theorem 1.1 is a special case of Theorem 1.2 by setting $\beta = 1/2$ and $N = \lceil \alpha n \rceil$. The proof of Theorem 1.2 is given in Section 3.

Theorem 1.1 is very naturally related to the question of estimating the vertex index of a not necessarily symmetric n-dimensional convex body. The vertex index of a symmetric convex body K in $\mathbb{R}^n$ was introduced in [6] as follows:

$$\mathrm{vi}(K) = \inf\Big\{ \sum_{j=1}^N \|y_j\|_K : K \subseteq \mathrm{conv}(\{y_1, \dots, y_N\}) \Big\},$$

where $\|\cdot\|_K$ is the norm with unit ball K in $\mathbb{R}^n$. This index is closely related to the illumination parameter of a convex body, introduced by K. Bezdek in [5], and to a well-known conjecture of Boltyanski and Hadwiger about covering of an n-dimensional convex body by 2^n smaller positively homothetic copies (see [6] and [12]). Bezdek and Litvak proved that

$$\frac{c_1 n^{3/2}}{\mathrm{ovr}(K)} \leqslant \mathrm{vi}(K) \leqslant c_2 n^{3/2},$$

where $c_1, c_2 > 0$ are absolute constants and $\mathrm{ovr}(K)$ is the outer volume ratio of K (see Section 2 for the definition). To the best of our knowledge the notion of vertex index has not been studied in the not necessarily symmetric case. A way to define it for an arbitrary convex body K in $\mathbb{R}^n$ is to consider first any $z \in \mathrm{int}(K)$ and to set

$$\mathrm{vi}_z(K) = \inf\Big\{\sum_{j=1}^{N} p_{K,z}(y_j) : K \subseteq \mathrm{conv}(\{y_1,\dots,y_N\})\Big\},$$

where

$$p_{K,z}(x) = p_{K-z}(x) = \inf\{t>0 : x \in t(K-z)\}$$

is the Minkowski functional of K with respect to z. Then, one may define the (generalized) vertex index of K by

$$\mathrm{vi}(K) = \mathrm{vi}_{\mathrm{bar}(K)}(K),$$

where $\mathrm{bar}(K)$ is the center of mass of K. With this definition, we clearly have $\mathrm{vi}(K) = \mathrm{vi}(K - \mathrm{bar}(K))$, and hence we may restrict our attention to centered convex bodies (i.e. convex bodies whose center of mass is at the origin). In Section 4 we establish some elementary properties of this index and using Theorem 1.1 we obtain the following general estimate.

Theorem 1.3. *There exist two absolute constants $c_1, c_2 > 0$ such that for every $n \geqslant 2$ and for every centered convex body K in $\mathbb{R}^n$,*

$$\frac{c_1 n^{3/2}}{\mathrm{ovr}(\mathrm{conv}(K,-K))} \leqslant \mathrm{vi}(K) \leqslant c_2 n^2.$$

The lower bound of Theorem 1.3 is not sharp, even in the symmetric case. Gluskin and Litvak [13] have proved that for every $n \geqslant 1$ there exists a symmetric convex body K in $\mathbb{R}^n$ such that

$$\mathrm{ovr}(K) \geqslant c\sqrt{\frac{n}{\log(2n)}} \quad \text{and} \quad \mathrm{vi}(K) \geqslant c n^{3/2}.$$

It would be interesting to provide alternative lower bounds for $\mathrm{vi}(K)$ and of course it would be also interesting to decide whether, in the non-symmetric case, the upper bound $\mathrm{vi}(K) \leqslant Cn^2$ of Theorem 1.3 is sharp or not.

2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We use the same notation $|X|$ for the cardinality of a finite set X. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The letters $c, c', c_1, c_2, \dots$ denote absolute positive constants which may change from line to line.

We refer to the book of Schneider [25] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its center of mass

$$\mathrm{bar}(K) = \frac{1}{|K|}\int_K x\,dx$$

is at the origin. The support function of K is defined by $h_K(y) = \max\{\langle x,y\rangle : x \in K\}$. The circumradius of K is the radius of the smallest ball which is centered at the origin and contains K, i.e. $R(K) = \max\{\|x\|_2 : x \in K\}$.

If $0 \in \mathrm{int}(K)$ then the polar body K° of K is defined by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1 \text{ for all } x \in K\},$$

and the Minkowski functional of K is defined by

$$p_K(x) = \inf\{t > 0 : x \in tK\}.$$

Recall that p_K is subadditive and positively homogeneous.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n. John's theorem ([16], see also [1, Chapter 2]) states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $v_1, \ldots, v_m \in \mathrm{bd}(K) \cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers $a_1, \ldots, a_m$ such that

$$\sum_{j=1}^{m} a_j v_j = 0,$$

and the identity operator I_n is decomposed in the form

$$I_n = \sum_{j=1}^{m} a_j v_j \otimes v_j, \tag{2.1}$$

where $(v_j \otimes v_j)(y) = \langle v_j, y\rangle v_j$. We say that a convex body K is in Löwner's position if the ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n. One can check that this holds true if and only if K° is in John's position (see [1, Lemma 2.1.8]); in particular, we have a decomposition of the identity similar to (2.1). The outer volume ratio of a convex body K in $\mathbb{R}^n$ is the quantity

$$\mathrm{ovr}(K) = \inf\left\{ \left(\frac{|\mathcal{E}|}{|K|}\right)^{1/n} : \mathcal{E} \text{ is an ellipsoid and } K \subseteq \mathcal{E} \right\}.$$

If K is in Löwner's position then $(|B_2^n|/|K|)^{1/n} = \mathrm{ovr}(K)$.

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. It is known that every convex body has an isotropic affine image, and if K is isotropic then

$$cL_K\, B_2^n \subseteq K \subseteq (n+1)L_K\, B_2^n,$$

where $c > 0$ is an absolute constant. A simple proof of the left hand side inclusion is given in [9, Section 3.2.1], while the right hand side inclusion was proved in [17]. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 1$. Bourgain proved in [7] that $L_n \leqslant c\sqrt[4]{n}\,\log n$, while Klartag [18] obtained the bound $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [19]. We refer the reader to the article of V. Milman and Pajor [21] and to the book [9] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. The L_q-centroid body $Z_q(K)$ of K is the centrally symmetric convex body with support function

$$h_{Z_q(K)}(y) = \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}.$$

Note that K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in SL(n)$ then $Z_q(T(K)) = T(Z_q(K))$. From Hölder's inequality it follows that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all

$1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}(K, -K)$. Using Borell's lemma (see [9, Chapter 1]) one can check that

$$Z_q(K) \subseteq \bar{c}_1 \frac{q}{p} Z_p(K) \tag{2.2}$$

for all $1 \leqslant p < q$, where $\bar{c}_1 > 0$ is an absolute constant. In particular, if K is isotropic then $R(Z_q(K)) \leqslant \bar{c}_1 q L_K$. One can also check that if K is centered, then $Z_q(K) \supseteq c_2 Z_\infty(K)$ for all $q \geqslant n$. For a proof of all these assertions see [9, Chapter 5]. The class of L_q-centroid bodies of K was introduced (with a different normalization) by Lutwak, Yang and Zhang in [20]. An asymptotic approach to this family was developed by Paouris in [22] and [23].

For the proof of Theorem 1.2 we generalize arguments from [10] where L_q-centroid bodies are used in order to describe the asymptotic shape of the absolute convex hull of N random points chosen from a convex body. The use of one-sided L_q-centroid bodies allows one to consider the convex hull itself.

3 Random approximation of convex bodies

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ we consider the one-sided L_q-centroid body $Z_q^+(K)$ of K with support function

$$h_{Z_q^+(K)}(y) = \left(2 \int_K \langle x, y\rangle_+^q \, dx\right)^{1/q},$$

where $a_+ = \max\{a, 0\}$. In a dual form, the one-sided L_q-centroid bodies were introduced in [15]. When K is symmetric, it is clear that $Z_q^+(K) = Z_q(K)$. In any case, we easily verify that

$$Z_q^+(K) \subseteq 2^{1/q} Z_q(K).$$

Note that $Z_q^+(K) \subseteq 2^{1/q} K$ for all $q \geqslant 1$. One can check that if $1 \leqslant q \leqslant r < \infty$ then

$$\left(\frac{2}{e}\right)^{\frac{1}{q} - \frac{1}{r}} Z_q^+(K) \subseteq Z_r^+(K) \subseteq \frac{Cr}{q} \left(\frac{2e-2}{e}\right)^{\frac{1}{q} - \frac{1}{r}} Z_q^+(K), \tag{3.1}$$

where $C > 0$ is an absolute constant. This double inclusion is stated as (2.3) in [14, Section 2] and is the analogue of (2.2). One can verify it following the proof of (2.2) and using Grünbaum's lemma (see [1, Proposition 1.5.16]). The next lemma is also due to Guédon and E. Milman.

Lemma 3.1. *There exists an absolute constant $\bar{c}_0 > 0$ such that, for every isotropic convex body K in $\mathbb{R}^n$,*

$$Z_2^+(K) \supseteq \bar{c}_0 L_K B_2^n.$$

Equivalently, for any $\theta \in S^{n-1}$,

$$h_{Z_2^+(K)}(\theta) = \left(2 \int_K \langle x, y\rangle_+^2 \, dx\right)^{1/2} \geqslant \bar{c}_0 L_K.$$

Finally, we need the next lemma, which appears in [14] (see also [9, Theorem 13.2.7]).

Lemma 3.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $\theta \in S^{n-1}$,*

$$\left(\frac{2}{e^2}\right)^{1/q} \left(\frac{\Gamma(n)\Gamma(q+1)}{\Gamma(n+q+1)}\right)^{1/q} h_K(\theta) \leqslant h_{Z_q^+(K)}(\theta) \leqslant 2^{1/q} h_K(\theta).$$

Proof. We sketch the proof of the left hand side inequality. Let

$$H_\theta^+ = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \geqslant 0\}, \quad H_\theta(t) = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = t\},$$

and

$$f_\theta(t) = |K \cap H_\theta(t)|.$$

First observe that, by the Brunn-Minkowski inequality, $f_\theta^{\frac{1}{n-1}}$ is concave on its support, and hence we have

$$f_\theta(t) \geqslant \Big(1 - \frac{t}{h_K(\theta)}\Big)^{n-1} f_\theta(0)$$

for all $t \in [0, h_K(\theta)]$. Therefore,

$$\begin{aligned} h^q_{Z_q^+(K)}(\theta) =& 2\int_0^{h_K(\theta)} t^q f_\theta(t)dt \geqslant 2\int_0^{h_K(\theta)} t^q \Big(1 - \frac{t}{h_K(\theta)}\Big)^{n-1} f_\theta(0)dt \\ &= 2f_\theta(0) h_K^{q+1}(\theta) \int_0^1 s^q(1-s)^{n-1}ds \\ &= \frac{\Gamma(n)\Gamma(q+1)}{\Gamma(q+n+1)} 2f_\theta(0)h_K^{q+1}(\theta). \end{aligned}$$

Observe that

$$2f_\theta(0)h_K(\theta) = \frac{f_\theta(0)}{\|f_\theta\|_\infty} 2\|f_\theta\|_\infty h_K(\theta) \geqslant \frac{f_\theta(0)}{\|f_\theta\|_\infty} (2|K \cap H_\theta^+|).$$

We know that $\|f_\theta\|_\infty \leqslant e f_\theta(0)$ by a result of Fradelizi (see e.g. [9, Theorem 2.2.2]) and that $|K \cap H_\theta^+| \geqslant e^{-1}$ by Grünbaum's lemma (see [1, Proposition 1.5.16]). Combining the above we get the result. □

Theorem 1.2 (and thus Theorem 1.1) will follow from the next fact, which generalizes the work of Dafnis, Giannopoulos and Tsolomitis [10] to the not necessarily symmetric setting.

Theorem 3.3. *Let $\beta \in (0,1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on β and absolute constants $c_1, c_2 > 0$ with the following property: if K is a centered convex body of volume 1 in $\mathbb{R}^n$, $N \geqslant \alpha n$, and $x_1, \ldots, x_N$ are independent random points uniformly distributed in K then for $q = c_1 \beta \log(N/n)$ the inclusion*

$$\text{conv}(\{x_1, \ldots, x_N\}) \supseteq c_2 Z_q^+(K) \tag{3.2}$$

holds with probability greater than $1 - e^{-N^{1-\beta}n^\beta}$.

Our proof of (3.2) uses the family of one-sided L_q-centroid bodies of K. In particular, we need the following estimate (the idea of the proof can be traced back in [10]; see also [24]).

Lemma 3.4. *There exists an absolute constant $C > 1$ with the following property: for every $n \geqslant 1$, every centered convex body K of volume 1 in $\mathbb{R}^n$ and every $q \geqslant 2$,*

$$\inf_{\theta \in S^{n-1}} \mu_K\left(\left\{x : \langle x, \theta \rangle > \tfrac{1}{2} h_{Z_q^+(K)}(\theta)\right\}\right) \geqslant C^{-q},$$

where μ_K is the Lebesgue measure on K.

Proof. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, let $q \geqslant 2$ and let $\theta \in S^{n-1}$. We apply the Paley-Zygmund inequality

$$\mathbb{P}\left(g \geqslant t\mathbb{E}(g)\right) \geqslant (1-t)^2 \frac{[\mathbb{E}(g)]^2}{\mathbb{E}(g^2)} \tag{3.3}$$

for the non-negative random variable

$$g_\theta(x) = 2\langle x, \theta\rangle_+^q$$

on (K, μ_K). Applying (3.1) with $r = 2q$ we see that

$$\mathbb{E}\,(g_\theta^2) = h^{2q}_{Z^+_{2q}(K)}(\theta) \leqslant C_1^q h^{2q}_{Z^+_q(K)}(\theta) = C_1^q\,[\mathbb{E}\,(g_\theta)]^2,$$

where $C_1 > 0$ is an absolute constant. From (3.3) we get

$$\begin{aligned}\mu_K(\{x : \langle x,\theta\rangle > t\, h_{Z^+_q(K)}(\theta)\}) &= \mu_K(\{x : \langle x,\theta\rangle > t\,[\mathbb{E}\,(g_\theta)]^{1/q}\}) = \mu_K(\{x : \langle x,\theta\rangle_+ > t\,[\mathbb{E}\,(g_\theta)]^{1/q}\}) \\ &= \mu_K(\{x : \langle x,\theta\rangle_+^q > t^q\,\mathbb{E}\,(g_\theta)\}) = \mu_K(\{x : g_\theta(x) > 2t^q\,\mathbb{E}\,(g_\theta)\}) \\ &\geqslant (1-2t^q)^2\,\frac{[\mathbb{E}\,(g_\theta)]^2}{\mathbb{E}\,(g_\theta^2)} \geqslant \frac{(1-2t^q)^2}{C_1^q}\end{aligned}$$

for every $t \in (0, 2^{-\frac{1}{q}})$. Choosing $t = \frac{1}{2}$ we get the lemma with $C = 4C_1$. □

Proof of Theorem 3.3. Let $q \geqslant 2$ and consider the random polytope $C_N := \mathrm{conv}\{x_1,\dots,x_N\}$. With probability equal to one, C_N has non-empty interior and, for every $J = \{j_1,\dots,j_n\} \subset \{1,\dots,N\}$, the points $x_{j_1},\dots,x_{j_n}$ are affinely independent. Write H_J for the affine subspace determined by $x_{j_1},\dots,x_{j_n}$ and H_J^+, H_J^- for the two closed halfspaces whose bounding hyperplane is H_J.

If $\frac{1}{2}Z_q^+(K) \not\subseteq C_N$, then there exists $x \in \frac{1}{2}Z_q^+(K) \setminus C_N$, and hence, there is a facet of C_N defining some affine subspace H_J as above that satisfies the following: either $x \in H_J^-$ and $C_N \subset H_J^+$, or $x \in H_J^+$ and $C_N \subset H_J^-$. Observe that, for every J, the probability of each of these two events is bounded by

$$\Big(\sup_{\theta\in S^{n-1}} \mu_K\Big(\Big\{x : \langle x,\theta\rangle \leqslant \tfrac{1}{2} h_{Z_q^+(K)}(\theta)\Big\}\Big)\Big)^{N-n} \leqslant \big(1 - C^{-q}\big)^{N-n},$$

where $C > 0$ is the constant in Lemma 3.4. It follows that

$$\mathbb{P}\left(\tfrac{1}{2}Z_q^+(K) \not\subseteq C_N\right) \leqslant 2\binom{N}{n}(1-C^{-q})^{N-n}.$$

Since $\binom{N}{n} \leqslant \left(\frac{eN}{n}\right)^n$, this probability is smaller than $\exp(-N^{1-\beta}n^\beta)$ if

$$\left(\frac{2eN}{n}\right)^n (1-C^{-q})^{N-n} < \left(\frac{2eN}{n}\right)^n e^{-C^{-q}(N-n)} < \exp(-N^{1-\beta}n^\beta),$$

and the second inequality is satisfied if

$$\frac{N}{n} - 1 > C^q\left[\left(\frac{N}{n}\right)^{1-\beta} + \log\left(\frac{2eN}{n}\right)\right]. \tag{3.4}$$

We choose $q = \frac{\beta}{2\log C}\log\left(\frac{N}{n}\right)$ and $\alpha_1(\beta) := C^{4/\beta}$. Note that if $N \geqslant \alpha_1(\beta)n$ then $q \geqslant 2$ and that (3.4) becomes

$$\frac{N}{n} - 1 > \left(\frac{N}{n}\right)^{1-\frac{\beta}{2}} + \left(\frac{N}{n}\right)^{\frac{\beta}{2}}\log\left(\frac{2eN}{n}\right). \tag{3.5}$$

Since

$$\lim_{t\to+\infty}\left[t - 1 - t^{1-\frac{\beta}{2}} - t^{\frac{\beta}{2}}\log(2et)\right] = +\infty,$$

we may find $\alpha_2(\beta)$ such that (3.5) is satisfied for all $N \geqslant \alpha_2(\beta)n$. Setting $\alpha = \max\{\alpha_1(\beta),\alpha_2(\beta)\}$ we see that the assertion of the theorem is satisfied with probability greater that $1 - e^{-N^{1-\beta}n^\beta}$ for all $N \geqslant \alpha n$, with $c_1 = \frac{1}{2\log C}$ and $c_2 = \frac{1}{2}$. □

Proof of Theorem 1.2. Let $\beta \in (0,1)$ and let $\alpha = \alpha(\beta)$ be the constant from Theorem 3.3. Let $\alpha n \leqslant N \leqslant e^n$ and let $x_1, \dots, x_N$ be independent random points uniformly distributed in K. Applying Lemma 3.2 with $q = n$ we see that $h_{Z_n^+(K)} \geqslant c_1 h_K(\theta)$ for all $\theta \in S^{n-1}$, and hence

$$Z_n^+(K) \supseteq c_1 K,$$

where $c_1 > 0$ is an absolute constant. From Theorem 3.3 we know that if $q = c_2 \beta \log(N/n)$ (note also that $q \leqslant n$) then

$$C_N = \mathrm{conv}(\{x_1, \dots, x_N\}) \supseteq c_3 Z_q^+(K)$$

with probability greater than $1 - \exp(-N^{1-\beta} n^{\beta})$, where $c_2, c_3 > 0$ are absolute constants. From (3.1) we see that

$$Z_n^+(K) \subseteq \frac{c_4 n}{q} \left(\frac{2e-2}{e}\right)^{\frac{1}{q}-\frac{1}{n}} Z_q^+(K) \subseteq \frac{2c_4 n}{q} Z_q^+(K),$$

where $c_4 > 0$ is an absolute constant. Combining the above we get that

$$C_N = \mathrm{conv}(\{x_1, \dots, x_N\}) \supseteq \frac{c_5 q}{n} K \supseteq \frac{c_6 \beta \log(N/n)}{n} K$$

with probability greater than $1 - \exp(-N^{1-\beta} n^{\beta})$, where $c_5, c_6 > 0$ are absolute constants. □

4 Generalized vertex index

Let K be a convex body in $\mathbb{R}^n$. From the definition of the vertex index that we gave in the introduction, we may clearly assume that K is centered, and then

$$\mathrm{vi}(K) = \inf\Big\{ \sum_{j=1}^{N} p_K(y_j) : K \subseteq \mathrm{conv}(\{y_1, \dots, y_N\}) \Big\},$$

where p_K is the Minkowski functional of K. Since every origin symmetric convex body is centered, our definition coincides with the one given by Bezdek and Litvak in [6] for the symmetric case.

It is also easy to check that the vertex index is invariant under invertible linear transformations. For every convex body K in $\mathbb{R}^n$ and any $T \in GL(n)$ one has

$$\mathrm{vi}(T(K)) = \mathrm{vi}(K).$$

Another useful observation is that the vertex index is stable under a variant of the Banach-Mazur distance. Recall that the Banach-Mazur distance between two convex bodies K and L in $\mathbb{R}^n$ is the quantity

$$d(K,L) = \inf\{t > 0 : T(L+y) \subseteq K + x \subseteq t(T(L+y))\},$$

where the infimum is over all $T \in GL(n)$ and $x, y \in \mathbb{R}^n$. Given two centered convex bodies K and L, we set

$$\tilde{d}(K,L) = \inf\{t > 0 : T(L) \subseteq K \subseteq tT(L)\},$$

where the infimum is over all $T \in GL(n)$. Note that if K and L are symmetric convex bodies then $\tilde{d}(K,L) = d(K,L)$. With this definition we easily check that if K and L are centered convex bodies in $\mathbb{R}^n$ then

$$\mathrm{vi}(K) \leqslant \tilde{d}(K,L)\, \mathrm{vi}(L).$$

The main result of this section is the upper bound in Theorem 1.3.

Proof of Theorem 1.3. We may assume that K is isotropic. By Theorem 1.1 we can find $N \leqslant \alpha n$ and $x_1, \dots, x_N \in K$ such that

$$K \subseteq Cn\, \mathrm{conv}(\{x_1, \dots, x_N\}),$$

where $\alpha, C > 0$ are absolute constants. We set $y_j = Cnx_j$, $1 \leqslant j \leqslant N$. Then, $K \subseteq \mathrm{conv}(\{y_1, \ldots, y_N\})$ and $p_K(y_j) = Cnp_K(x_j) \leqslant Cn$, therefore

$$\mathrm{vi}(K) \leqslant \sum_{j=1}^{N} p_K(y_j) \leqslant CnN \leqslant C\alpha n^2.$$

The result follows with $C_1 = C\alpha$.

For the lower bound we just check that the argument of [6] remains valid in the not necessarily symmetric case. By the linear invariance of the vertex index we may assume that B_2^n is the ellipsoid of minimal volume which contains $\mathrm{conv}(K, -K)$. In other words, $K \subseteq \mathrm{conv}(K, -K) \subseteq B_2^n$ and

$$\left(\frac{|B_2^n|}{|\mathrm{conv}(K, -K)|} \right)^{1/n} = \mathrm{ovr}(\mathrm{conv}(K, -K)).$$

For any $N \in \mathbb{N}$ and $y_1, \ldots, y_N$ such that $K \subseteq \mathrm{conv}(\{y_1, \ldots, y_N\})$ we consider the absolute convex hull $Q = \mathrm{conv}(\{\pm y_1, \ldots, \pm y_N\}) \supseteq \mathrm{conv}(K, -K)$ of $y_1, \ldots, y_N$. Then,

$$Q^\circ = \{x \in \mathbb{R}^n : |\langle x, y_j \rangle| \leqslant 1 \text{ for all } j = 1, \ldots, N\},$$

and a result of Ball and Pajor [2] provides the lower bound

$$|Q^\circ| \geqslant \left(\frac{n}{\sum_{j=1}^{N} \|y_j\|_2} \right)^{1/n}$$

for its volume. Using the Blaschke-Santaló inequality we get

$$|\mathrm{conv}(K, -K)| \leqslant |Q| \leqslant \frac{|B_2^n|^2}{|Q^\circ|} \leqslant |B_2^n|^2 \left(\frac{\sum_{j=1}^{N} \|y_j\|_2}{n} \right)^n.$$

It follows that

$$1 \leqslant \left(\frac{|B_2^n|}{|\mathrm{conv}(K, -K)|} \right)^{1/n} |B_2^n|^{1/n} \frac{\sum_{j=1}^{N} \|y_j\|_2}{n} \leqslant \frac{\mathrm{ovr}(\mathrm{conv}(K, -K)))}{cn^{3/2}} \sum_{j=1}^{N} \|y_j\|_2$$

for some absolute constant $c > 0$. Since $K \subseteq B_2^n$, we have $\|y_j\|_2 \leqslant p_K(y_j)$ for all $j = 1, \ldots, N$. Therefore,

$$\sum_{j=1}^{N} p_K(y_j) \geqslant \frac{cn^{3/2}}{\mathrm{ovr}(\mathrm{conv}(K, -K))},$$

and taking the infimum over all N and all such N-tuples $(y_1, \ldots, y_N)$ we get the lower bound for $\mathrm{vi}(K)$. $\square$

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