Asymptotic shape of the convex hull of isotropic log-concave random vectors

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Abstract

Let $x_1, \dots, x_N$ be independent random points distributed according to an isotropic log-concave measure μ on $\mathbb{R}^n$, and consider the random polytope

$$K_N := \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}.$$

We provide sharp estimates for the quermaßintegrals and other geometric parameters of K_N in the range $cn \leqslant N \leqslant \exp(n)$; these complement previous results from [13] and [14] that were given for the range $cn \leqslant N \leqslant \exp(\sqrt{n})$. One of the basic new ingredients in our work is a recent result of E. Milman that determines the mean width of the centroid body $Z_q(\mu)$ of μ for all $1 \leqslant q \leqslant n$.

1 Introduction

The purpose of this work is to add new information on the asymptotic shape of random polytopes whose vertices have a log-concave distribution. Without loss of generality we shall assume that this distribution is also isotropic. Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its center of mass is at the origin, and its inertia matrix is a multiple of the identity: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \tag{1.1}$$

for every θ in the Euclidean unit sphere S^{n-1}. More generally, a log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if its center of mass is at the origin and its inertia matrix is the identity; in this case, the isotropic constant of μ is defined as

$$L_\mu := \sup_{x \in \mathbb{R}^n} \big(f_\mu(x)\big)^{1/n}, \tag{1.2}$$

where f_μ is the density of μ with respect to the Lebesgue measure. Note that a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic if and only if the log-concave probability measure μ_K with density $x \mapsto L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic.

A very well-known open question in the theory of isotropic measures is the hyperplane conjecture, which asks if there exists an absolute constant $C > 0$ such that

$$L_n := \sup\{L_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C \tag{1.3}$$

for all $n \geqslant 1$. Bourgain proved in [9] that $L_n \leqslant c\sqrt[4]{n}\log n$ (more precisely, he showed that $L_K \leqslant c\sqrt[4]{n}\log n$ for every isotropic symmetric convex body K in $\mathbb{R}^n$), while Klartag [18] obtained the bound $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's estimate appears in [20].

The study of the asymptotic shape of random polytopes whose vertices have a log-concave distribution was initiated in [13] and [14]. Given an isotropic log-concave measure μ on $\mathbb{R}^n$, for every $N \geqslant n$ we consider N independent random points $x_1, \dots, x_N$ distributed according to μ and define the random polytope

$K_N := \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}$. The main idea in these works was to compare K_N with the L_q-centroid body of μ for a suitable value of q; roughly speaking, K_N is close to the body $Z_{\log(2N/n)}(\mu)$ with high probability. Recall that the L_q-centroid bodies $Z_q(\mu)$, $q \geqslant 1$, are defined through their support function $h_{Z_q(\mu)}$, which is given by

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y\rangle\|_{L_q(\mu)} = \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^q d\mu(x)\right)^{1/q}. \tag{1.4}$$

These bodies incorporate information about the distribution of linear functionals with respect to μ. The L_q-centroid bodies were introduced, under a different normalization, by Lutwak and Zhang in [23], while in [29] for the first time, and in [30] later on, Paouris used geometric properties of them to acquire detailed information about the distribution of the Euclidean norm with respect to μ.

It was proved in [13] that, given any isotropic log-concave measure μ on $\mathbb{R}^n$ and any $cn \leqslant N \leqslant e^n$, the random polytope K_N defined by N independent random points $x_1, \ldots, x_N$ which are distributed according to μ satisfies, with high probability, the inclusion

$$K_N \supseteq c_1 Z_{\log(N/n)}(\mu) \tag{1.5}$$

(for the precise statement see Fact 3.2). Then, using the fact that the volume of the L_q-centroid bodies satisfies the lower bounds $|Z_q(\mu)|^{1/n} \geqslant c_2\sqrt{q/n}$ if $q \leqslant \sqrt{n}$ and $|Z_q(\mu)|^{1/n} \geqslant c_3 L_\mu^{-1}\sqrt{q/n}$ if $\sqrt{n} \leqslant q \leqslant n$ (see Section 2), we see that for $n \leqslant N \leqslant e^{\sqrt{n}}$ we have

$$|K_N|^{1/n} \geqslant c_4 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, \tag{1.6}$$

while in the range $e^{\sqrt{n}} \leqslant N \leqslant e^n$ we have

$$|K_N|^{1/n} \geqslant c_5 L_\mu^{-1} \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \tag{1.7}$$

with probability exponentially close to 1. On the other hand, one can check that for every $\alpha > 1$ and $q \geqslant 1$,

$$\mathbb{E}\left[\sigma_n(\{\theta : h_{K_N}(\theta) \geqslant \alpha h_{Z_q(\mu)}(\theta)\})\right] \leqslant N\alpha^{-q}, \tag{1.8}$$

where σ_n is the rotationally invariant probability measure on the Euclidean unit sphere S^{n-1}. This estimate is sufficient for some sharp upper bounds. First, for all $n \leqslant N \leqslant \exp(n)$ one has

$$\mathbb{E}\left[w(K_N)\right] \leqslant c_6\, w(Z_{\log N}(\mu)), \tag{1.9}$$

where the mean width $w(C)$ of a convex body C in $\mathbb{R}^n$ containing the origin, is defined as twice the average of its support function on S^{n-1}:

$$w(C) = \int_{S^{n-1}} h_C(\theta)\, d\sigma_n(\theta).$$

Second, one has

$$|K_N|^{1/n} \leqslant c_7 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \tag{1.10}$$

with probability greater than $1 - \frac{1}{N}$, where $C > 0$ is an absolute constant.

In [14] these results were extended to the full family of quermaßintegrals $W_{n-k}(K_N)$ of K_N. These are defined through Steiner's formula

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_{n-k}(K) t^{n-k}, \tag{1.11}$$

where $W_{n-k}(K)$ is the mixed volume $V(K,k;B_2^n,n-k)$. It is more convenient to express the estimates using a normalized variant of $W_{n-k}(K)$: for every $1\leqslant k\leqslant n$ we set

$$Q_k(K)=\left(\frac{W_{n-k}(K)}{\omega_n}\right)^{1/k}=\left(\frac{1}{\omega_k}\int_{G_{n,k}}|P_F(K)|\,d\nu_{n,k}(F)\right)^{1/k}, \tag{1.12}$$

where the last equality follows from Kubota's integral formula (see Section 2 for background information on mixed volumes). Then, one has the following results on the expectation of $Q_k(K_N)$ for all values of k:

Theorem 1.1 (Dafnis, Giannopoulos and Tsolomitis, [14]). *If $n^2\leqslant N\leqslant \exp(cn)$ then for every $1\leqslant k\leqslant n$ we have*

$$L_\mu^{-1}\sqrt{\log N}\lesssim \mathbb{E}\left[Q_k(K_N)\right]\lesssim w(Z_{\log N}(K)). \tag{1.13}$$

In the range $n^2\leqslant N\leqslant\exp(\sqrt{n})$ one has an asymptotic formula: for every $1\leqslant k\leqslant n$,

$$\mathbb{E}\left[Q_k(K_N)\right]\simeq\sqrt{\log N}. \tag{1.14}$$

All these estimates remain valid for $n^{1+\delta}\leqslant N\leqslant n^2$, where $\delta\in(0,1)$ is fixed, if we allow the constants to depend on δ. Working in the range $N\simeq n$ is possible, but requires some additional attention (see e.g. [5] for the case of mean width).

A more careful analysis (which can be found in [14, Theorem 1.2]) shows that if $n^2\leqslant N\leqslant\exp(\sqrt{n})$ then, for any $s\geqslant 1$, a random K_N satisfies, with probability greater than $1-N^{-s}$,

$$Q_k(K_N)\leqslant c_1(s)\sqrt{\log N} \tag{1.15}$$

for all $1\leqslant k\leqslant n$ and, with probability greater than $1-\exp(-\sqrt{n})$,

$$Q_k(K_N)\geqslant c_8\sqrt{\log N} \tag{1.16}$$

for all $1\leqslant k\leqslant n$, where $c_1(s)>0$ depends only on s, and $c_8>0$ is an absolute constant.

A natural question that arises is whether these results can be extended to the full range $cn\leqslant N\leqslant\exp(n)$ of values of N. If one decides to follow the approach of [13] and [14] then there are two main obstacles. The first one is that the lower bound $|Z_q(\mu)|^{1/n}\geqslant c\sqrt{q/n}$ is currently known only in the range $q\leqslant\sqrt{n}$. In fact, proving the same for larger values of q would lead to improved estimates on L_n (for example, see the computation after Lemma 2.2 in [20]). The second one was that, until recently, a sharp estimate on the mean width of $Z_q(\mu)$ was known only for $q\leqslant\sqrt{n}$; G. Paouris proved in [29] that for every isotropic log-concave measure μ on $\mathbb{R}^n$ and any $q\leqslant\sqrt{n}$ one has

$$w\big(Z_q(K)\big)\leqslant c_9\sqrt{q}. \tag{1.17}$$

Recently, E. Milman [25] obtained the same upper bound (modulo logarithmic terms) for q beyond $\sqrt{n}$.

Theorem 1.2 (E. Milman, [25]). *For every isotropic log-concave measure μ on $\mathbb{R}^n$ and for all $q\in[\sqrt{n},n]$ we have*

$$w(Z_q(\mu))\leqslant c_{10}\sqrt{q}\ \log^2(1+q). \tag{1.18}$$

An immediate consequence of this result is that it provides a new bound for the mean width of an origin symmetric isotropic convex body K in $\mathbb{R}^n$. In this case it is known that $Z_n(K)\supseteq cK$, and we conclude that

$$w(K)\leqslant C_1\sqrt{n}\ \log^2(1+n)L_K \tag{1.19}$$

improving the earlier known bound $w(K)\leqslant C_2n^{3/4}L_K$ of Hartzoulaki, from her PhD thesis [17]. We note here that not all of the logarithmic terms in (1.19) can be removed, as the example of $B_1^n/|B_1^n|^{1/n}$ shows.

Using E. Milman's theorem we can show the following.

Theorem 1.3. *Let $x_1,\ldots,x_N$ be independent random points distributed according to an isotropic log-concave measure μ on $\mathbb{R}^n$, and consider the random polytope $K_N := \mathrm{conv}\{\pm x_1,\ldots,\pm x_N\}$. If $\exp(\sqrt{n}) \leqslant N \leqslant \exp(cn)$ then for every $1 \leqslant k \leqslant n$ we have*

$$L_\mu^{-1}\sqrt{\log N} \lesssim \mathbb{E}\left[Q_k(K_N)\right] \lesssim \sqrt{\log N}\big(\log\log N\big)^2. \tag{1.20}$$

Next we provide estimates for $Q_k(K_N)$ for "most" K_N:

Theorem 1.4. *Let $x_1,\ldots,x_N$ be independent random points distributed according to an isotropic log-concave measure μ on $\mathbb{R}^n$, and consider the random polytope $K_N := \mathrm{conv}\{\pm x_1,\ldots,\pm x_N\}$. For all $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ and $s \geqslant 1$ we have*

$$Q_k(K_N) \leqslant c_2(s)\sqrt{\log N}\ (\log\log N)^2, \tag{1.21}$$

for all $1 \leqslant k < n$, with probability greater than $1 - N^{-s}$.

We also provide estimates on the volume radius of a random projection $P_F(K_N)$ of K_N onto $F \in G_{n,k}$ (in terms of n, k and N) in the range $e^{\sqrt{n}} \leqslant N \leqslant e^n$; these extend the sharp estimate $\mathrm{v.rad}(P_F(K_N)) \simeq \sqrt{\log N}$ that was obtained in [14] for the case $N \leqslant e^{\sqrt{n}}$.

Theorem 1.5. *If $\exp(\sqrt{n}) \leqslant N \leqslant e^{cn}$ and $s \geqslant 1$, then a random K_N satisfies with probability greater than $1 - \max\{N^{-s}, e^{-c_{11}\sqrt{N}}\}$ the following: for every $1 \leqslant k \leqslant n$ there exists a subset $M_{n,k}$ of $G_{n,k}$ with $\nu_{n,k}(M_{n,k}) \geqslant 1 - e^{-c_{12}k}$ such that*

$$\begin{aligned} c_{13}L_\mu^{-1}\sqrt{\log N} \leqslant \mathrm{v.rad}(P_F(K_N)) := \left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \\ \leqslant c_3(s)\sqrt{\log N}\big(\log\log N\big)^2 \end{aligned} \tag{1.22}$$

for all $F \in M_{n,k}$.

In Section 4 we provide an alternative proof of an estimate of Alonso-Gutiérrez, Dafnis, Hernández-Cifre and Prochno from [3] on the k-th mean outer radius

$$\tilde{R}_k(K_N) = \int_{G_{n,k}} R(P_F(K_N))\, d\nu_{n,k}(F) \tag{1.23}$$

of a random K_N, as a function of N, n and k.

Theorem 1.6. *Let $x_1,\ldots,x_N$ be independent random points distributed according to an isotropic log-concave measure μ on $\mathbb{R}^n$, and consider the random polytope $K_N := \mathrm{conv}\{\pm x_1,\ldots,\pm x_N\}$. If $n \leqslant N \leqslant \exp(\sqrt{n})$ then, for all $1 \leqslant k \leqslant n$ and $s > 0$ one has*

$$c_4(s)\max\left\{\sqrt{k}, \sqrt{\log(N/n)}\right\} \leqslant \tilde{R}_k(K_N) \leqslant c_5(s)\max\left\{\sqrt{k}, \sqrt{\log N}\right\} \tag{1.24}$$

with probability greater than $1 - N^{-s}$, where $c_4(s), c_5(s)$ are positive constants depending only on s.

We provide a formula for $\tilde{R}_k(K_N)$ which is valid for all $cn \leqslant N \leqslant \exp(n)$. This allows us to recover (and explain) the sharp estimate of Theorem 1.6 for "small" values of N and to obtain its analogue for "large" values of N; see Theorem 4.5.

In Section 5 we obtain estimates on the regularity of the covering numbers and the dual covering numbers of a random K_N. In a certain range of values of N, these allow us to conclude that a random K_N is in α-regular M-position with $\alpha \sim 1$ (see Section 5 for definitions and terminology).

Theorem 1.7. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Then, assuming that $n^2 \leqslant N \leqslant \exp\big((n\log n)^{2/5}\big)$, we have that a random K_N satisfies with probability greater than $1-N^{-1}$ the entropy estimates*

$$\max\{\log N(K_N, tr_N B_2^n), \log N(r_N B_2^n, tK_N)\} \leqslant c_{14}\frac{n(\log n)^2\log(1+t)}{t}$$

for every $t\geqslant 1$, where $r_N=\sqrt{\log N}$ and $c_{14}>0$ is an absolute constant.

As an application we estimate the average diameter of k-dimensional sections of a random K_N, defined by

$$\tilde{D}_k(K_N) = \int_{G_{n,k}} R(K_N\cap F)\,d\nu_{n,k}(F). \tag{1.25}$$

The discussion shows that the behavior of $\tilde{D}_k(K_N)$ is not always the same as that of $\tilde{R}_k(K_N)$. In order to give an idea of the results, let us mention here the following simplified version.

Theorem 1.8. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$ and $a,b\in(0,1)$.*

(i) *If $k\leqslant bn$ then a random K_N satisfies with probability $1-N^{-1}$*

$$\tilde{D}_k(K_N)\leqslant c_b\sqrt{\log N} \quad \textit{if } n^2\leqslant N\leqslant \exp(\sqrt{n})$$

and

$$\tilde{D}_k(K_N)\leqslant c_b\sqrt{\log N}(\log\log N)^2 \quad \textit{if } \exp(\sqrt{n})\leqslant N\leqslant \exp(n).$$

(ii) *If $k\geqslant an$ and $N\leqslant \exp((n\log n)^{2/5})$ then a random K_N satisfies with probability $1-\exp(-\sqrt{n})$*

$$c_a\frac{\sqrt{\log N}}{\log^3 n}\leqslant \tilde{D}_k(K_N).$$

where c_a, c_b are positive constants that depend only on a and b respectively.

We conclude this paper with a brief discussion of the interesting (open) question whether the isotropic constant of a random K_N is bounded by a constant independent from n and N. The first class of random polytopes K_N in $\mathbb{R}^n$ for which uniform bounds were established was the class of Gaussian random polytopes. Klartag and Kozma proved in [19] that if $N>n$ and if $G_1,\dots,G_N$ are independent standard Gaussian random vectors in $\mathbb{R}^n$, then the isotropic constant of the random polytope $K_N=\mathrm{conv}\{\pm G_1,\dots,\pm G_N\}$ is bounded by an absolute constant $C>0$ with probability greater than $1-Ce^{-cn}$. The same idea works in the case where the vertices x_j of K_N are distributed according to an isotropic ψ_2-measure μ; the bound then depends only on the ψ_2-constant of μ. Alonso-Gutiérrez [2] and Dafnis, Giannopoulos and Guédon [12] have applied the same more or less method to obtain a positive answer in the case where the vertices of K_N are chosen from the unit sphere or an unconditional isotropic convex body respectively. We show that, in the general isotropic log-concave case, the method of Klartag and Kozma gives the bound $O(\sqrt{\log(2N/n)})$ if $N\leqslant\exp(\sqrt{n})$ (a proof along the same lines and an extension to random perturbations of random polytopes appear in [4]).

2 Notation and background material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ_n for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $1\leqslant k\leqslant n$ and $F\in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F=B_2^n\cap F$ and $S_F=S^{n-1}\cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Similarly, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$. We also write $\overline{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\overline{A} := \frac{A}{|A|^{1/n}}$.

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $-x \in C$ whenever $x \in C$. We say that C is centered if it has center of mass at the origin i.e. $\int_C \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. For each $-\infty < p < \infty$, $p \neq 0$, we define the p-mean width of C by

$$w_p(C) := \left(\int_{S^{n-1}} h_C^p(\theta) d\sigma_n(\theta) \right)^{1/p}. \tag{2.1}$$

The mean width of C is the quantity $w(C) = w_1(C)$. The radius of C is defined as $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } x \in C\}. \tag{2.2}$$

Finally, if C is a symmetric convex body in $\mathbb{R}^n$ and $\|\cdot\|_C$ is the norm induced to $\mathbb{R}^n$ by C, we set

$$M(C) = \int_{S^{n-1}} \|x\|_C d\sigma_n(x)$$

and write $b(C)$ for the smallest positive constant b with the property $\|x\|_C \leqslant b\|x\|_2$ for all $x \in \mathbb{R}^n$. From V. Milman's proof of Dvoretzky's theorem (see [6, Chapter 5]) we know that if $k \leqslant cn(M(C)/b(C))^2$ then for most $F \in G_{n,k}$ we have $C \cap F \simeq \frac{1}{M(C)} B_F$.

2.1 Quermaßintegrals

Let $\mathcal{K}_n$ denote the class of non-empty compact convex subsets of $\mathbb{R}^n$. The relation between volume and the operations of addition and multiplication of compact convex sets by nonnegative reals is described by Minkowski's fundamental theorem: If $K_1, \ldots, K_m \in \mathcal{K}_n$, $m \in \mathbb{N}$, then the volume of $t_1 K_1 + \cdots + t_m K_m$ is a homogeneous polynomial of degree n in $t_i \geqslant 0$:

$$|t_1 K_1 + \cdots + t_m K_m| = \sum_{1 \leqslant i_1, \ldots, i_n \leqslant m} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} \cdots t_{i_n}, \tag{2.3}$$

where the coefficients $V(K_{i_1}, \ldots, K_{i_n})$ can be chosen to be invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is called the mixed volume of the n-tuple $(K_{i_1}, \ldots, K_{i_n})$.

Steiner's formula is a special case of Minkowski's theorem; if K is a convex body in $\mathbb{R}^n$ then the volume of $K + tB_2^n$, $t > 0$, can be expanded as a polynomial in t:

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_{n-k}(K) t^{n-k}, \tag{2.4}$$

where $W_{n-k}(K) := V(K, k; B_2^n, n-k)$ is the $(n-k)$-th quermaßintegral of K. It will be convenient for us to work with a normalized variant of $W_{n-k}(K)$: for every $1 \leqslant k \leqslant n$ we set

$$Q_k(K) = \left(\frac{1}{\omega_k} \int_{G_{n,k}} |P_F(K)| \, d\nu_{n,k}(F) \right)^{1/k}. \tag{2.5}$$

Note that $Q_1(K) = w(K)$. Kubota's integral formula

$$W_{n-k}(K) = \frac{\omega_n}{\omega_k} \int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F) \tag{2.6}$$

shows that

$$Q_k(K) = \left(\frac{W_{n-k}(K)}{\omega_n}\right)^{1/k}. \tag{2.7}$$

The Aleksandrov-Fenchel inequality states that if $K,\ L,\ K_3,\dots,K_n \in \mathcal{K}_n$, then

$$V(K,L,K_3,\dots,K_n)^2 \geqslant V(K,K,K_3,\dots,K_n)V(L,L,K_3,\dots,K_n). \tag{2.8}$$

This implies that the sequence $(W_0(K),\dots,W_n(K))$ is log-concave: we have

$$W_j^{k-i} \geqslant W_i^{k-j} W_k^{j-i} \tag{2.9}$$

if $0 \leqslant i < j < k \leqslant n$. Taking into account (2.7) we conclude that $Q_k(K)$ is a decreasing function of k. For the theory of mixed volumes we refer to [33].

2.2 L_q-centroid bodies of isotropic log-concave measures

We denote by $\mathcal{P}_n$ the class of all Borel probability measures on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_n$ is denoted by f_μ. We say that $\mu \in \mathcal{P}_n$ is centered if, for all $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \langle x,\theta\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x,\theta\rangle f_\mu(x)dx = 0. \tag{2.10}$$

A measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for all compact subsets A and B of $\mathbb{R}^n$ and all $\lambda \in (0,1)$. A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set and the restriction of $\log f$ to it is concave. Borell has proved in [8] that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_n$ and its density f_μ is log-concave. Note that if K is a convex body of volume 1 in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{2.11}$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}. \tag{2.12}$$

Note that L_μ is an affine invariant of μ and does not depend on the choice of the Euclidean structure. We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered and $\mathrm{Cov}(\mu)$ is the identity matrix.

Recall that if μ is a log-concave probability measure on $\mathbb{R}^n$ and if $q \geqslant 1$ then the L_q-centroid body $Z_q(\mu)$ of μ is the symmetric convex body with support function

$$h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x,y\rangle|^q d\mu(x)\right)^{1/q}. \tag{2.13}$$

Observe that μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$. From Hölder's inequality it follows that $Z_1(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leqslant p \leqslant q < \infty$. Conversely, using Borell's lemma (see [28, Appendix III]), one can check that

$$Z_q(\mu) \subseteq c_1 \frac{q}{p} Z_p(\mu) \tag{2.14}$$

for all $1 \leqslant p < q$. In particular, if μ is isotropic, then $R(Z_q(\mu)) \leqslant c_2 q$.

For any $\alpha \geqslant 1$ and any $\theta \in S^{n-1}$ we define the ψ_α-norm of $x \mapsto \langle x, \theta\rangle$ as follows:

$$\|\langle\cdot,\theta\rangle\|_{\psi_\alpha} := \inf\left\{ t>0 : \int_{\mathbb{R}^n} \exp\left(\left(\frac{|\langle x,\theta\rangle|}{t}\right)^\alpha\right) d\mu(x) \leqslant 2\right\}, \tag{2.15}$$

provided that the set on the right hand side is non-empty. We say that μ satisfies a ψ_α-estimate with constant $b_\alpha = b_\alpha(\theta)$ in the direction of θ if we have

$$\|\langle\cdot,\theta\rangle\|_{\psi_\alpha} \leqslant b_\alpha \|\langle\cdot,\theta\rangle\|_2.$$

We say that μ is a ψ_α-measure with constant $B_\alpha > 0$ if

$$\sup_{\theta\in S^{n-1}} \frac{\|\langle\cdot,\theta\rangle\|_{\psi_\alpha}}{\|\langle\cdot,\theta\rangle\|_2} \leqslant B_\alpha.$$

From Borell's lemma it follows that every log-concave measure is a ψ_1-measure with constant C, where C is an absolute positive constant.

From [29] and [30] one knows that the "q-moments"

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q dx\right)^{1/q}, \quad q \in (-n, +\infty)\setminus\{0\}, \tag{2.16}$$

of the Euclidean norm with respect to an isotropic log-concave probability measure μ on $\mathbb{R}^n$ are equivalent to $I_2(\mu) = \sqrt{n}$ as long as $|q| \leqslant \sqrt{n}$. Two main consequences of this fact are: (i) Paouris' deviation inequality

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 \geqslant c_3 t\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right) \tag{2.17}$$

for every $t \geqslant 1$, where $c_3 > 0$ is an absolute constant, and (ii) Paouris' small ball probability estimate: for any $0 < \varepsilon < \varepsilon_0$, one has

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 < \varepsilon\sqrt{n}\}) \leqslant \varepsilon^{c_4\sqrt{n}}, \tag{2.18}$$

where $\varepsilon_0, c_4 > 0$ are absolute constants.

The next theorem summarizes our knowledge on the mean width of $Z_q(\mu)$. The first statement was proved by Paouris in [29], while the second one is E. Milman's Theorem 1.2.

Theorem 2.1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. If $1 \leqslant q \leqslant \sqrt{n}$, then*

$$w(Z_q(\mu)) \simeq \sqrt{q}. \tag{2.19}$$

Moreover, for all $q \in [\sqrt{n}, n]$ we have

$$w(Z_q(\mu)) \leqslant c_5\sqrt{q}\,\log^2(1+q). \tag{2.20}$$

The next theorem summarizes our knowledge on the volume radius of $Z_q(\mu)$. The first statement follows from the results of [29] and [20], while the left hand-side in the second one was obtained in [24] and the right hand-side in [29].

Theorem 2.2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. If $1 \leqslant q \leqslant \sqrt{n}$ then*

$$|Z_q(\mu)|^{1/n} \simeq \sqrt{q/n}, \tag{2.21}$$

while if $\sqrt{n} \leqslant q \leqslant n$ then

$$c_6 L_\mu^{-1}\sqrt{q/n} \leqslant |Z_q(\mu)|^{1/n} \leqslant c_7\sqrt{q/n}. \tag{2.22}$$

The reader may find a detailed exposition of the theory of isotropic log-concave measures in the book [11].

3 Estimates for the Quermaßintegrals

We start with the *proof of Theorem* 1.3. Recall that the equivalence $\mathbb{E}\left[Q_k(K_N)\right] \simeq \sqrt{\log N}$ in the range $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ was proved in [14] (see Theorem 1.1). What is new is the right hand-side estimate in (1.20). However, in [14] it was proved that $\mathbb{E}[Q_k(K_N)] \leqslant w(Z_{\log N}(K))$ for the full range of N. So the result follows immediately by applying Theorem 2.1.

To prove Theorem 1.4 we will need Lemma 4.2 from [14] which holds true in the more general setting of isotropic log-concave random vectors.

Lemma 3.1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $n^2 \leqslant N \leqslant \exp(cn)$ and for every $q \geqslant \log N$ and $r \geqslant 1$, we have*

$$\int_{S^{n-1}} \frac{h_{K_N}^q(\theta)}{h_{Z_q(\mu)}^q(\theta)}\, d\sigma_n(\theta) \leqslant (c_1 r)^q \tag{3.1}$$

with probability greater than $1 - r^{-q}$, where $c_1 > 0$ is an absolute constant.

Proof of Theorem 1.4. Let $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$. Applying Hölder's inequality we get

$$\begin{aligned} w(K_N) &= \int_{S^{n-1}} h_{K_N}(\theta)\, d\sigma_n(\theta) \\ &\leqslant \left(\int_{S^{n-1}} \big(h_{Z_q(\mu)}(\theta)\big)^p\, d\sigma_n(\theta)\right)^{1/p} \left(\int_{S^{n-1}} \left(\frac{h_{K_N}(\theta)}{h_{Z_q(\mu)}(\theta)}\right)^q d\sigma_n(\theta)\right)^{1/q} \\ &= w_p\big(Z_q(\mu)\big) \left(\int_{S^{n-1}} \left(\frac{h_{K_N}(\theta)}{h_{Z_q(\mu)}(\theta)}\right)^q d\sigma_n(\theta)\right)^{1/q}, \end{aligned}$$

where p is the conjugate exponent of q. If we now choose $q = \log N \geqslant \sqrt{n}$ and use Lemma 3.1 we arrive at

$$w(K_N) \leqslant c_1 r w_p\big(Z_q(\mu)\big)$$

with probability greater than $1 - r^{-q}$. Since $q = \log N$ it follows that $p < 2$ and thus $w_p(Z_q(\mu))$ is equivalent to $w(Z_q(\mu))$ (see [6, Chapter 5]). Using this and applying Theorem 1.2 we conclude that

$$w(K_N) \leqslant c_2 r \sqrt{\log N}\big(\log\log N\big)^2$$

with probability greater than $1 - r^{-\log N}$. Choosing $r = e$ we complete the proof of (1.21). □

We can also give estimates on the volume radius of a random projection $P_F(K_N)$ of K_N onto $F \in G_{n,k}$ in terms of n, k and N. In [14] it was shown that if $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then, a random K_N satisfies with probability greater than $1 - N^{-s}$ the following: for every $1 \leqslant k \leqslant n$,

$$c_3 \sqrt{\log N} \leqslant \text{v.rad}(P_F(K_N)) \leqslant c_4(s)\sqrt{\log N} \tag{3.2}$$

with probability greater than $1 - e^{-c_5 k}$ with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$. We extend this result to the case $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$.

For the proof we will use Theorem 1.1 from [13], which was already mentioned in the introduction. We formulate it in the more general setting of isotropic log-concave random vectors (the probability estimate in the statement makes use of [1, Theorem 3.13]: if $\gamma > 1$ and $\Gamma : \ell_2^n \to \ell_2^N$ is the random operator $\Gamma(y) = (\langle x_1, y\rangle, \dots \langle x_N, y\rangle)$ defined by the vertices $x_1, \dots, x_N$ of K_N then $\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geqslant \gamma\sqrt{N}) \leqslant \exp(-c_0\gamma\sqrt{N})$ for all $N \geqslant c\gamma n$—see [13] for the details).

Fact 3.2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$ and let $x_1, \ldots, x_N$ be independent random vectors distributed according to μ, with $N \geqslant c_1 n$ where $c_1 > 1$ is an absolute constant. Then, for all $q \leqslant c_2 \log(N/n)$ we have that*

$$K_N \supseteq c_3\, Z_q(\mu) \tag{3.3}$$

with probability greater than $1 - \exp(-c_4\sqrt{N})$.

Proof of Theorem 1.5. For the upper bound we use (1.21) and Kubota's formula to get

$$\left(\frac{1}{\omega_k}\int_{G_{n,k}} |P_F(K_N)|\, d\nu_{n,k}(F)\right)^{1/k} \leqslant c_6(s)\sqrt{\log N}\,\big(\log\log N\big)^2 L_K.$$

Applying now Markov's inequality we get that with probability greater than $1 - t^{-k}$ with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$ we have

$$\left(\frac{|P_F(K_N)|}{\omega_k}\right)^{1/k} \leqslant c_6(s) t\sqrt{\log N}\big(\log\log N\big)^2.$$

Choosing $t = e$ proves the result.

For the lower bound integrating in polar coordinates and using Hölder's inequality we have

$$\begin{aligned} \int_{G_{n,k}} \frac{|P_F^\circ(K_N)|}{\omega_k}\, d\nu_{n,k}(F) &= \int_{G_{n,k}}\int_{S_F} \frac{1}{h^k_{P_F(K_N)}(\theta)} d\sigma_F(\theta)\, d\nu_{n,k}(F) \\ &= \int_{G_{n,k}}\int_{S_F} \frac{1}{h^k_{K_N}(\theta)} d\sigma_F(\theta)\, d\nu_{n,k}(F) \\ &\leqslant \left(\int_{G_{n,k}}\int_{S_F} \frac{1}{h^n_{K_N}(\theta)} d\sigma_F(\theta)\, d\nu_{n,k}(F)\right)^{k/n} \\ &= \left(\int_{S^{n-1}} \frac{1}{h^n_{K_N}(\theta)} d\sigma_n(\theta)\right)^{k/n} \\ &= \left(\frac{|K_N^\circ|}{\omega_n}\right)^{k/n}. \end{aligned} \tag{3.4}$$

Apply now the Blaschke-Santaló inequality and the fact that $K_N \supseteq c_7 Z_{\log N}(\mu)$ (with probability greater than $1 - \exp(-c\sqrt{N})$ (notice that $\log N \simeq \log N/n$ for the range of N we use)) to get

$$\left(\frac{|K_N^\circ|}{\omega_n}\right)^{k/n} \leqslant \left(\frac{\omega_n}{|K_N|}\right)^{k/n} \leqslant \left(\frac{\omega_n}{|c_7 Z_{\log N}(\mu)|}\right)^{k/n}. \tag{3.5}$$

Since $\log N$ is greater than $\sqrt{n}$ we can apply the inequality $|Z_{\log N}(K)|^{1/n} \geqslant cL_\mu^{-1}\sqrt{(\log N)/n}$ to arrive at

$$\int_{G_{n,k}} \frac{|P_F^\circ(K_N)|}{\omega_k}\, d\nu_{n,k}(F) \leqslant \left(\frac{c_8 L_\mu}{\sqrt{\log N}}\right)^k. \tag{3.6}$$

Finally, we apply Markov's inequality and the reverse Santaló inequality of Bourgain and V. Milman [10] to complete the proof. $\square$

4 Mean outer radii

For any convex body C in $\mathbb{R}^n$ and any $1 \leqslant k \leqslant n$, the k-th mean outer radius of C is defined by

$$\tilde{R}_k(C) = \int_{G_{n,k}} R(P_F(C))\, d\nu_{n,k}(F). \tag{4.1}$$

Alonso-Gutiérrez, Dafnis, Hernández-Cifre and Prochno studied in [3] the order of growth of $\tilde{R}_k(K_N)$ as a function of N, n and k. Their main result is Theorem 1.6: If $n \leqslant N \leqslant \exp(\sqrt{n})$ then, for all $1 \leqslant k \leqslant n$ and $s > 0$ one has

$$c_1(s) \max\left\{\sqrt{k}, \sqrt{\log(N/n)}\right\} \leqslant \tilde{R}_k(K_N) \leqslant c_2(s) \max\left\{\sqrt{k}, \sqrt{\log N}\right\} \tag{4.2}$$

with probability greater than $1 - N^{-s}$, where $c_1(s), c_2(s)$ are positive constants depending only on s.

In this section we give an alternative (and simpler) proof of this result. We also extend the estimates to the range $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$. Our approach is based on the next general fact, which is a standard application of concentration of measure on the Euclidean sphere (see [6, Section 5.7] for the details). If C is a symmetric convex body in $\mathbb{R}^n$ then, for any $1 \leqslant k < n$ and any $s > 1$ there exists a subset $\Gamma_{n,k} \subset G_{n,k}$ with measure greater than $1 - e^{-c_1 s^2 k}$ such that the orthogonal projection of C onto any subspace $F \in \Gamma_{n,k}$ satisfies

$$R(P_F(C)) \leqslant w(C) + c_2 s\sqrt{k/n}R(C), \tag{4.3}$$

where $c_1 > 0, c_2 > 1$ are absolute constants. In fact, one has that the reverse inequality $R(P_F(C)) \geqslant c\max\{w(C), \sqrt{k/n}R(C)\}$ holds for most $F \in G_{n,k}$. To see this, first note that if $x \in C$ and $\|x\|_2 = R(C)$ then, for most $F \in G_{n,k}$ we have $\|P_F(x)\|_2 \geqslant c\sqrt{k/n}\|x\|_2$, and hence $R(P_F(C)) \geqslant c\sqrt{k/n}R(C)$; integrating with respect to $\nu_{n,k}$ we get $\tilde{R}_k(C) \geqslant c\sqrt{k/n}R(C)$. On the other hand, if $\sqrt{k/n}R(C) \leqslant c'w(C)$ for a small enough absolute constant $0 < c' < 1$ then V. Milman's proof of Dvoretzky's theorem shows that most k-dimensional projections of C are isomorphic Euclidean balls of radius $w(C)$, which implies that $\tilde{R}_k(C) \geqslant cw(C)$. These observations lead to the next asymptotic formula.

Proposition 4.1. *Let C be a symmetric convex body in $\mathbb{R}^n$. For any $1 \leqslant k \leqslant n$ one has*

$$\tilde{R}_k(C) \simeq w(C) + \sqrt{k/n}R(C). \tag{4.4}$$

We will exploit this formula for a random K_N. Because of (4.4) we only need to estimate $w(K_N)$ and $R(K_N)$ for a random K_N. This is done in Proposition 4.2 and Proposition 4.4 below. Essential ingredients are the deviation and small ball probability estimates (2.17) and (2.18) of Paouris, as well as Fact 3.2.

We start with the case $N \leqslant \exp(\sqrt{n})$.

Proposition 4.2. *If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then, for any $s \geqslant 1$, a random K_N satisfies*

$$c_1\sqrt{\log N} \leqslant w(K_N) \leqslant c_2 s\sqrt{\log N}$$

and

$$c_3\sqrt{n} \leqslant R(K_N) \leqslant c_4 s\sqrt{n}$$

with probability greater than $1 - \max\{N^{-s}, e^{-c\sqrt{n}}\}$.

Proof. In the proof of Theorem 1.4 we saw that, for any $n \leqslant N \leqslant \exp(n)$,

$$w(K_N) \leqslant c_1 s w\big(Z_{\log N}(\mu)\big) \tag{4.5}$$

with probability greater than $1 - N^{-s}$. Assuming that $N \leqslant \exp(\sqrt{n})$ we have that $\log N \leqslant \sqrt{n}$; then Theorem 2.1 and (4.5) show that

$$w(K_N) \leqslant c_2 s\sqrt{\log N} \tag{4.6}$$

with probability greater than $1 - N^{-s}$. For the lower bound we use Fact 3.2: we know that for all $N \geqslant c_3 n$ we have

$$K_N \supseteq c_4\, Z_{\log(N/n)}(\mu) \tag{4.7}$$

with probability greater than $1 - \exp(-c_5\sqrt{N})$. It follows that if $N \leqslant \exp(\sqrt{n})$ then

$$w(K_N) \geqslant c_4 w(Z_{\log(N/n)}(\mu)) \geqslant c_6\sqrt{\log(N/n)}$$

with probability greater than $1 - \exp(-c_7\sqrt{N})$.

For the radius of K_N, applying (2.17) we see that, for any $t \geqslant 2$,

$$R(K_N) = \max_{1\leqslant j\leqslant N} \|x_j\|_2 \leqslant c_8 t\sqrt{n} \tag{4.8}$$

with probability greater than $1 - N\exp(-t\sqrt{n}) \geq 1 - \exp(-(t-1)\sqrt{n}) \geqslant 1 - N^{-(t-1)}$. For the lower bound, if $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ we use (2.18) to write

$$\begin{aligned}\mathrm{Prob}(R(K_N) \leqslant \varepsilon_0\sqrt{n}) &= \mathrm{Prob}\left(\max_{1\leqslant j\leqslant N} \|x_j\|_2 \leqslant \varepsilon_0\sqrt{n}\right)\\ &= \left[\mu(\{x \in \mathbb{R}^n : \|x\|_2 < \varepsilon_0\sqrt{n}\})\right]^N \leqslant e^{-c_9\sqrt{n}N},\end{aligned}$$

which shows that $R(K_N) \geqslant \varepsilon_0\sqrt{n}$ with probability greater than $1 - e^{-c_9\sqrt{n}N}$.

□

Remark 4.3. In fact, for the proof of the lower bound $R(K_N) \geqslant c\sqrt{n}$ we do not really need the small ball probability estimate of Paouris. Latała has proved in [22] that if μ is a log-concave probability measure on $\mathbb{R}^n$ then, for any norm $\|\cdot\|$ on $\mathbb{R}^n$ and any $0 \leqslant t \leqslant 1$ one has

$$\mu(\{x : \|x\| \leqslant t\mathbb{E}_\mu(\|x\|)\}) \leqslant Ct, \tag{4.9}$$

where $C > 0$ is an absolute constant. If we assume that μ is isotropic then we easily see that $\mathbb{E}_\mu(\|x\|_2) \leqslant \sqrt{n}$, and hence, choosing a small enough absolute constant ε_0 we have by (4.9) that

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 < \varepsilon_0\sqrt{n}\}) \leqslant e^{-1}.$$

This information is enough for our purposes.

Proof of Theorem 1.6. Let $N \leqslant \exp(\sqrt{n})$. From (4.4) and Proposition 4.2 we get that K_N satisfies with probability greater than $1 - \max\{N^{-s}, e^{-c\sqrt{n}}\}$ the following: for any $1 \leqslant k \leqslant n$

$$\begin{aligned}\tilde{R}_k(K_N) &= \int_{G_{n,k}} R(P_F(K_N))\, d\nu_{n,k}(F) \simeq w(K_N) + \sqrt{k/n}R(K_N)\\ &\geqslant c_1\left(\sqrt{\log(N/n)} + \sqrt{k/n}\,\sqrt{n}\right) \simeq \max\left\{\sqrt{\log(N/n)}, \sqrt{k}\right\}\end{aligned}$$

and similarly,

$$\begin{aligned}\tilde{R}_k(K_N) &= \int_{G_{n,k}} R(P_F(K_N))\, d\nu_{n,k}(F) \simeq w(K_N) + \sqrt{k/n}R(K_N)\\ &\leqslant c_2(s)\left(\sqrt{\log N} + \sqrt{k/n}\,\sqrt{n}\right) \leqslant 2c_2(s)\max\left\{\sqrt{\log N}, \sqrt{k}\right\},\end{aligned}$$

as in [3]. □

The next proposition will allow us to handle the case $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$.

Proposition 4.4. *If* $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ *then, for any* $s \geqslant 1$*, a random* K_N *satisfies*

$$c_1 L_\mu^{-1}\sqrt{\log N} \leqslant w(K_N) \leqslant c_2 s\sqrt{\log N}(\log\log N)^2$$

and

$$c_3\max\{\sqrt{n}, R(Z_{\log N}(\mu))\} \leqslant R(K_N) \leqslant c_3 s\,\log N$$

with probability greater than $1-\max\{N^{-s}, e^{-c\sqrt{n}}\}$.

Proof. Applying again (4.5) in the range $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ we have that

$$w(K_N) \leqslant c_2 s\sqrt{\log N}(\log\log N)^2 \tag{4.10}$$

from Theorem 1.2. For the lower bound we use again Fact 3.2, Urysohn's inequality and (2.22) from Theorem 2.2 to write

$$w(K_N) \geqslant c_4 w(Z_{\log N}(\mu)) \geqslant c_4(|Z_{\log N}(\mu)|/|B_2^n|)^{1/n} \geqslant c_6 L_\mu^{-1}\sqrt{\log N}$$

with probability greater than $1-\exp(-c_5\sqrt{N})$.

For the radius of K_N we first use the estimate $R(K_N) \leqslant ct\sqrt{n}$ from (4.8) with $t \simeq s\log N/\sqrt{n}$ to obtain the bound $c\log N$ with probability greater than $1-N^{-s}$. For the lower bound, we show that $R(K_N) \geqslant c\sqrt{n}$ exactly as in the proof of Proposition 4.2, and we also use the bound $R(K_N) \geqslant R(Z_{\log N}(\mu))$. □

Using Proposition 4.4 and Proposition 4.1 as in the proof of Theorem 1.6, we arrive at the following estimate:

Theorem 4.5. *Let* $x_1,\dots,x_N$ *be independent random points distributed according to an isotropic log-concave measure* μ *on* $\mathbb{R}^n$*, and consider the random polytope* $K_N := \mathrm{conv}\{\pm x_1,\dots,\pm x_N\}$*. If* $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ *then, for any* $s \geqslant 1$ *and for all* $1 \leqslant k \leqslant n$ *one has*

$$\begin{aligned} c\max\left\{L_\mu^{-1}\sqrt{\log N}, \sqrt{k}, \sqrt{k/n}R(Z_{\log N}(\mu))\right\} \\ \leqslant \tilde{R}_k(K_N) \leqslant Cs\max\{\sqrt{\log N}(\log\log N)^2, \sqrt{k/n}\log N\}\end{aligned}$$

with probability greater than $1-N^{-s}$*, where* $c, C>0$ *are absolute constants.*

In full generality one cannot expect something significantly better: for example, if $\mu = \mu_1^n$ is the uniform measure on $B_1^n/|B_1^n|$ then $R(Z_{\log N}(\mu_1^n)) \simeq \log N$, and for large values of N (i.e. exponential in N) we get

$$\tilde{R}_k(K_N) \simeq \sqrt{k/n}\log N.$$

On the other hand, if μ satisfies a ψ_2 estimate with constant b then we know that $L_\mu \leqslant C_1 b$ (see [20]) and we also know that $I_n(\mu) \leqslant cb\sqrt{n}$ (see [29]), which implies that $w(K_N) \leqslant R(K_N) \leqslant C_2 b\sqrt{n}$. Moreover, $Z_{\log N}(\mu) \subseteq b\sqrt{\log N}B_2^n$. Thus, in this case (which e.g. includes the case of the standard Gaussian measure) we get:

Theorem 4.6. *Let* $x_1,\dots,x_N$ *be independent random points distributed according to an isotropic log-concave measure* μ *on* $\mathbb{R}^n$ *which satisfies a* ψ_2*-estimate with constant* b*, and consider the random polytope* $K_N := \mathrm{conv}\{\pm x_1,\dots,\pm x_N\}$*. If* $n \leqslant N \leqslant \exp(n)$ *and* $s \geqslant 1$ *then* K_N *satisfies with probability greater than* $1-N^{-s}$

$$c_1 b^{-1}\max\left\{\sqrt{k}, \sqrt{\log(N/n)}\right\} \leqslant \tilde{R}_k(K_N) \leqslant c_2(s) b\max\left\{\sqrt{k}, \sqrt{\log N}\right\} \tag{4.11}$$

for all $1 \leqslant k \leqslant n$*, where* $c_2(s)$ *is a positive constant depending only on* s*.*

5 Entropy estimates and diameter of sections

For every pair of convex bodies A and B in $\mathbb{R}^n$, the covering number $N(A,B)$ of A by B is defined to be the smallest number of translates of B whose union covers A. A fundamental theorem of V. Milman states that there exists an absolute constant $\beta>0$ such that every symmetric convex body K in $\mathbb{R}^n$ has a linear image $\tilde{K}$ which satisfies $|\tilde{K}|=|B_2^n|$ and

$$\max\big\{N(\tilde{K},B_2^n),N(B_2^n,\tilde{K}),N(\tilde{K}^\circ,B_2^n),N(B_2^n,\tilde{K}^\circ)\big\}\leqslant \exp(\beta n). \tag{5.1}$$

A convex body which satisfies the above is said to be in M-position with constant β. Pisier has offered in [31] a refined version of this result: for every $0<\alpha<2$ and every symmetric convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K}_\alpha$ of K such that

$$\max\big\{N(\tilde{K}_\alpha,tB_2^n),N(B_2^n,t\tilde{K}_\alpha),N(\tilde{K}_\alpha^\circ,tB_2^n),N(B_2^n,t\tilde{K}_\alpha^\circ)\big\}\leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right) \tag{5.2}$$

for every $t\geqslant 1$, where $c(\alpha)$ depends only on α, and $c(\alpha)=O\big((2-\alpha)^{-\alpha/2}\big)$ as $\alpha\to 2$. One says that $\tilde{K}_\alpha$ is an α-regular M-position of K (we refer to [6, Chapter 8] and [32] for a detailed exposition of these results).

In this section we will first show that if μ is an isotropic log-concave measure on $\mathbb{R}^n$ then, for a considerably large range of values of N, a random K_N is in α-regular M-position with $\alpha\sim 1$. To this end, it is convenient to set $r_N=\sqrt{\log N}$: recall that if $n^2\leq N\leq \exp(\sqrt{n})$ then $\text{v.rad}(K_N)\simeq r_N$ for a random K_N (in the case $N\geqslant \exp(\sqrt{n})$ one has the weaker estimate $c_1L_\mu^{-1}r_N\leqslant \text{v.rad}(K_N)\leqslant c_2r_N$). We provide estimates for the covering numbers $N(K_N,tr_NB_2^n)$ and $N(r_NB_2^n,tK_N)$ for a random K_N and for all $t\geqslant 1$; by the duality of entropy theorem of Artstein-Avidan, V. Milman and Szarek [7], these also determine the covering numbers $N(r_NK_N^\circ,tB_2^n)$ and $N(B_2^n,tr_NK_N^\circ)$, thus completing the proof of the four required entropy estimates in (5.2).

Proposition 5.1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Then a random K_N satisfies with probability greater than $1-N^{-1}$ the entropy estimate*

$$\log N(K_N,tr_NB_2^n)\leqslant \begin{cases} \frac{cn}{t^2} & \text{if } n^2\leqslant N\leqslant \exp(\sqrt{n})\\ \frac{cn\log^4 n}{t^2} & \text{if } \exp(\sqrt{n})\leqslant N\leqslant \exp(cn).\end{cases}$$

for every $t\geqslant 1$, where $c>0$ is an absolute constant.

Proof. We simply recall that a random K_N satisfies $w(K_N)\leqslant c_1\sqrt{\log N}\simeq r_N$ for "small" N, and $w(K_N)\leqslant c_2\sqrt{\log N}(\log\log N)^2\simeq r_N(\log\log N)^2$ for "large" N, by Proposition 4.2 and Proposition 4.4 respectively. The bound for $N(K_N,tr_NB_2^n)$ is then a direct consequence of Sudakov's inequality

$$\log N(C,tB_2^n)\leqslant cn(w(C)/t)^2$$

which is true for every convex body C in $\mathbb{R}^n$ and every $t>0$ (see e.g. [6, Chapter 4]). □

We turn to estimates for the dual covering numbers $N(r_NB_2^n,tK_N)$. We will make use of the following fact (see [16] and [11, Proposition 9.2.8] or [15] for the stronger statement below): If μ is an isotropic log-concave measure on $\mathbb{R}^n$, then for any $2\leqslant q\leqslant\sqrt{n}$ and for any $1\leqslant t\leqslant \min\left\{\sqrt{q},c_1\frac{n\log q}{q^2}\right\}$ we have

$$\log N\big(\sqrt{q}B_2^n,tZ_q(\mu)\big)\leqslant c_2\frac{n(\log q)^2\log t}{t}, \tag{5.3}$$

where $c_1,c_2>0$ are absolute constants. Moreover, if $q\leqslant (n\log n)^{2/5}$ then (5.3) holds true for all $t\geqslant 1$. Analogous estimates are available for larger values of q, but they are weaker and do not seem to be final; so, we prefer to restrict ourselves to the next case.

Proposition 5.2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Then, assuming that $n^2 \leqslant N \leqslant \exp\big((n\log n)^{2/5}\big)$, we have that a random K_N satisfies with probability greater than $1-\exp(-c_1\sqrt{N})$ the entropy estimate*

$$\log N(r_N B_2^n, tK_N) \leqslant c_2 \frac{n(\log n)^2 \log(1+t)}{t}$$

for every $t \geqslant 1$, where $c_1, c_2 > 0$ are absolute constants.

Proof. It is an immediate consequence of the fact that $K_N \supseteq c_3 Z_{\log N}(\mu)$ with probability greater than $1-\exp(-c_1\sqrt{N})$. Then, we clearly have

$$\log N(r_N B_2^n, tK_N) \leqslant \log N(r_N B_2^n, c_3 t Z_{\log N}(\mu)),$$

and the result follows from (5.3). □

Proof of Theorem 1.7. By Proposition 5.2

$$\log N(r_N B_2^n, tK_N) \leq c\frac{n\log^2 n \log(1+t)}{t}.$$

By Proposition 5.1, since

$$N \leq \exp\left((n\log n)^{2/5}\right) \leq \exp(\theta\sqrt{n}),$$

for a suitable absolute constant $\theta > 0$, we have

$$\log N(K_N, tr_N B_2^n) \leq \frac{cn}{t}$$

(here we can compensate for the extra factor θ in the exponent since for the proof of Proposition 5.1 we can use the fact that $Z_{\theta\sqrt{n}}(\mu) \subseteq \theta Z_{\sqrt{n}}(\mu)$). Combining the above bounds we get the result. □

Remark 5.3. Following the reasoning of [14] one can also check that there exist absolute positive constants c_1, c_2, c_3 and c_4 so that for every $0 < t < 1$ a random K_N satisfies with probability greater than $1 - N^{-1}$ the next entropy estimates:
(i) If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then

$$c_1 n \log\frac{c_2}{t} \leqslant \log N(K_N, tr_N B_2^n) \leqslant c_3 n \log\frac{c_4}{t}, \tag{5.4}$$

(ii) If $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ then

$$c_1 n \log\frac{c_2}{t} \leqslant \log N(K_N, t\tilde{r}_N B_2^n) \leqslant c_3 n \log\frac{c_4\big(\log\log N\big)^2}{t}, \tag{5.5}$$

where $\tilde{r}_N := \text{v.rad}(K_N)$ satisfies $c_5 L_\mu^{-1} r_N \leqslant \tilde{r}_N \leqslant c_6 r_N$.

As an application we provide estimates for the average diameter of k-dimensional sections of a random K_N. This parameter can be defined for any convex body C in $\mathbb{R}^n$ and any $1 \leqslant k \leqslant n$ as follows:

$$\tilde{D}_k(C) = \int_{G_{n,k}} R(C\cap F)\, d\nu_{n,k}(F). \tag{5.6}$$

We shall use the next lemma that (in the case $\alpha = 2$) can be essentially found in the article [26] of V. Milman (see also [11, Lemma 9.2.5]):

Lemma 5.4. *Let C be a symmetric convex body in $\mathbb{R}^n$ and assume that*

$$\log N(C, tB_2^n) \leqslant \frac{\gamma n}{t^\alpha} \tag{5.7}$$

for all $t \geqslant 1$ and some constants $\alpha > 0$ and $\gamma \geqslant 1$. Then, for every integer $1 \leqslant d < n$, a subspace $H \in G_{n,d}$ satisfies

$$C \cap H^\perp \subseteq c_1 \alpha^{-1} \left(\frac{\gamma n}{d}\right)^{1/\alpha} \log\left(\frac{n}{d}\right) B_{H^\perp} \tag{5.8}$$

with probability greater than $1 - \exp(-c_2 d)$, where $c_1, c_2 > 0$ are absolute constants.

From Proposition 5.1 we know that a random $r_N^{-1} K_N$ satisfies the assumption of Lemma 5.4 with $\gamma \simeq 1$ if $N \leqslant \exp(\sqrt{n})$ and $\gamma \simeq \log^4 n$ if $N \geqslant \exp(\sqrt{n})$. Therefore, for any $k < n$ we have that if $N \leqslant \exp(\sqrt{n})$ then a k-dimensional section of K_N has radius

$$R(K_N \cap F) \leqslant c_1 \sqrt{\log N} \sqrt{\frac{n}{n-k}} \log\left(\frac{n}{n-k}\right), \tag{5.9}$$

while if $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ then the bound becomes

$$R(K_N \cap F) \leqslant c_1 \sqrt{\log N} (\log n)^2 \sqrt{\frac{n}{n-k}} \log\left(\frac{n}{n-k}\right), \tag{5.10}$$

both with probability greater than $1 - \exp(-c_2(n-k))$, where $c_1, c_2 > 0$ are absolute constants. From Proposition 4.2 and Proposition 4.4 we also know that a random K_N has radius

$$R(K_N) \leqslant c \max\{\sqrt{n}, \log N\},$$

and the same bound is clearly true for all its sections $K_N \cap F$. Therefore, if $n \exp(-c_2(n-k)) \leqslant 1$ (which is true provided that $k < n - c_3 \log n$) integration on $G_{n,k}$ shows that the bounds (5.9) and (5.10) hold for $\tilde{D}_k(K_N)$ as well. Taking into account the fact that $\tilde{D}_k(K_N) \leqslant \tilde{R}_k(K_N)$ we conclude the following.

Proposition 5.5. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Then a random K_N satisfies with probability greater than $1 - N^{-1}$ the following:*

(i) *If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then:*

1. *If $k \leqslant \log N$ then $\tilde{D}_k(K_N) \leqslant c_1 \sqrt{\log N}$.*
2. *If $k \geqslant \log N$ then $\tilde{D}_k(K_N) \leqslant c_1 \min\left\{\sqrt{k}, \sqrt{\log N}\sqrt{\frac{n}{n-k}} \log\left(\frac{n}{n-k}\right)\right\}$,*

(ii) *If $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ then:*

1. *If $k \leqslant n(\log\log N)^4/\log N$ then $\tilde{D}_k(K_N) \leqslant c_2 \sqrt{\log N} (\log\log N)^2$.*
2. *If $k \geqslant n(\log\log N)^4/\log N$ then*

$$\tilde{D}_k(K_N) \leqslant c_2 \min\left\{\sqrt{k/n}\log N, \sqrt{\log N}(\log n)^2 \sqrt{\frac{n}{n-k}} \log\left(\frac{n}{n-k}\right)\right\},$$

where $c_1, c_2 > 0$ are absolute constants.

Remark 5.6. An alternative way to estimate the average radius of $K_N \cap F$ on $G_{n,k}$ for some values of k is given by the next theorem of Klartag and Vershynin from [21]: If $1 \leqslant k \leqslant c_1 n (M(C)/b(C))^2$, then

$$\frac{c_2}{M(C)} \leqslant \left(\int_{G_{n,k}} R(C \cap F)^k \, d\nu_{n,k}(F)\right)^{1/k} \leqslant \frac{c_3}{M(C)}, \tag{5.11}$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

Note that a random K_N satisfies $K_N \supset Z_2(\mu) = B_2^n$ and integration in polar coordinates combined with Hölder's inequality shows that

$$M(K_N) \geqslant \frac{1}{\text{v.rad}(K_N)} \simeq \frac{1}{\sqrt{\log N}}.$$

Therefore, we may apply (5.11) to K_N: for all $1 \leqslant k \leqslant cn/\log N$ we have

$$\tilde{D}_k(K_N) \leqslant \left(\int_{G_{n,k}} R(C\cap F)^k \, d\nu_{n,k}(F)\right)^{1/k} \leqslant \frac{c_3}{M(C)} \leqslant c_4\sqrt{\log N}. \tag{5.12}$$

We pass now to lower bounds for $\tilde{D}_k(K_N)$. In fact, we will give a lower bound which is valid for the radius of *every* section $K_N \cap F$, $F \in G_{n,k}$. We need the next lemma.

Lemma 5.7. *Let C be a symmetric convex body in $\mathbb{R}^n$ and assume that*

$$\log N(B_2^n, tC) \leqslant \frac{\gamma n}{t^\alpha} \tag{5.13}$$

for all $t \geqslant 1$ and some constants $\alpha > 0$ and $\gamma \geqslant 1$. Then, for every $1 \leqslant k < n$ and any subspace $F \in G_{n,k}$ we have

$$R(C\cap F) \geqslant c\alpha\gamma^{-1/\alpha}(k/n)^{1/\alpha}. \tag{5.14}$$

where $c > 0$ is an absolute constant.

Proof. Let $1 \leqslant k < n$ and consider any $F \in G_{n,k}$. By the duality of entropy theorem of S. Artstein-Avidan, V. Milman, and S. Szarek (see [7]) the projection $P_F(C^\circ)$ of C° onto F satisfies

$$N(P_F(C^\circ), tB_F) \leqslant N(C^\circ, tB_2^n) \leqslant \exp(\frac{\gamma n}{k}\frac{k}{t^\alpha}), \tag{5.15}$$

for every $t \geqslant 1$. We apply Lemma 5.4 for the body $P_F(C^\circ)$ (with $\gamma' = \gamma n/k$): there exists $H \in G_{k,\lfloor k/2\rfloor}(F)$ such that

$$P_F(C^\circ) \cap H \subseteq c_1\alpha(\gamma n/k)^{1/\alpha} B_H. \tag{5.16}$$

Taking polars in H we see that $P_H(C\cap F) \supseteq c_1\alpha(k/\gamma n)^{1/\alpha}B_H$. Using the fact that for every symmetric convex body A in $\mathbb{R}^k$ and every $H \in G_{k,s}$ we have $M(A\cap H) \leqslant \sqrt{k/s}M(A)$ (see [6, Chapter 5]) we get

$$\begin{aligned} w(C\cap F) = M((C\cap F)^\circ) &\geqslant \frac{1}{\sqrt{2}}M((C\cap F)^\circ \cap H) = \frac{1}{\sqrt{2}}w(P_H(C\cap F)) \\ &\geqslant c_2\alpha(k/\gamma n)^{1/\alpha}. \end{aligned}$$

The same lower bound holds for $R(C\cap F)$. □

From Proposition 5.2 we know that if e.g. $n^2 \leqslant N \leqslant \exp\big((n\log n)^{2/5}\big)$ then a random K_N satisfies with probability greater than $1 - \exp(-c_1\sqrt{N})$ the entropy estimate

$$\log N(B_2^n, tr_N^{-1}K_N) \leqslant c_2\frac{n(\log n)^2\log(1+t)}{t}$$

for every $t \geqslant 1$, where $c_1, c_2 > 0$ are absolute constants. Notice that the interesting range for t is up to n (otherwise $tr_N^{-1}K_N$ contains B_2^n) so, we may apply Lemma 5.7 with $C = r_N^{-1}K_N$, $\gamma = \log^3 n$ and $\alpha = 1$ to get:

Proposition 5.8. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. If $n^2 \leqslant N \leqslant \exp\big((n\log n)^{2/5}\big)$ then a random K_N satisfies with probability greater than $1-\exp(-c_1\sqrt{N})$ the following: for every $1 \leqslant k < n$ and any subspace $F \in G_{n,k}$,*

$$R(K_N \cap F) \geqslant c\sqrt{\log N}\frac{k}{n\log^3 n}, \tag{5.17}$$

where $c > 0$ is an absolute constant. The same bound holds for $\tilde{D}_k(K_N)$.

Remark 5.9. The question to give an upper bound for $M(K_N)$ seems open and interesting. Let us note that the analogous question for $Z_q(\mu)$ is still open. The best known result appears in [15] (see also [16]): For any isotropic log-concave probability measure μ on $\mathbb{R}^n$ and any $2 \leqslant q \leqslant q_0 := (n\log n)^{2/5}$ one has

$$M(Z_q(\mu)) \leqslant C\frac{\sqrt{\log q}}{\sqrt[4]{q}}. \tag{5.18}$$

This estimate does not seem to be optimal; note that since $K_N \supseteq cZ_{\log N}(\mu)$ we also have

$$M(K_N) \leqslant C\frac{\sqrt{\log\log N}}{\sqrt[4]{\log N}} \tag{5.19}$$

for a random K_N, at least in the range $\log N \leqslant (n\log n)^{2/5}$.

6 Remarks on the isotropic constant

In this last section we apply directly the method of Klartag and Kozma in order to estimate the isotropic constant L_{K_N} of a random K_N. The starting point is the inequality

$$|K_N|^{2/n} n L_{K_N}^2 \leqslant \frac{1}{|K_N|}\int_{K_N} \|x\|_2^2\,dx \tag{6.1}$$

(it is well-known that this holds for any symmetric convex body in $\mathbb{R}^n$; see e.g. [27] or [11, Chapter 3]). Assuming that $N \leqslant \exp(\sqrt{n})$ we know by (1.6) that

$$|K_N|^{1/n} \geqslant c_1\frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \tag{6.2}$$

with probability greater than $1-\exp(-c_2\sqrt{n})$.

We write $\mathcal{F}(K_N)$ for the family of facets of K_N and we denote by $[y_1,\dots,y_n]$ the convex hull of $y_1,\dots,y_n$. Observe that, with probability equal to 1, all the facets of K_N are simplices and that, for all $1 \leqslant j \leqslant n$, x_j and $-x_j$ cannot belong to the same facet of K_N. Following [19, Lemma 2.5] one can show the next lemma.

Lemma 6.1. *Let $F_1,\dots,F_M$ be the facets of K_N. Then,*

$$\frac{1}{|K_N|}\int_{K_N}\|x\|_2^2dx \leqslant \frac{n}{n+2}\max_{1\leqslant s\leqslant M}\frac{1}{|F_s|}\int_{F_s}\|u\|_2^2du. \tag{6.3}$$

Let $y_1,\dots,y_n \in \mathbb{R}^n$ and define $F = [y_1,\dots,y_n]$. Then, $F = T(\Delta^{n-1})$ where $\Delta^{n-1} = [e_1,\dots,e_n]$ and $T_{ij} = \langle y_j, e_i\rangle =: y_{ji}$. Assume that $\det T \neq 0$. Then,

$$\begin{aligned}\frac{1}{|F|}\int_F \|u\|_2^2du &= \frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}}\|Tu\|_2^2du \\ &= \frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}}\sum_{i=1}^n\left(\sum_{j=1}^n y_{ji}u_j\right)^2 du.\end{aligned}$$

Using the fact that

$$\frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}}(u_{j_1}u_{j_2})\,du=\frac{1+\delta_{j_1,j_2}}{n(n+1)}, \tag{6.4}$$

we see that

$$\frac{1}{|F|}\int_F \|u\|_2^2du=\frac{1}{n(n+1)}\sum_{i=1}^n\left(\sum_{j=1}^n y_{ji}^2+\left(\sum_{j=1}^n y_{ji}\right)^2\right), \tag{6.5}$$

from where one can conclude that

$$\frac{1}{|F|}\int_F \|u\|_2^2du\leqslant\frac{2}{n(n+1)}\max_{\varepsilon_j=\pm1}\|\varepsilon_1y_1+\cdots+\varepsilon_ny_n\|_2^2. \tag{6.6}$$

Next we use a Bernstein type inequality (for a proof, see e.g. [6, Theorem 3.5.16]):

Lemma 6.2. *Let $g_1,\dots,g_n$ be independent random variables with $\mathbb{E}\,(g_j)=0$ on some probability space (Ω,μ). Assume that $\|g_j\|_{\psi_1}\leqslant A$ for all $1\leqslant j\leqslant n$ and some constant $A>0$. Then,*

$$\mathbb{P}\left\{\left|\sum_{j=1}^n a_jg_j\right|\geqslant t\right\}\leqslant 2\exp\left(-c\min\left\{\frac{t^2}{A^2\|a\|_2^2},\frac{t}{A\|a\|_\infty}\right\}\right) \tag{6.7}$$

for every $t>0$.

We first fix $\theta\in S^{n-1}$ and a choice of signs $\varepsilon_j=\pm1$, and apply Lemma 6.2 to the random variables $g_j(y_1,\dots,y_n)=\langle\varepsilon_jy_j,\theta\rangle$ on $\Omega=(\mathbb{R}^n,\mu)^n$. Since μ is isotropic, we know that $\|g_j\|_{\psi_1}\leqslant C$. Choosing $\alpha=C_0\log(2N/n)$ we get

$$\mathbb{P}\left\{|\langle\varepsilon_1y_1+\cdots+\varepsilon_ny_n,\theta\rangle|>\alpha n\right\}\leqslant 2\exp(-c\alpha n). \tag{6.8}$$

Consider a 1/2-net $\mathcal{N}$ for S^{n-1} with cardinality $|\mathcal{N}|\leqslant 5^n$. Then, with probability greater than $1-\exp(-c_2\alpha n)$ we have

$$|\langle\varepsilon_1y_1+\cdots+\varepsilon_ny_n,\theta\rangle|\leqslant\alpha n \tag{6.9}$$

for every $\theta\in\mathcal{N}$ and every choice of signs $\varepsilon_j=\pm1$. Using a standard successive approximation argument, and taking into account all 2^n possible choices of signs $\varepsilon_j=\pm1$, we get that, with probability greater than $1-\exp(-c_3\alpha n)$,

$$\max_{\varepsilon_j=\pm1}\|\varepsilon_1y_1+\cdots+\varepsilon_ny_n\|_2\leqslant C_1\alpha n. \tag{6.10}$$

Now, we use the fact that

$$|\mathcal{F}(K_N)|\leqslant\binom{2N}{n}\leqslant\exp(c_3\alpha n/2) \tag{6.11}$$

provided that C_0 is large enough. Therefore, taking also Lemma 6.1 and (6.6) into account, we see that, with probability greater than

$$1-|\mathcal{F}(K_N)|\exp(-c_3\alpha n)\geqslant 1-\exp(-c_4\alpha n),$$

we have

$$\frac{1}{|K_N|}\int_{K_N}\|x\|_2^2dx\leqslant C_2\alpha^2=C_3\log^2(2N/n), \tag{6.12}$$

where $C_3 > 0$ is an absolute constant. From (6.1) and (6.2) we get (with probability greater than $1 - \exp(-c\sqrt{n})$)

$$L_{K_N}^2 \leqslant \frac{c_4}{\log(2N/n)} \frac{1}{|K_N|} \int_{K_N} \|x\|_2^2 \, dx \leqslant C_5 \log(2N/n) \tag{6.13}$$

and hence $L_{K_N} \leqslant C_6 \sqrt{\log(2N/n)}$.

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