Asymptotic shape of a random polytope in a convex body

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Abstract

Let K be an isotropic convex body in $\mathbb{R}^n$ and let $Z_q(K)$ be the L_q–centroid body of K. For every $N > n$ consider the random polytope $K_N := \mathrm{conv}\{x_1, \dots, x_N\}$ where $x_1, \dots, x_N$ are independent random points, uniformly distributed in K. We prove that a random K_N is "asymptotically equivalent" to $Z_{[\ln(N/n)]}(K)$ in the following sense: there exist absolute constants $\rho_1, \rho_2 > 0$ such that, for all $\beta \in (0, \frac{1}{2}]$ and all $N \geq N(n, \beta)$, one has:

(i) $K_N \supseteq c(\beta) Z_q(K)$ for every $q \leq \rho_1 \ln(N/n)$, with probability greater than $1 - c_1 \exp(-c_2 N^{1-\beta} n^{\beta})$.

(ii) For every $q \geq \rho_2 \ln(N/n)$, the expected mean width $\mathbb{E}\,[w(K_N)]$ of K_N is bounded by $c_3\, w(Z_q(K))$.

As an application we show that the volume radius $|K_N|^{1/n}$ of a random K_N satisfies the bounds $c_4 \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \leq |K_N|^{1/n} \leq c_5 L_K \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}$ for all $N \leq \exp(n)$.

1 Introduction

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ we define the L_q–centroid body $Z_q(K)$ of K by its support function:

$$h_{Z_q(K)}(x) = \|\langle \cdot, x\rangle\|_q := \left(\int_K |\langle y, x\rangle|^q dy \right)^{1/q}. \tag{1.1}$$

The aim of this article is to provide some precise quantitative information on the "asymptotic shape" of a random polytope $K_N = \mathrm{conv}\{x_1, \dots, x_N\}$ spanned by N independent random points $x_1, \dots, x_N$ uniformly distributed in K. Our approach is to compare K_N with the L_q-centroid body $Z_q(K)$ of K for $q \simeq \ln(N/n)$.

The origin of our work is in the study of the behavior of symmetric random ± 1–polytopes, the absolute convex hulls of random subsets of the discrete cube $E_2^n = \{-1, 1\}^n$. The natural way to produce these random polytopes is to fix $N > n$ and to consider the convex hull $K_{n,N} = \mathrm{conv}\{\pm \vec{X}_1, \dots, \pm \vec{X}_N\}$ of N independent random points $\vec{X}_1, \dots, \vec{X}_N$, uniformly distributed over E_2^n. It turns out (see [9]) that a random polytope $K_{n,N}$ has the largest possible volume among all ± 1–polytopes with N vertices, at every scale of n and N. This is a consequence of the following fact: If $n \geq n_0$ and if $N \geq n(\ln n)^2$, then

$$K_{n,N} \supseteq c\left(\sqrt{\ln(N/n)}B_2^n \cap B_\infty^n\right) \tag{1.2}$$

with probability greater than $1 - e^{-n}$, where $c > 0$ is an absolute constant, B_2^n is the Euclidean unit ball in $\mathbb{R}^n$ and $B_\infty^n = [-1,1]^n$.

In [16], Litvak, Pajor, Rudelson, and Tomczak–Jaegermann worked in a more general setting which contains the previous Bernoulli model and the Gaussian model; let $K_{n,N}$ be the absolute convex hull of the rows of the random matrix $\Gamma_{n,N} = (\xi_{ij})_{1\leq i\leq N,\, 1\leq j\leq n}$, where ξ_{ij} are independent symmetric random variables satisfying certain conditions ($\|\xi_{ij}\|_{L^2} \geq 1$ and $\|\xi_{ij}\|_{L^{\psi_2}} \leq \rho$ for some $\rho \geq 1$, where $\|\cdot\|_{L^{\psi_2}}$ is the Orlicz norm corresponding to the function $\psi_2(t) = e^{t^2} - 1$). For this larger class of random polytopes, the estimates from [9] were generalized and improved in two ways: the paper [16] provides estimates for all $N \geq (1+\delta)n$, where $\delta > 0$ can be as small as $1/\ln n$, and establishes the following inclusion: for every $0 < \beta < 1$,

$$K_{n,N} \supseteq c(\rho)\left(\sqrt{\beta \ln(N/n)}B_2^n \cap B_\infty^n\right) \tag{1.3}$$

with probability greater than $1 - \exp(-c_1 n^\beta N^{1-\beta}) - \exp(-c_2 N)$. The proof in [16] is based on a lower bound of the order of $\sqrt{N}$ for the smallest singular value of the random matrix $\Gamma_{n,N}$ with probability greater than $1 - \exp(-cN)$.

In a sense, both works correspond to the study of the size of a random polytope $K_N = \mathrm{conv}\{x_1, \ldots, x_N\}$ spanned by N independent random points $x_1, \ldots, x_N$ uniformly distributed in the unit cube $Q_n := [-1/2, 1/2]^n$. The connection of the estimates (1.2) and (1.3) with L_q–centroid bodies comes from the following observation.

Remark. For $x \in \mathbb{R}^n$ and $t > 0$, define

$$K_{1,2}(x,t) := \inf\left\{\|u\|_1 + t\|x-u\|_2 :\ u \in \mathbb{R}^n\right\}. \tag{1.4}$$

If we write $(x_j^*)_{j\leq n}$ for the decreasing rearrangement of $(|x_j|)_{j\leq n}$ we have Holmstedt's approximation formula

$$\frac{1}{c}K_{1,2}(x,t) \leq \sum_{j=1}^{[t^2]} x_j^* + t\left(\sum_{j=[t^2]+1}^{n} (x_j^*)^2\right)^{1/2} \leq K_{1,2}(x,t) \tag{1.5}$$

where $c > 0$ is an absolute constant (see [14]). Now, for any $\alpha \geq 1$ define $C(\alpha) = \alpha B_2^n \cap B_\infty^n$. Then,

$$h_{C(\alpha)}(\theta) = K_{1,2}(\theta, \alpha) \tag{1.6}$$

for every $\theta \in S^{n-1}$. On the other hand,

$$\|\langle\cdot,\theta\rangle\|_{L^q(Q_n)} \simeq \sum_{j\leq q}\theta_j^* + \sqrt{q}\left(\sum_{q<j\leq n}(\theta_j^*)^2\right)^{1/2} \tag{1.7}$$

for every $q \geq 1$ (see, for example, [6]). In other words,

$$C(\sqrt{q}) \simeq Z_q(Q_n) \tag{1.8}$$

where $Z_q(K)$ is the L_q-centroid body of K. This shows that (1.3) or (1.2) can be written in the form

$$K_{n,N} \supseteq c(\rho) Z_{\beta\ln(N/n)}(Q_n). \tag{1.9}$$

This observation leads us to consider a random polytope $K_N = \mathrm{conv}\{x_1,\dots,x_N\}$ spanned by N independent random points $x_1,\dots,x_N$ uniformly distributed in an isotropic convex body K and try to compare K_N with $Z_q(K)$ for a suitable value $q = q(N,n) \simeq \ln(N/n)$. Our first main result states that an analogue of (1.9) holds true in full generality.

Theorem 1.1 *Let $\beta \in (0,1/2]$ and $\gamma > 1$. If*

$$N \geq N(\gamma,n) = c\gamma n, \tag{1.10}$$

where $c > 0$ is an absolute constant, for every isotropic convex body K in $\mathbb{R}^n$ we have

$$K_N \supseteq c_1\, Z_q(K) \text{ for all } q \leq c_2\beta\ln(N/n), \tag{1.11}$$

with probability greater than

$$1 - \exp\left(-c_3 N^{1-\beta} n^{\beta}\right) - \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K\sqrt{N}), \tag{1.12}$$

where $\Gamma : \ell_2^n \to \ell_2^N$ is the random operator $\Gamma(y) = (\langle x_1, y\rangle, \dots \langle x_N, y\rangle)$ defined by the vertices $x_1,\dots,x_N$ of K_N.

The proof of Theorem 1.1 is given in Section 2, where we also collect what is known about the probability $\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K\sqrt{N})$ which appears in (1.12).

It should be emphasized that a reverse inclusion of the form $K_N \subseteq c_4\, Z_q(K)$ cannot be expected with probability close to 1, unless q is of the order of n. This follows by a simple volume argument which makes use of the upper estimate of Paouris (see [20]) for the volume of $Z_q(K)$ and is presented in Section 3. However, one can easily see that K_N is "weakly sandwiched" between $Z_{q_i}(K)$ $(i = 1, 2)$, where $q_i \simeq \ln(N/n)$, in the following sense:

Proposition 1.2 *For every $\alpha > 1$ one has*

$$\mathbb{E}\left[\sigma(\theta : (h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta))\right] \leq N\alpha^{-q}. \tag{1.13}$$

This shows that if $q \geq c_5 \ln(N/n)$ then, for most $\theta \in S^{n-1}$, one has $h_{K_N}(\theta) \leq c_6 h_{Z_q(K)}(\theta)$. It follows that several geometric parameters of K_N, e.g. the mean width, are controlled by the corresponding parameter of $Z_{[\ln(N/n)]}(K)$.

As an application, we discuss the volume radius of K_N: Let K be a convex body of volume 1 in $\mathbb{R}^n$. The question to estimate the expected volume radius

$$\mathbb{E}(K,N) = \int_K \cdots \int_K |\mathrm{conv}(x_1,\dots,x_N)|^{1/n} dx_N \cdots dx_1 \tag{1.14}$$

of K_N was studied in [12] where it was proved that for every isotropic convex body K in $\mathbb{R}^n$ and every $N \geq n+1$,

$$\mathbb{E}(B(n),N) \leq \mathbb{E}(K,N) \leq cL_K \frac{\ln(2N/n)}{\sqrt{n}}, \tag{1.15}$$

where $B(n)$ is a ball of volume 1. This estimate is rather weak for large values of N: a strong conjecture is that

$$\mathbb{E}(K,N) \simeq \min\left\{ \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}, 1 \right\} L_K \tag{1.16}$$

for every $N \geq n+1$. This was verified in [10] in the unconditional case, where it was also shown that the general problem is related to the "ψ_2-behavior" of linear functionals on isotropic convex bodies. Using a recent result of G. Paouris [21] on the negative moments of the support function of $h_{Z_q(K)}$ we can settle the question for the full range of values of N.

Theorem 1.3 *For every $N \leq \exp(n)$, one has*

$$c_4 \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \leq |K_N|^{1/n} \leq c_5 L_K \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \tag{1.17}$$

with probability greater than $1 - \frac{1}{N}$, where $c_4, c_5 > 0$ are absolute constants.

Notation and terminology. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also write $\overline{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\overline{A} := \frac{A}{|A|^{1/n}}$.

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $-x \in C$ whenever $x \in C$. We say that C has center of mass at the origin if $\int_C \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. The mean width of C is defined by

$$w(C) = \int_{S^{n-1}} h_C(\theta) \sigma(d\theta). \tag{1.18}$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$, and the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in C\}. \tag{1.19}$$

Whenever we write $a \simeq b$, we mean that there exist universal constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters $c, c', c_1, c_2 > 0$ etc., denote universal positive constants which may change from line to line.

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, center of mass at the origin, and there is a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.20}$$

for every θ in the Euclidean unit sphere S_2^{n-1}. For every convex body K in $\mathbb{R}^n$ there exists an affine transformation T of $\mathbb{R}^n$ such that $T(K)$ is isotropic. Moreover, if we ignore orthogonal transformations, this isotropic image is unique, and hence, the isotropic constant L_K is an invariant of the affine class of K. We refer to [18] and [8] for more information on isotropic convex bodies.

2 The main inclusion

In this Section we prove Theorem 1.1. Let K be an isotropic convex body in $\mathbb{R}^n$. For every $q \geq 1$ consider the L_q–centroid body $Z_q(K)$ of K; recall that

$$h_{Z_q(K)}(x) = \|\langle \cdot, x\rangle\|_q := \left(\int_K |\langle y, x\rangle|^q dy\right)^{1/q}. \tag{2.1}$$

Since $|K| = 1$, we readily see that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \text{conv}\{K, -K\}$. On the other hand, one has the reverse inclusions

$$Z_q(K) \subseteq \frac{cq}{p} Z_p(K) \tag{2.2}$$

for every $1 \leq p < q < \infty$, as a consequence of the ψ_1–behavior of $y \mapsto \langle y, x\rangle$. Observe that $Z_q(K)$ is always symmetric, and $Z_q(TK) = T(Z_q(K))$ for every $T \in SL(n)$ and $q \in [1, \infty]$. Also, if K has its center of mass at the origin, then $Z_q(K) \supseteq cZ_\infty(K)$ for all $q \geq n$, where $c > 0$ is an absolute constant. We refer to [8] for proofs of these assertions and further information.

Lemma 2.1 *Let $0 < t < 1$. For every $\theta \in S^{n-1}$ one has*

$$\mathbb{P}\left(\{x \in K : |\langle x, \theta\rangle| \geq t\|\langle \cdot, \theta\rangle\|_q\}\right) \geq \frac{(1 - t^q)^2}{C^q}. \tag{2.3}$$

Proof. We apply the Paley-Zygmund inequality

$$\mathbb{P}\left(g(x) \geq t^q \mathbb{E}\left(g\right)\right) \geq (1-t^q)^2 \frac{[\mathbb{E}\left(g\right)]^2}{\mathbb{E}\left(g^2\right)} \tag{2.4}$$

for the function $g(x) = |\langle x, \theta\rangle|^q$. Since, by (2.2),

$$\mathbb{E}\left(g^2\right) = \mathbb{E}\,|\langle x, \theta\rangle|^{2q} \leq C^q \left(\mathbb{E}\,|\langle x, \theta\rangle|^q\right)^2 = C^q \left[\mathbb{E}\left(g\right)\right]^2 \tag{2.5}$$

for some absolute constant $C > 0$, the lemma is proved. □

Lemma 2.2 *For every $\sigma \subseteq \{1, \ldots, N\}$ and any $\theta \in S^{n-1}$ one has*

$$\mathbb{P}\left(\{\vec{X} = (x_1, \ldots, x_N) \in K^N \ : \ \max_{j\in\sigma} |\langle x_j, \theta\rangle| \leq \frac{1}{2}\|\langle\cdot, \theta\rangle\|_q\}\right) \leq \exp\left(-|\sigma|/(4C^q)\right), \tag{2.6}$$

where $C > 0$ is an absolute constant.

Proof. Applying Lemma 2.1 with $t = 1/2$ we see that

$$\begin{aligned} \mathbb{P}\left(\max_{j\in\sigma} |\langle x_j, \theta\rangle| \leq \frac{1}{2}\|\langle\cdot, \theta\rangle\|_q\right) &= \prod_{j\in\sigma} \mathbb{P}\left(|\langle x_j, \theta\rangle| \leq \frac{1}{2}\|\langle\cdot, \theta\rangle\|_q\right) \\ &\leq \left(1 - \frac{1}{4C^q}\right)^{|\sigma|} \\ &\leq \exp\left(-|\sigma|/(4C^q)\right), \end{aligned}$$

since $1 - v < e^{-v}$ for every $v > 0$. □

Proof of Theorem 1.1. Let $\Gamma : \ell_2^n \to \ell_2^N$ be the random operator defined by

$$\Gamma(y) = (\langle x_1, y\rangle, \ldots, \langle x_N, y\rangle). \tag{2.7}$$

We modify an idea from [16]. Define $m = [8(N/n)^{2\beta}]$ and $k = [N/m]$. Fix a partition $\sigma_1, \ldots, \sigma_k$ of $\{1, \ldots, N\}$ with $m \leq |\sigma_i|$ for all $i = 1, \ldots, k$ and define the norm

$$\|u\|_0 = \frac{1}{k}\sum_{i=1}^{k} \|P_{\sigma_i}(u)\|_\infty. \tag{2.8}$$

Since

$$h_{K_N}(z) = \max_{1\leq j\leq N} |\langle x_j, z\rangle| \geq \|P_{\sigma_i}\Gamma(z)\|_\infty \tag{2.9}$$

for all $z \in \mathbb{R}^n$ and $i = 1, \ldots, k$, we observe that

$$h_{K_N}(z) \geq \|\Gamma(z)\|_0. \tag{2.10}$$

If $z \in \mathbb{R}^n$ and $\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle\cdot, z\rangle\|_q$, then, Markov's inequality implies that there exists $I \subset \{1, \dots, k\}$ with $|I| > k/2$ such that $\|P_{\sigma_i}\Gamma(z)\|_\infty < \frac{1}{2}\|\langle\cdot, z\rangle\|_q$, for all $i \in I$. It follows that, for fixed $z \in S^{n-1}$ and $\alpha \geq 1$,

$$\begin{aligned}
&\mathbb{P}\left(\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle\cdot, z\rangle\|_q\right) \\
&\qquad \leq \sum_{|I|=[(k+1)/2]} \mathbb{P}\left(\|P_{\sigma_i}\Gamma(z)\|_\infty < \frac{1}{2}\|\langle\cdot, z\rangle\|_q, \text{ for all } i \in I\right) \\
&\qquad \leq \sum_{|I|=[(k+1)/2]} \prod_{i \in I} \mathbb{P}\left(\|P_{\sigma_i}\Gamma(z)\|_\infty < \frac{1}{2}\|\langle\cdot, z\rangle\|_q\right) \\
&\qquad \leq \sum_{|I|=[(k+1)/2]} \prod_{i \in I} \exp\left(-|\sigma_i|/(4C^q)\right) \\
&\qquad \leq \binom{k}{[(k+1)/2]} \exp\left(-c_1 km/C^q\right) \\
&\qquad \leq \exp\left(k \ln 2 - c_1 km/C^q\right).
\end{aligned}$$

Choosing

$$q \simeq \beta \ln(N/n) \tag{2.11}$$

we see that

$$\mathbb{P}\left(\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle\cdot, z\rangle\|_q\right) \leq \exp\left(-c_2 N^{1-\beta} n^\beta\right). \tag{2.12}$$

Let $S = \{z : \|\langle\cdot, z\rangle\|_q/2 = 1\}$ and consider a δ-net U of S with cardinality $|U| \leq (3/\delta)^n$. For every $u \in U$ we have

$$\mathbb{P}\left(\|\Gamma(u)\|_0 < \frac{1}{2}\right) \leq \exp\left(-c_2 N^{1-\beta} n^\beta\right), \tag{2.13}$$

and hence,

$$\mathbb{P}\left(\bigcup_{u \in U}\left\{\|\Gamma(u)\|_0 < \frac{1}{2}\right\}\right) \leq \exp\left(n \ln(3/\delta) - c_2 N^{1-\beta} n^\beta\right). \tag{2.14}$$

Fix $\gamma > 1$ and set

$$\Omega_\gamma = \{\Gamma : \|\Gamma : \ell_2^n \to \ell_2^N\| \leq \gamma L_K \sqrt{N}\}. \tag{2.15}$$

Since $Z_q(K) \supseteq c L_K\, B_2^n$, we have

$$\|\Gamma(z)\|_0 \leq \frac{1}{\sqrt{k}}\|\Gamma(z)\|_2 \leq c\gamma L_K \sqrt{N/k}\|z\|_2 \leq c\gamma\sqrt{N/k}\|\langle\cdot, z\rangle\|_q \tag{2.16}$$

for all $z \in \mathbb{R}^n$ and all Γ in Ω_γ.

Let $z \in S$. There exists $u \in U$ such that $\frac{1}{2}\|\langle \cdot, z-u\rangle\|_q < \delta$, which implies that

$$\|\Gamma(u)\|_0 \leq \|\Gamma(z)\|_0 + c\gamma\delta\sqrt{N/k} \tag{2.17}$$

on Ω_γ. Now, choose $\delta = \sqrt{k/N}/(4c\gamma)$. Then,

$$\begin{aligned}\mathbb{P}\left(\{\Gamma \in \Omega_\gamma :\ \exists z \in \mathbb{R}^n : \|\Gamma(z)\|_0 \leq \|\langle \cdot, z\rangle\|_q/8\}\right)& \\ &= \mathbb{P}\left(\{\Gamma \in \Omega_\gamma :\ \exists z \in S : \|\Gamma(z)\|_0 \leq 1/4\}\right) \\ &\leq \mathbb{P}\left(\{\Gamma \in \Omega_\gamma :\ \exists u \in U : \|\Gamma(u)\|_0 \leq 1/2\}\right) \\ &\leq \exp\left(n\ln(12c\gamma\sqrt{N/k}) - c_2N^{1-\beta}n^\beta\right) \\ &\leq \exp\left(-c_3N^{1-\beta}n^\beta\right)\end{aligned}$$

provided that N is large enough. Since $h_{K_N}(z) \geq \|\Gamma(z)\|_0$ for every $z \in \mathbb{R}^n$, we get that $K_N \supseteq cZ_q(K)$ with probability greater than $1 - \exp\left(-c_4N^{1-\beta}n^\beta\right) - \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K\sqrt{N})$.

We now analyze the restriction for N; we need $n\ln(12c_4\gamma\sqrt{N/k}) \leq CN^{1-\beta}n^\beta$ for some suitable constant $C > 0$. Assuming

$$N \geq 12c\gamma n, \tag{2.18}$$

and since $\beta \in (0, \frac{1}{2}]$, using the definitions of k and m we see that it is enough to guarantee

$$\ln(N/n) \leq C\sqrt{N/n},$$

which is valid if $N/n \geq c_5$ for a suitable absolute constant $c_5 > 0$. We get the result taking (2.18) into account. □

Remark 2.3 The statement of Theorem 1.1 raises the question to estimate the probability

$$\mathbb{P}(\Omega_\gamma) = \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K\sqrt{N}). \tag{2.19}$$

In [16] it was proved that if $\mathit{\Gamma}_{n,N} = (\xi_{ij})_{1\leq i\leq N,\, 1\leq j\leq n}$ is a random matrix, where ξ_{ij} are independent symmetric random variables satisfying $\|\xi_{ij}\|_{L^2} \geq 1$ and $\|\xi_{ij}\|_{L^{\psi_2}} \leq \rho$ for some $\rho \geq 1$, then $\mathbb{P}(\Omega_\gamma) \leq \exp(-c(\rho,\gamma)N)$. In our case, Γ is a random $N \times n$ matrix whose rows are N uniform random points from an isotropic convex body K in $\mathbb{R}^n$. Then, the question is to estimate the probability that, N random points $x_1, \ldots, x_N$ from K satisfy

$$\frac{1}{N}\sum_{j=1}^{N}\langle x_j, \theta\rangle^2 \leq \gamma^2 L_K^2 \tag{2.20}$$

for all $\theta \in S^{n-1}$. This is related to the following well-studied question of Kannan, Lovász and Simonovits [15] which has its origin in the problem of finding a fast algorithm for the computation of the volume of a given convex body: given $\delta, \varepsilon \in (0,1)$, find the smallest positive integer $N_0(n,\delta,\varepsilon)$ so that if $N \geq N_0$ then with probability greater than $1-\delta$ one has

$$(1-\varepsilon)L_K^2 \leq \frac{1}{N}\sum_{j=1}^{N}\langle x_j,\theta\rangle^2 \leq (1+\varepsilon)L_K^2 \tag{2.21}$$

for all $\theta \in S^{n-1}$. In [15] it was proved that one can have $N_0 \simeq c(\delta,\varepsilon)n^2$, which was later improved to $N_0 \simeq c(\delta,\varepsilon)n(\ln n)^3$ by Bourgain [2] and to $N_0 \simeq c(\delta,\varepsilon)n(\ln n)^2$ by Rudelson [24]. One can actually check (see [11]) that this last estimate can be obtained by Bourgain's argument if we also use Alesker's concentration inequality. For subsequent developments, see see, for instance, [20], [13], [17] and [1].

Here, we are only interested in the upper bound of (2.21); actually, we need an isomorphic version of this upper estimate, since we are allowed to choose an absolute constant $\gamma \gg 1$ in (2.20). An application of the main result of [17] to the isotropic case gives such an estimate: If $N \geq c_1 n\ln^2 n$, then

$$\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma L_K\sqrt{N}) \leq \exp\left(-c_2\gamma\left(\frac{N}{(\ln N)(n\ln n)}\right)^{1/4}\right). \tag{2.22}$$

A slightly better estimate can be extracted from the work of Guédon and Rudelson in [13]. It should be emphasized that this term does not allow us to fully exploit the second term $\exp\left(-c_3N^{1-\beta}n^{\beta}\right)$ in the probability estimate of Theorem 1.1. However, it is not clear if it is optimal.

Remark 2.4 G. Paouris and E. Werner [22] have recently studied the relation between the family of L_q-centroid bodies and the family of floating bodies of a convex body K. Given $\delta \in (0,\frac{1}{2}]$, the floating body K_δ of K is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume δ from K. It was observed in [18] that K_δ is isomorphic to an ellipsoid as long as δ stays away from 0. In [22] it is proved that

$$c_1 Z_{\ln(1/\delta)}(K) \subseteq K_\delta \subseteq c_2 Z_{\ln(1/\delta)}(K) \tag{2.23}$$

where $c_1, c_2 > 0$ are absolute constants. From Theorem 1.1 it follows that if K is isotropic and if, for example, $N \geq n^2$ then

$$K_N \supseteq c_3 K_{1/N} \tag{2.24}$$

with probability greater than $1 - o_n(1)$, where $c_3 > 0$ is an absolute constant. This fact should be compared with the following well-known result from [3]: for any convex body K in $\mathbb{R}^n$ one has $c|K_{1/N}| \leq \mathbb{E}\,|K_N| \leq c_n|K_{1/N}|$ (where the constant on the left is absolute and the right hand side inequality holds true with a constant c_n depending on the dimension, for N large enough; the critical value of N is exponential in n).

2.1 Unconditional case

In this subsection we consider separately the case of unconditional convex bodies: we assume that K is centrally symmetric and that, after a linear transformation, the standard orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is a 1-unconditional basis for $\|\cdot\|_K$, i.e. for every choice of real numbers $t_1,\dots,t_n$ and every choice of signs $\varepsilon_j=\pm1$,

$$\big\|\varepsilon_1t_1e_1+\cdots+\varepsilon_nt_ne_n\big\|_K=\big\|t_1e_1+\cdots+t_ne_n\big\|_K. \tag{2.25}$$

Then, a diagonal operator brings K to the isotropic position. It is also known that the isotropic constant of an unconditional convex body K satisfies $L_K\simeq 1$.

Bobkov and Nazarov have proved that $K\supseteq c_2Q_n$, where $Q_n=[-\frac12,\frac12]^n$ (see [4]). The following argument of R. Latala (private communication) shows that the family of L_q-centroid bodies of the cube Q_n is extremal in the sense that $Z_q(K)\supseteq cZ_q(Q_n)$ for all $q\geq 1$, where $c>0$ is an absolute constant: Let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ be independent and identically distributed ±1 random variables, defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, with distribution $\mathbb{P}(\varepsilon_i=1)=\mathbb{P}(\varepsilon_i=-1)=\frac12$. For every $\theta\in S^{n-1}$, by the unconditionality of K, Jensen's inequality and the contraction principle, one has

$$\begin{aligned}
\|\langle\cdot,\theta\rangle\|_{L^q(K)} &= \left(\int_K\left|\sum_{i=1}^n\theta_ix_i\right|^q dx\right)^{1/q}=\left(\int_\Omega\int_K\left|\sum_{i=1}^n\theta_i\varepsilon_i|x_i|\right|^q dx\,d\mathbb{P}(\varepsilon)\right)^{1/q}\\
&\geq \left(\int_\Omega\left|\sum_{i=1}^n\theta_i\varepsilon_i\int_K|x_i|\,dx\right|^q d\mathbb{P}(\varepsilon)\right)^{1/q}=\left(\int_\Omega\left|\sum_{i=1}^n t_i\theta_i\varepsilon_i\right|^q d\mathbb{P}(\varepsilon)\right)^{1/q}\\
&\geq \left(\int_{Q_n}\left|\sum_{i=1}^n t_i\theta_iy_i\right|^q dy\right)^{1/q}=\|\langle\cdot,(t\theta)\rangle\|_{L^q(Q_n)},
\end{aligned}$$

where $t_i=\int_K|x_i|\,dx$ and $t\theta=(t_1\theta_1,\dots,t_n\theta_n)$. Since $t_i\simeq 1$ for all $i=1,\dots,n$, from (1.7) we readily see that

$$\|\langle\cdot,\theta\rangle\|_{L^q(K)}\geq\|\langle\cdot,(t\theta)\rangle\|_{L^q(Q_n)}\geq c\|\langle\cdot,\theta\rangle\|_{L^q(Q_n)}.$$

In view of (1.8), this observation and Theorem 1.1 show that, if K is unconditional, then a random K_N contains $Z_{\ln(N/n)}(Q_n)$:

Theorem 2.5 *Let $\beta\in(0,1/2]$ and $\gamma>1$. There exists an absolute constant $c>0$ so that if*

$$N\geq N(\gamma,n)=c\gamma n, \tag{2.26}$$

and if $K_N=\mathrm{conv}\{x_1,\dots,x_N\}$ is a random polytope spanned by N independent random points $x_1,\dots,x_N$ uniformly distributed in an unconditional isotropic convex body K in $\mathbb{R}^n$, then we have

$$K_N\supseteq c_1\,C(\alpha)=c_1\left(\alpha B_2^n\cap B_\infty^n\right)\quad \textit{for all } \alpha\leq c_2\sqrt{\beta\ln(N/n)}, \tag{2.27}$$

with probability greater than

$$1-\exp\left(-c_3N^{1-\beta}n^{\beta}\right)-\mathbb{P}(\|\Gamma:\ell_2^n\to\ell_2^N\|\geq\gamma\sqrt{N}), \tag{2.28}$$

where $\Gamma:\ell_2^n\to\ell_2^N$ *is the random operator* $\Gamma(y)=(\langle x_1,y\rangle,\dots\langle x_N,y\rangle)$ *defined by the vertices* $x_1,\dots,x_N$ *of* K_N.

Next, we outline a direct proof of Theorem 2.5 (in which L_q-centroid bodies are not involved): For $k\in\mathbb{N}$ and $y\in\mathbb{R}^n$, define

$$\|y\|_{P(k)}:=\sup\left\{\sum_{i=1}^{k}\left(\sum_{j\in B_i}y_j^2\right)^{1/2}:\ \bigcup_{i=1}^{k}B_i=[n],\ B_i\cap B_j=\emptyset\ (i\neq j)\right\}, \tag{2.29}$$

where we write $[n]$ for the set $\{1,2,\dots,n\}$. Montgomery–Smith has shown (see [19]) that: For any $y\in\mathbb{R}^n$ and $k\in\mathbb{N}$, one has

$$\mathbb{P}\left(\sum_{i=1}^{n}\varepsilon_iy_i\geq\lambda\|y\|_{P(k)}\right)\geq\left(\frac{1}{3}\right)^{k}(1-2\lambda^2)^{2k}\qquad(0\leq\lambda\leq1/\sqrt{2}). \tag{2.30}$$

Also, for $y\in\mathbb{R}^n$, one has

$$\|y\|_{P(t^2)}\leq K_{1,2}(y,t)\leq\sqrt{2}\|y\|_{P(t^2)} \tag{2.31}$$

when $t^2\in\mathbb{N}$, from where one concludes the following:

Lemma 2.6 *There exists a constant* $c>0$ *such that, for all* $y\in\mathbb{R}^n$ *and any* $t>0$,

$$\mathbb{P}\left(\sum_{i=1}^{n}\varepsilon_iy_i\geq\lambda K_{1,2}(y,t)\right)\geq e^{-\phi(\lambda)t^2}, \tag{2.32}$$

where $\phi(\lambda)=4\ln\bigl(3(1-2\lambda^2)^{-2}\bigr)$ *for* $0<\lambda<1/\sqrt{2}$.

P. Pivovarov [23] has recently obtained the following result: There exists an absolute constant $C\geq1$ such that for any unconditional isotropic convex body K in $\mathbb{R}^n$, the spherical measure of the set of $\theta\in S^{n-1}$ such that

$$\mathbb{P}\left(|\langle x,\theta\rangle|\geq t\right)\geq\exp(-Ct^2)$$

whenever $C\leq t\leq\frac{\sqrt{n}}{C\ln n}$, is at least $1-2^{-n}$. The proof of the next Lemma follows more or less the same lines.

Lemma 2.7 *Let* K *be an isotropic unconditional convex body in* $\mathbb{R}^n$. *For every* $\theta\in S^{n-1}$ *and any* $\alpha\geq1$ *we have*

$$\mathbb{P}_x\left(\langle x,\theta\rangle\geq h_{C(\alpha)}(\theta)\right)\geq c_1e^{-c_2\alpha^2}. \tag{2.33}$$

Proof. For $\theta = (\theta_i)_{i=1}^n \in \mathbb{S}^{n-1}$, $x \in K$ and $0 < s < 1/\sqrt{2}$ define the set

$$K_s(\theta) = \{x \in K \,:\, K_{1,2}(\theta, \alpha) \leq s K_{1,2}(x\theta, \alpha)\}, \tag{2.34}$$

where by "$x\theta$" we mean the vector with coordinates $x_i\theta_i$ and s is to be chosen. We have:

$$\begin{aligned}
\mathbb{P}_x\left(\sum_{i=1}^n x_i\theta_i \geq h_{C(\alpha)}(\theta)\right) &= \mathbb{P}_x\left(\sum_{i=1}^n \varepsilon_i x_i\theta_i \geq h_{C(\alpha)}(\theta)\right) \\
&= \int_K \mathbb{P}_\varepsilon\left(\sum_{i=1}^n \varepsilon_i (x_i\theta_i) \geq h_{C(\alpha)}(\theta)\right) dx \\
&= \int_K \mathbb{P}_\varepsilon\left(\sum_{i=1}^n \varepsilon_i (x_i\theta_i) \geq K_{1,2}(\theta, \alpha)\right) dx \\
&\geq \int_{K_s(\theta)} \mathbb{P}_\varepsilon\left(\sum_{i=1}^n \varepsilon_i (x_i\theta_i) \geq s K_{1,2}(x\theta, \alpha)\right) dx \\
&\geq e^{-\phi(s)\alpha^2}\, |K_s(\theta)|,
\end{aligned}$$

by Lemma 2.6.

Assume first that $m := \alpha^2$ is an integer and let $B_1, B_2, \ldots, B_m$ be a partition of the set $\{1, 2, \ldots, n\}$ so that

$$K_{1,2}(\theta, \alpha) = \sum_{i=1}^m \left(\sum_{j \in B_i} |\theta_j|^2\right)^{1/2} =: A. \tag{2.35}$$

Consider the seminorm

$$f(x) = \sum_{i=1}^m \left(\sum_{j \in B_i} |x_j\theta_j|^2\right)^{1/2}. \tag{2.36}$$

Then, using the reverse Hölder inequality $c_1\|f\|_{L^2(K)} \leq \|f\|_{L^1(K)}$ and the fact that $L_K \simeq 1$, we get

$$\begin{aligned}
\int_K K_{1,2}(x\theta, \alpha)\, dx &\geq \int_K \sum_{i=1}^m \left(\sum_{j \in B_i} |x_j\theta_j|^2\right)^{1/2} \\
&\geq c_1 \sum_{i=1}^m \left(\sum_{j \in B_i} |\theta_j|^2 \int_K |x_j|^2\right)^{1/2} \\
&\geq cA.
\end{aligned}$$

We now apply the Paley-Zygmund inequality to get

$$|K_s(\theta)| = \mathbb{P}_x\left(f > sA\right) \geq \frac{(\mathbb{E}|f|^2 - (sA)^2)^2}{\mathbb{E}\left[f^4\right]}. \tag{2.37}$$

Choosing $s = \frac{1}{2\sqrt{2}} \min\{c, 1\}$ we get

$$|K_s(\theta)| \geq \frac{cA^4}{\mathbb{E}\,[f^4]},$$

for a suitable new absolute constant $c > 0$. On the other hand, we can estimate $\mathbb{E}\,[f^4]$ from above, by the reverse Hölder inequality:

$$\begin{aligned}(\mathbb{E}\,[f^4])^{1/4} &\leq 4c\mathbb{E}|f| = 4c \sum_{i=1}^{m} \mathbb{E} \left(\sum_{j \in B_i} |x_j \theta_j|^2 \right)^{1/2} \\ &\leq 4cL_K \sum_{i=1}^{m} \left(\sum_{j \in B_i} |\theta_j|^2 \right)^{1/2} \leq 4cA.\end{aligned}$$

As a result, $|K_s(\theta)| \geq c$. Returning to the estimate

$$\mathbb{P}_x \left(\sum_{i=1}^{n} x_i \theta_i \geq h_{C(\alpha)}(\theta) \right) \geq e^{-\phi(s)\alpha^2} |K_s(\theta)|, \tag{2.38}$$

we get:

$$\mathbb{P}_x \left(\sum_{i=1}^{n} x_i \theta_i \geq h_{C(\alpha)}(\theta) \right) \geq ce^{-c\alpha^2}. \tag{2.39}$$

This proves the Lemma for $\alpha^2 \in \mathbb{N}$ and the result follows easily for every α. □

Proof of Theorem 2.5. Now, using the procedure of the proof of Theorem 1.1 we complete the proof of Theorem 2.5. □

Remark 2.8 Regarding the probability $\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq \gamma\sqrt{N})$, in the unconditional case Aubrun has proved in [1] that for every $\rho > 1$ and $N \geq \rho n$, one has

$$\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq c_1(\rho)\sqrt{N}) \leq \exp(-c_2(\rho)n^{1/5}). \tag{2.40}$$

In particular, one can find $c, C > 0$ so that, if $N \geq Cn$, then

$$\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \geq C\sqrt{N}) \leq \exp(-cn^{1/5}). \tag{2.41}$$

This allows us to use Theorem 2.5 with a probability estimate $1 - \exp(-cn^c)$ for values of N which are proportional to n.

3 Weakly sandwiching K_N

We proceed to the question whether the inclusion given by Theorem 1.1 is sharp. It was already mentioned in the Introduction that we cannot expect a reverse inclusion of the form $K_N \subseteq c_4 Z_q(K)$ with probability close to 1, unless if q is of the order of n. To see this, observe that, for any $\alpha > 0$,

$$\begin{aligned} \mathbb{P}\big(K_N \subseteq \alpha Z_q(K)\big) &= \mathbb{P}\big(x_1, x_2, \dots, x_N \in \alpha Z_q(K)\big) \\ &= \Big(\mathbb{P}\big(x \in \alpha Z_q(K)\big)\Big)^N \\ &\leq |\alpha Z_q(K)|^N. \end{aligned}$$

It was proved in [20] that, for every $q \leq n$, the volume of $Z_q(K)$ is bounded by $(c\sqrt{q/n}L_K)^n$. This leads immediately to the estimate

$$\mathbb{P}\big(K_N \subseteq \alpha Z_q(K)\big) \leq (c\alpha\sqrt{q/n}L_K)^{nN}, \tag{3.1}$$

where $c > 0$ is an absolute constant. Assume that K has bounded isotropic constant and we want to keep $\alpha \simeq 1$. Then, (3.1) shows that, independently from the value of N, we have to choose q of the order of n so that it might be possible to show that $\mathbb{P}\big(K_N \subseteq \alpha Z_q(K)\big)$ is really close to 1. Actually, if $q \sim n$ then this is always the case, because $Z_n(K) \supseteq cK$.

Lemma 3.1 *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $N > n$. Fix $\alpha > 1$. Then, for every $\theta \in S^{n-1}$ one has*

$$\mathbb{P}\left(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)\right) \leq N\alpha^{-q}. \tag{3.2}$$

Proof. Markov's inequality shows that

$$\mathbb{P}(\alpha,\theta) := \mathbb{P}\left(x \in K : |\langle x,\theta\rangle| \geq \alpha\|\langle\cdot,\theta\rangle\|_q\right) \leq \alpha^{-q}. \tag{3.3}$$

Then,

$$\mathbb{P}\left(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)\right) = \mathbb{P}\left(\max_{j\leq N} |\langle x_j,\theta\rangle| \geq \alpha\|\langle\cdot,\theta\rangle\|_q\right) \leq N\,\mathbb{P}(\alpha,\theta)$$

and the result follows. □

Lemma 3.2 *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $N > n$. For every $\alpha > 1$ one has*

$$\mathbb{E}\left[\sigma(\theta : (h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta))\right] \leq N\alpha^{-q}. \tag{3.4}$$

Proof. Immediate: observe that

$$\mathbb{E}\left[\sigma(\theta : (h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta))\right] = \int_{S^{n-1}} \mathbb{P}\left(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)\right) d\sigma(\theta)$$

by Fubini's theorem. □

The estimate of Lemma 3.2 is already enough to show that if $q \geq c \ln N$ then, on the average, $h_{K_N}(\theta) \leq c h_{Z_q(K)}(\theta)$ with probability greater than $1 - N^{-c}$. In particular, the mean width of a random K_N is bounded by the mean width of $Z_{\ln(N/n)}(K)$:

Proposition 3.3 *Let K be an isotropic convex body in $\mathbb{R}^n$. If $q \geq 2 \ln N$ then*

$$\mathbb{E}\left[w(K_N)\right] \leq c\, w(Z_q(K)), \tag{3.5}$$

where $c > 0$ is an absolute constant.

Proof. We write

$$w(K_N) \leq \int_{A_N} h_{K_N}(\theta)\, d\sigma(\theta) + c\sigma(A_N^c) n L_K, \tag{3.6}$$

where $A_N = \{\theta : h_{K_N}(\theta) \leq \alpha h_{Z_q(K)}(\theta)\}$. Then,

$$w(K_N) \leq \alpha \int_{A_N} h_{Z_q(K)}(\theta)\, d\sigma(\theta) + c\sigma(A_N^c) n L_K, \tag{3.7}$$

and hence, by Lemma 3.2,

$$\mathbb{E}\, w(K_N) \leq \alpha w(Z_q(K)) + cNn\alpha^{-q} L_K. \tag{3.8}$$

Since $w(Z_q(K)) \geq w(Z_2(K)) = L_K$, we get

$$\mathbb{E}\, w(K_N) \leq (\alpha + cNn\alpha^{-q}) w(Z_q(K)). \tag{3.9}$$

The result follows if we choose $\alpha = e$. □

3.1 Volume radius of K_N

Next, we discuss the volume radius of K_N. A lower bound follows by comparison with the Euclidean ball. It was proved in [12, Lemma 3.3] that if K is a convex body in $\mathbb{R}^n$ with volume 1, then

$$\mathbb{P}\left(|K_N| \geq t\right) \geq \mathbb{P}(|[\overline{B}_2^n]_N| \geq t) \tag{3.10}$$

for every $t > 0$. Therefore, it is enough to consider the case of B_2^n. In [10] it is shown that there exist $c_1 > 1$ and $c_2 > 0$ such that if $N \geq c_1 n$ and $x_1, \ldots, x_N$ are independent random points uniformly distributed in $\overline{B}_2^n$, then

$$[\overline{B}_2^n]_N \supseteq c_2 \min\left\{\frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}, 1\right\} \overline{B}_2^n \tag{3.11}$$

with probability greater than $1-\exp(-n)$. It follows that if $N \geq c_1 n$ then, with probability greater than $1-\exp(-n)$ we have

$$|K_N|^{1/n} \geq c_2 \min\left\{\frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}, 1\right\}, \tag{3.12}$$

where $c_1 > 1$ and $c_2 > 0$ are absolute constants.

The case $n < N < c_1 n$ was studied in [7] where it was proved that (3.11) continues to hold true with probability greater than $1-\exp(-cn/\ln n)$, where $c > 0$ is an absolute constant. Combining this fact with (3.10), we see that (3.12) is valid for all $N > n$.

We now pass to the upper bound; Proposition 3.3, combined with Urysohn's inequality, yields the following:

Proposition 3.4 *Let K be an isotropic convex body in $\mathbb{R}^n$. If $N > n$ and $q \geq 2\ln N$, then*

$$\mathbb{E}(K,N) \leq \frac{c_1 \mathbb{E}\left[w(K_N)\right]}{\sqrt{n}} \leq \frac{c_2\, w(Z_q(K))}{\sqrt{n}}, \tag{3.13}$$

where $c_1, c_2 > 0$ are absolute constants.

Proposition 3.4 reduces, in a sense, the question to that of giving upper bounds for $w(Z_q(K))$. It is proved in [20] that, if $q = \ln N \leq \sqrt{n}$ then $w(Z_q(K)) \leq c\sqrt{q}L_K$. It follows that

$$\mathbb{E}(K,N) \leq c\frac{\sqrt{\ln(N/n)}L_K}{\sqrt{n}}, \tag{3.14}$$

which is the conjectured estimate for $N \leq e^{\sqrt{n}}$. For $q = \ln N > \sqrt{n}$ we know that $w(Z_q(K)) \leq \frac{qL_K}{\sqrt[4]{n}}$ since $Z_q(K) \subseteq (q/\sqrt{n})\, Z_{\sqrt{n}}(K)$. This is most probably a non-optimal bound.

However, we can further exploit the simple estimate of Lemma 3.1 to obtain a sharp estimate for larger values of N. We will make use of the following facts:

Fact 1. Let A be a symmetric convex body in $\mathbb{R}^n$. For any $1 \leq q < n$, set

$$w_{-q}(A) = \left(\int_{S^{n-1}} \frac{1}{h_A^q(\theta)}\, d\sigma(\theta)\right)^{-1/q}. \tag{3.15}$$

An application of Hölder's inequality shows that

$$\left(\frac{|A^\circ|}{|B_2^n|}\right)^{1/n} = \left(\int_{S^{n-1}} \frac{1}{h_A^n(\theta)}\, d\sigma(\theta)\right)^{1/n} \geq \left(\int_{S^{n-1}} \frac{1}{h_A^q(\theta)}\, d\sigma(\theta)\right)^{1/q} = \frac{1}{w_{-q}(A)}. \tag{3.16}$$

From the Blaschke–Santaló inequality, it follows that

$$|A|^{1/n} \leq |B_2^n|^{1/n} w_{-q}(A) \leq \frac{c_1 w_{-q}(A)}{\sqrt{n}}. \tag{3.17}$$

Fact 2. A recent result of G. Paouris (see [21, Proposition 5.4]) shows that if A is an isotropic convex body in $\mathbb{R}^n$ then, for any $1 \le q < n/2$,

$$w_{-q}(Z_q(A)) \simeq \frac{\sqrt{q}}{\sqrt{n}}\, I_{-q}(A) \tag{3.18}$$

where

$$I_p(A) = \left(\int_A \|x\|_2^p \, dx\right)^{1/p}, \qquad p > -n. \tag{3.19}$$

Fact 3. Let K be an isotropic convex body in $\mathbb{R}^n$, let $N > n^2$ and $q = 2\ln(2N)$. We write

$$\begin{aligned} [w_{-q/2}(Z_q(K))]^{-q} &= \left(\int_{S^{n-1}} \frac{1}{h^{q/2}_{Z_q(K)}(\theta)}\, d\sigma(\theta)\right)^2 \\ &\le \left(\int_{S^{n-1}} \frac{1}{h^q_{K_N}(\theta)}\, d\sigma(\theta)\right)\left(\int_{S^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(K)}(\theta)}\, d\sigma(\theta)\right). \end{aligned}$$

Observe that $K_N \subseteq K \subseteq (n+1)L_K$ and $Z_q(K) \supseteq Z_2(K) \supseteq L_K\, B_2^n$, and hence, $h_{K_N}(\theta) \le (n+1)h_{Z_q(K)}(\theta)$ for all $\theta \in S^{n-1}$. Therefore,

$$\int_{S^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(K)}(\theta)}\, d\sigma(\theta) = \int_0^{n+1} qt^{q-1}\left[\sigma\big(\theta :\, h_{K_N}(\theta) \ge t h_{Z_q(K)}(\theta)\big)\right] dt. \tag{3.20}$$

Fact 4. Taking expectations in (3.20) and using Lemma 3.2, we see that, for every $a > 1$,

$$\begin{aligned} \mathbb{E}\left[\int_{S^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(K)}(\theta)}\, d\sigma(\theta)\right] &\le a^q + \int_a^{n+1} qt^{q-1} N t^{-q}\, dt \\ &= a^q + qN\ln\left(\frac{n+1}{a}\right). \end{aligned}$$

Choosing $a = 2e$ and using the fact that $e^q = (2N)^2$ by the choice of q, we see that

$$\mathbb{E}\left[\int_{S^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(K)}(\theta)}\, d\sigma(\theta)\right] \le c_2^q \tag{3.21}$$

where $c_2 > 0$ is an absolute constant. Then, Markov's inequality implies that

$$\int_{S^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(K)}(\theta)}\, d\sigma(\theta) \le (c_2 e)^q \tag{3.22}$$

with probability greater than $1 - e^{-q}$. Going back to Fact 3, we conclude that $[w_{-q/2}(Z_q(K))]^{-q} \leq c_3^q [w_{-q}(K_N)]^{-q}$, i.e.

$$w_{-q}(K_N) \leq c_4 w_{-q/2}(Z_q(K)) \tag{3.23}$$

with probability greater than $1 - e^{-q}$.

Proof of Theorem 1.3. Assume that K_N satisfies (3.23) and set $S_N = K_N - K_N$. From Fact 1 we have

$$|K_N|^{1/n} \leq |S_N|^{1/n} \leq \frac{c_1}{\sqrt{n}} w_{-q}(S_N) = \frac{2c_1}{\sqrt{n}} w_{-q}(K_N). \tag{3.24}$$

Now, Fact 4 shows that

$$|K_N|^{1/n} \leq \frac{c_5}{\sqrt{n}} w_{-q/2}(Z_q(K)) \tag{3.25}$$

with probability greater than $1 - e^{-q}$. Since $Z_q(K) \subseteq cZ_{q/2}(K)$, using Fact 2 we write

$$w_{-q/2}(Z_q(K)) \leq c_6 w_{-q/2}(Z_{q/2}(K)) \leq \frac{c_7\sqrt{q}}{\sqrt{n}} I_{-q/2}(K). \tag{3.26}$$

Since K is isotropic, we have $I_{-q/2}(K) \leq I_2(K) = \sqrt{n}\, L_K$, which implies

$$w_{-q/2}(Z_q(K)) \leq c_7\sqrt{q}\, L_K. \tag{3.27}$$

Putting everything together, we have

$$|K_N|^{1/n} \leq \frac{c\sqrt{q}}{\sqrt{n}} L_K \simeq \frac{\sqrt{\ln(N/n)} L_K}{\sqrt{n}}, \tag{3.28}$$

with probability greater than $1 - e^{-q} \geq 1 - \frac{1}{N}$. This completes the proof. □

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