A note on norms of signed sums of vectors

Giorgos Chasapis and Nikos Skarmogiannis

Abstract

Our starting point is an improved version of a result of D. Hajela related to a question of Komlós: we show that if $f(n)$ is a function such that $\lim_{n \to \infty} f(n) = \infty$ and $f(n) = o(n)$, there exists $n_0 = n_0(f)$ such that for every $n \geqslant n_0$ and any $S \subseteq \{-1, 1\}^n$ with cardinality $|S| \leqslant 2^{n/f(n)}$ one can find orthonormal vectors $x_1, \dots, x_n \in \mathbb{R}^n$ that satisfy

$$\|\epsilon_1 x_1 + \dots + \epsilon_n x_n\|_\infty \geqslant c\sqrt{\log f(n)}$$

for all $(\epsilon_1, \dots, \epsilon_n) \in S$. We obtain analogous results in the case where $x_1, \dots, x_n$ are independent random points uniformly distributed in the Euclidean unit ball B_2^n or any symmetric convex body, and the ℓ_∞^n-norm is replaced by an arbitrary norm on $\mathbb{R}^n$.

1 Introduction

Given a pair K, D of symmetric convex bodies in $\mathbb{R}^n$, the parameter $\beta(K, D)$ is the smallest $r > 0$ such that for any $x_1, \dots, x_n \in K$ there exist signs $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ for which

$$\epsilon_1 x_1 + \dots + \epsilon_n x_n \in rD.$$

A general lower bound for $\beta(K, D)$ was proved by Banaszczyk; in [2] he showed that if K and D are two symmetric convex bodies in $\mathbb{R}^n$ then

$$(1.1) \qquad \beta(K, D) \geqslant c\sqrt{n}(|K|/|D|)^{1/n}$$

for an absolute constant $c > 0$, where $|K|$ denotes the volume of K. In what follows we write B_p^n for the unit ball of $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, $1 \leqslant p \leqslant \infty$. A well-known theorem of Spencer [20] states that $\beta(B_\infty^n, B_\infty^n) \leqslant c\sqrt{n}$, where $c > 0$ is an absolute constant (the same result was proved independently by Gluskin in [10]): there exists an absolute constant $c > 0$ such that for any $n \geqslant 1$ and $x_1, \dots, x_n \in \mathbb{R}^n$ with $\|x_i\|_\infty \leqslant 1$, we may find $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ such that

$$(1.2) \qquad \|\epsilon_1 x_1 + \dots + \epsilon_n x_n\|_\infty \leqslant c\sqrt{n}.$$

From (1.1) one readily sees that this result is optimal up to absolute constants. A well-known question of Komlós (see [21] and [22]) asks if the sequence $\beta(B_2^n, B_\infty^n)$ is bounded. Since $B_\infty^n \subseteq \sqrt{n}B_2^n$, a positive answer to this question immediately implies Spencer's bound. The best known estimate is due to Banaszczyk [3]: there exists an absolute constant $c > 0$ such that for any $n \geqslant 1$ and $x_1, \dots, x_n \in \mathbb{R}^n$ with $\|x_i\|_2 \leqslant 1$ one may find $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ such that

$$(1.3) \qquad \|\epsilon_1 x_1 + \dots + \epsilon_n x_n\|_\infty \leqslant c\sqrt{\log n}.$$

In fact, Banaszczyk proved a more general theorem: if K is a convex body in $\mathbb{R}^n$ with Gaussian measure $\gamma_n(K) \geqslant 1/2$ then $\beta(B_2^n, K) \leqslant C$, where $C > 0$ is an absolute constant; this implies (1.3) because $\gamma_n(rB_\infty^n) \geqslant 1/2$ for all $r \geqslant c\sqrt{\log n}$. While the method of [3] is non-constructive, an algorithmic proof of the $O(\sqrt{\log n})$ bound in the above statement was recently given by Bansal, Dadush and Garg [4].

The starting point of this note is a result of Hajela [12] in the direction of providing a negative answer to the question of Komlós.

Theorem 1.1 (Hajela). *Let $f(n)$ be a function such that $\lim\limits_{n\to\infty} f(n) = \infty$ and $f(n) = o(n)$. For any $0 < \lambda < \frac{1}{2}$ there exists $n_0 = n_0(f, \lambda)$ such that for every $n \geqslant n_0$ and any $S \subseteq \{-1, 1\}^n$ with cardinality $|S| \leqslant 2^{n/f(n)}$ one can find orthonormal vectors $x_1, \ldots, x_n \in \mathbb{R}^n$ that satisfy*

$$\|\epsilon_1 x_1 + \cdots + \epsilon_n x_n\|_\infty \geqslant \exp\left(\frac{\lambda \log \log f(n)}{\log \log \log f(n)}\right)$$

for all $(\epsilon_1, \ldots, \epsilon_n) \in S$.

In fact, Hajela conjectures in [12] that the question of Komlós has a negative answer and that the estimate (1.3) that was later obtained by Banaszczyk should be optimal. Our first result is an improved version of Theorem 1.1.

Theorem 1.2. *There exists an absolute constant $c \in (0, 1)$ such that the following holds true: For every $n \geqslant 1$ and $\frac{1}{n} < \delta < 1$, and for any $S \subseteq \{-1, 1\}^n$ with $|S| \leqslant 2^{\delta n}$, there exist orthonormal vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$ such that*

$$\left\|\sum_{i=1}^n \epsilon_i x_i\right\|_\infty \geqslant c\sqrt{\log(e/\delta)}$$

for all $(\epsilon_1, \ldots, \epsilon_n) \in S$.

Theorem 1.2 implies a stronger version of Hajela's theorem. Let $f(n)$ be a function such that $\lim\limits_{n\to\infty} f(n) = \infty$ and $f(n) = o(n)$. Note that if we let $\delta = \delta(f, n) = ef(n)^{-1}$ in Theorem 1.2 then $\frac{1}{n} < \delta < 1$ for large enough n, resulting to the lower bound

$$\left\|\sum_{i=1}^n \epsilon_i x_i\right\|_\infty \geqslant c\sqrt{\log f(n)},$$

which improves upon the estimate of Theorem 1.1. In the proof of Theorem 1.2, presented in Section 2 below, initially we follow the idea of Hajela: the vectors $x_1, \ldots, x_n$ are obtained from a random rotation of the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. The improvement on the estimates is due to Lemma 2.1 which provides a stronger small-ball probability estimate for the $\|\cdot\|_\infty$-norm on the sphere.

The above method to lower-bound the ℓ_∞-norm of a signed sum of vectors has obvious limitations. Namely, one should consider a subset $S \subseteq \{-1, 1\}^n$ of cardinality $2^{o(n)}$ for Theorem 1.2 to obtain a lower bound of order greater than the one in (1.1). Thus, this line of thinking by itself does not seem adequate to provide a negative answer to the question of Komlós. Nevertheless, we find the link between small ball probability estimates and the norm of signed sums of vectors interesting in its own right, so in Section 3 we further explore this phenomenon in several aspects. Initially, the vectors $x_1, \ldots x_n$ are assumed to satisfy a lower bound of the form $\|\sum_{i=1}^n \epsilon_i x_i\|_2 \geqslant c\sqrt{n}$ for all choices of signs $\epsilon_i = \pm 1$, and the ℓ_∞^n-norm is replaced by an arbitrary norm on $\mathbb{R}^n$. In particular, given a symmetric convex body D in $\mathbb{R}^n$, we will prove that for $x_1, \ldots x_n$ as above and for any "big" subset $S \subseteq \{-1, 1\}^n$, the D-norm of $\sum_{i=1}^n \epsilon_i U x_i$ is "large" for every $(\epsilon_1, \ldots, \epsilon_n) \in S$, with overwhelming probability with respect to $U \in O(n)$. This "largeness" is determined by the Gaussian measure of dilates of D; to make this more precise, we give the following definition.

Definition 1.3. Given a symmetric convex body D in $\mathbb{R}^n$, and $\delta \in (0, 1)$, let

$$t_{D,\delta} := \max\{t > 0 : \gamma_n(2t \cdot m(D)D) \leqslant (2^\delta e)^{-n}\},$$

where $m(D)$ is the median of $\|\cdot\|_D$ with respect to the standard Gaussian measure γ_n on $\mathbb{R}^n$. In the case that D is the unit ball B_p^n of some ℓ_p^n, $p \in [1, \infty]$, we abbreviate $t_{p,\delta} := t_{B_p^n, \delta}$.

Note that, for any symmetric convex body D in $\mathbb{R}^n$ and $\delta \in (0, 1)$, $t_{D,\delta}$ satisfies the bounds

$$(1.4) \qquad\qquad c_1|D|^{-1/n} \leqslant t_{D,\delta} m(D) \leqslant \frac{1}{2}m(D),$$

for some absolute constant $c_1 > 0$ (for completeness, we provide a short justification of (1.4) in Section 3.1). Although the definition of $t_{D,\delta}$ might seem somewhat artificial at first sight, we trust that the idea behind it will become clear in the sequel (see the comments after Theorem 1.5).

Using the notation introduced above, our first result is the following statement.

Theorem 1.4. *Let D be a symmetric convex body in $\mathbb{R}^n$ and $\delta \in (0,1)$. For any $\tau > 0$, any n-tuple of vectors $x_1, \ldots, x_n$ with $\min_{\epsilon_i = \pm 1} \|\sum_{i=1}^n \epsilon_i x_i\|_2 \geqslant \tau\sqrt{n}$ and any $S \subseteq \{-1,1\}^n$ with $|S| \leqslant 2^{\delta n}$, there is some $\mathcal{U} \subseteq O(n)$ with $\nu_n(\mathcal{U}) \geqslant 1 - e^{-n}$ such that for every $U \in \mathcal{U}$,*

$$\left\| \sum_{i=1}^n \epsilon_i U(x_i) \right\|_D \geqslant \tau t_{D,\delta} m(D)$$

holds for all $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$.

We then consider the case where $x_1, \ldots, x_n$ are points in an arbitrary convex body K in $\mathbb{R}^n$. We observe that an alternative proof of (1.1) can be deduced from a more general result of Gluskin and V. Milman in [11]. Given a star body D in $\mathbb{R}^n$ with $0 \in \mathrm{int}(D)$ and some measurable subsets $V_1, \ldots, V_m$ of $\mathbb{R}^n$, we denote for the rest of this article by $\mathbb{P}$ the probability taken with respect to the product of the uniform probability measures $\mu_i(A) = \frac{|A \cap V_i|}{|V_i|}$. Using this notation, the aforementioned result of Gluskin and V. Milman can be stated as follows: Let $V_1, \ldots, V_m, D$ be as above, and such that $|D| = |V_1| = \cdots = |V_m|$. Then for any $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and any $0 < t < 1$ one has

$$\mathbb{P}\left((x_i)_{i=1}^m, x_i \in V_i : \left\| \sum_{i=1}^m \lambda_i x_i \right\|_D \leqslant t \Big(\sum_{i=1}^m \lambda_i^2 \Big)^{1/2} \right) \leqslant \left(t e^{\frac{1-t^2}{2}} \right)^n.$$

The proof of this estimate in [11] is based on (the sharp form of) the multivariate Young inequality, see [7] and [8].

As a next step, using Theorem 1.4 we obtain the following variant of the result of Gluskin and Milman, in the case that $V_i = B_2^n$ for all i.

Theorem 1.5. *Let $\delta \in (0,1)$, D be a symmetric convex body in $\mathbb{R}^n$ and $S \subseteq \{-1,1\}^n$ with $|S| \leqslant 2^{\delta n}$. Then*

$$\mathbb{P}\left((x_i)_{i=1}^n \subseteq B_2^n : \left\| \sum_{i=1}^n \epsilon_i x_i \right\|_D \leqslant \frac{1}{10} t_{D,\delta} m(D), \text{ for some } \epsilon \in S \right) \leqslant 3 e^{-n}.$$

This theorem may be viewed as an extension of Hajela's result: the ℓ_∞^n-norm is replaced by an arbitrary norm on $\mathbb{R}^n$ and the statement holds for a random choice of vectors in the Euclidean unit ball. In this context, the role of the parameter $t_{D,\delta}$ in the statement of our results becomes now more transparent: Since, by (1.4),

$$t_{D,\delta} m(D) \simeq t_{D,\delta} \sqrt{n} M(D) \simeq t_{D,\delta} M(D) \mathrm{vrad}(D) |D|^{-1/n},$$

where $M(D) = \int_{S^{n-1}} \|x\|_D \, d\sigma(x)$ and $\mathrm{vrad}(D) = (|D|/|B_2^n|)^{1/n}$, and since $M(D)\mathrm{vrad}(D) \geqslant 1$ (this is a simple consequence of Hölder's inequality), we see that Theorem 1.5 provides information stronger than the one from (1.1) provided that $M(D)\mathrm{vrad}(D) \gg 1$ and/or $t_{D,\delta} \simeq 1$. This is the case for the $\|\cdot\|_\infty$-norm and it would be interesting to provide further examples with sharp estimates.

The function $t \mapsto \gamma_n(2t \cdot m(D)D)$ that appears in the definition of $t_{D,\delta}$ above has been studied in [13], [14] and [18] and elsewhere. A sample of estimates is the following:

- In [13] it is proved that for every $0 < t < \frac{1}{2}$ one has

$$\gamma_n(\{x : \|x\|_D \leqslant t \cdot m(D)\}) \leqslant \frac{1}{2} t^{d(D)},$$

3

where

$$d(D) = \min\left\{n, -\log\gamma_n\left(\frac{m(D)}{2}D\right)\right\},$$

(see [18, Theorem 3.1] for this exact formulation of the result).

- In [14] it is proved that if $\gamma_n(D) \leqslant \frac{1}{2}$ then for every $0 < t < \frac{1}{2}$ one has $\gamma_n(tD) \leqslant (2t)^{r^2(D)/4}\gamma_n(D)$ where $r(D)$ is the inradius of D.

- In [18] it is proved that for every $0 < t < \frac{1}{2}$ one has $\gamma_n(\{x : \|x\|_D \leqslant t \cdot m(D)\}) \leqslant \frac{1}{2}t^{c/\beta(D)}$, where

$$\beta(D) = \frac{\mathrm{Var}_{\gamma_n}(\|\cdot\|_D)}{M^2(D)}$$

and $c > 0$ is an absolute constant.

In this note we discuss the estimates that can be derived from the above in special instances, as for example when D is an ℓ_p^n ball.

Finally, the argument in the proof of Theorem 1.5 can be further generalized to the case where $x_1, \ldots, x_n$ are independent random points chosen uniformly from any symmetric convex body.

Theorem 1.6. *There exists an absolute constant $c > 0$ such that the following holds: Let $\delta \in (0,1)$, D be a symmetric convex body in $\mathbb{R}^n$ and $S \subseteq \{-1,1\}^n$ with $|S| \leqslant 2^{\delta n}$. For any symmetric convex body K in $\mathbb{R}^n$ with $|K| = |B_2^n|$ we may find $\mathcal{U} \subseteq O(n)$ with $\nu_n(\mathcal{U}) > 1 - 3e^{-n/2}$ such that, for any $U \in \mathcal{U}$,*

$$\mathbb{P}\left((z_i)_{i=1}^n \subseteq U(K) \times \cdots \times U(K) : \left\|\sum_{i=1}^n \epsilon_i z_i\right\|_D \leqslant ct_{D,\delta}m(D), \text{ for some } \epsilon \in S\right) \leqslant e^{-n/2}.$$

2   An improved version of Hajela's result

In this section we present the proof of Theorem 1.2. In what follows, $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. We denote by S^{n-1} the Euclidean sphere in $\mathbb{R}^n$, and by σ the unique rotationally invariant probability measure on S^{n-1}. Recall that σ can be defined via the Haar probability measure ν_n on the orthogonal group $O(n)$ as follows: For every measurable $A \subseteq S^{n-1}$ we have

$$\sigma(A) = \nu_n(\{U \in O(n) : U(x) \in A\}),$$

for an arbitrary $x \in S^{n-1}$. We need a small ball probability estimate for the ℓ_∞^n-norm, which is given in the next lemma. It provides a bound whose behaviour for small values of t will play a crucial role in the sequel; in the case of $\|\cdot\|_\infty$ this type of behaviour has been observed in various works (see e.g. [23] and [17]). We provide a direct and short proof of the exact statement that we need.

Lemma 2.1. *There exists an absolute constant $c \in (0,1)$ such that, for any $n \geqslant 1$ and $\delta \in \left(\frac{1}{n}, \frac{1}{4}\right)$,*

$$\sigma\left(\left\{\theta \in S^{n-1} : \|\theta\|_\infty \leqslant \frac{c\sqrt{\log(e/\delta)}}{\sqrt{n}}\right\}\right) < 2^{-\delta n}.$$

Proof. We use the fact (see [13] for its simple proof) that if A is a symmetric convex body in $\mathbb{R}^n$ then

(2.1) $$\sigma(S^{n-1} \cap A) \leqslant 2\gamma_n(2\sqrt{n}A),$$

to write

$$\sigma\left(\left\{\theta \in S^{n-1} : \|\theta\|_\infty \leqslant \frac{s}{2\sqrt{n}}\right\}\right) \leqslant 2\gamma_n\left(\{x \in \mathbb{R}^n : \|x\|_\infty \leqslant s\}\right)$$

$$= 2(1 - 2(1 - \Phi(s)))^n \leqslant 2\exp(-2n(1 - \Phi(s))),$$

4

where, as usual, Φ stands for the cumulative distribution function of the standard normal distribution, that is, $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x}e^{-t^2/2}\,dt$. Using the inequality $(t+s)^2 \leqslant 2(t^2+s^2)$, we have that, for any $s > 0$,

$$1 - \Phi(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{(t+s)^2}{2}}\,dt \geqslant \frac{e^{-s^2}}{\sqrt{2\pi}}\int_0^\infty e^{-t^2}\,dt = \frac{e^{-s^2}}{2\sqrt{2}}.$$

It follows that

$$\sigma\left(\left\{\theta \in S^{n-1} : \|\theta\|_\infty \leqslant \frac{s}{2\sqrt{n}}\right\}\right) \leqslant 2\exp\left(-\frac{n}{\sqrt{2}}e^{-s^2}\right).$$

Finally, if $\delta \in \left(\frac{1}{n}, \frac{1}{4}\right)$ then choosing $s = \frac{1}{2}\sqrt{\log\left(\frac{e}{\delta}\right)}$ we get

$$2\exp\left(-\frac{n}{\sqrt{2}}e^{-s^2}\right) = 2\exp\left(-\frac{n}{\sqrt{2}}\left(\frac{\delta}{e}\right)^{1/4}\right) < 2^{-\delta n},$$

which proves the desired statement. $\qquad\square$

Proof of Theorem 1.2. The idea of the proof is the same as in Hajela's paper. We consider orthonormal systems obtained as random rotations of the standard basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$. Let $\delta \in (n^{-1}, 1/4)$. Since, for any $\epsilon = (\epsilon_1,\ldots,\epsilon_n) \in \{-1,1\}^n$, $n^{-1/2}\sum_{i=1}^{n}\epsilon_i e_i \in S^{n-1}$, by the definition of the measure σ on S^{n-1} it follows, taking $\alpha = c\sqrt{\log(e/\delta)}$ where c is the constant in Lemma 2.1, that

$$\nu_n\left(\left\{U \in O(n) : \left\|U\left(\sum_{i=1}^n \epsilon_i e_i\right)\right\|_\infty \leqslant c\sqrt{\log(e/\delta)}\right\}\right)$$

$$= \nu_n\left(\left\{U \in O(n) : \left\|U\left(\frac{\sum_{i=1}^n \epsilon_i e_i}{\sqrt{n}}\right)\right\|_\infty \leqslant \frac{c\sqrt{\log(e/\delta)}}{\sqrt{n}}\right\}\right)$$

$$= \sigma\left(\left\{\theta \in S^{n-1} : \|\theta\|_\infty \leqslant \frac{c\sqrt{\log(e/\delta)}}{\sqrt{n}}\right\}\right).$$

Applying Lemma 2.1 we get

$$\sigma\left(\left\{\theta \in S^{n-1} : \|\theta\|_\infty \leqslant \frac{c\sqrt{\log(e/\delta)}}{\sqrt{n}}\right\}\right) < 2^{-\delta n}.$$

This shows that

(2.2) $$\nu_n\left(\left\{U \in O(n) : \left\|U\left(\sum_{i=1}^n \epsilon_i e_i\right)\right\|_\infty \leqslant c\sqrt{\log(e/\delta)}\right\}\right) < 2^{-\delta n}.$$

Now let $S \subseteq \{-1,1\}^n$ with $S \leqslant 2^{\delta n}$. Then, by the union bound and (2.2),

$$\nu_n\left(\left\{U \in O(n) : \left\|\sum_{i=1}^n \epsilon_i U(e_i)\right\|_\infty \geqslant c\sqrt{\log(e/\delta)}, \text{ for every } \epsilon \in S\right\}\right) =$$

$$= 1 - \nu_n\left(\left\{U \in O(n) : \left\|\sum_{i=1}^n \epsilon_i U(e_i)\right\|_\infty \leqslant c\sqrt{\log(e/\delta)}, \text{ for some } \epsilon \in S\right\}\right)$$

$$\geqslant 1 - \sum_{\epsilon \in S}\nu_n\left(\left\{U \in O(n) : \left\|\sum_{i=1}^n \epsilon_i U(e_i)\right\|_\infty \leqslant c\sqrt{\log(e/\delta)}\right\}\right)$$

$$> 1 - |S| \cdot 2^{-\delta n} \geqslant 0.$$

We thus deduce that there exists some $U_0 \in O(n)$ such that, if we set $x_i = U_0(e_i)$, $i = 1, \ldots, n$, then

$$\left\| \sum_{i=1}^{n} \epsilon_i x_i \right\|_{\infty} \geqslant c \sqrt{\log(e/\delta)},$$

for any $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$, given that $1/n < \delta < 1/4$.

Finally, note that in the case $\delta \in (1/4, 1)$ the wanted result holds trivially, since for any $U \in O(n)$ and every $\epsilon \in \{-1, 1\}^n$,

$$\left\| \sum_{i=1}^{n} \epsilon_i U e_i \right\|_{\infty} \geqslant 1 \geqslant c \sqrt{\log(e/\delta)}$$

is valid, for a suitable absolute constant $c > 0$. □

3 Signed sums of random vectors from a convex body

The argument used for the proof of Theorem 1.2 motivates us to consider a more general setting. As a first step, we let the ℓ_{∞}^n norm be replaced by an arbitrary norm in $\mathbb{R}^n$ and relax our assumptions on the choice of the vectors $x_1, \ldots, x_n \in B_2^n$. As a second step, we will consider a further generalization, letting $x_1, \ldots, x_n$ be chosen uniformly and independently from B_2^n, or any convex body K.

3.1 Norms of signed sums of vectors

Let D be a symmetric convex body in $\mathbb{R}^n$. We consider the median $m(D)$ of $\|Z\|_D$ where $Z \sim N(0, I_n)$ is a standard Gaussian random vector in $\mathbb{R}^n$. Repeating the argument of the previous section we can show the following.

Proposition 3.1. *Let D be a symmetric convex body in $\mathbb{R}^n$ and $\tau > 0$. Assume that the vectors $x_1, \ldots, x_n \in \mathbb{R}^n$ satisfy*

$$\left\| \sum_{i=1}^{n} \zeta_i x_i \right\|_2 \geqslant \tau \sqrt{n},$$

for all $\zeta_i = \pm 1$. Then, for any $S \subseteq \{-1, 1\}^n$ with

$$|S| \cdot \gamma_n \left(\frac{2t}{\tau} m(D) D \right) < 1/2,$$

we may find $\mathcal{U} \subseteq O(n)$ with $\nu_n(\mathcal{U}) > 1 - 2|S| \cdot \gamma_n \left(\frac{2t}{\tau} m(D) D \right)$ such that any $U \in \mathcal{U}$ satisfies

$$\left\| \sum_{i=1}^{n} \epsilon_i U(x_i) \right\|_D \geqslant t \cdot m(D)$$

for all $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$.

Proof. Let $x_1, \ldots, x_n \in \mathbb{R}^n$ be as in the statement above, and fix $\epsilon_1, \ldots, \epsilon_n$. Normalizing by $\|\sum_{i=1}^{n} \epsilon_i x_i\|_2$ and using the fact that the latter is greater than $\tau \sqrt{n}$, we get

$$\nu_n \left(\left\{ U \in O(n) : \left\| \sum_{i=1}^{n} \epsilon_i U(x_i) \right\|_D \leqslant t \cdot m(D) \right\} \right)$$

$$\leqslant \nu_n \left(\left\{ U \in O(n) : \left\| U \left(\frac{\sum_{i=1}^{n} \epsilon_i x_i}{\|\sum_{i=1}^{n} \epsilon_i x_i\|_2} \right) \right\|_D \leqslant \frac{t \cdot m(D)}{\tau \sqrt{n}} \right\} \right)$$

$$= \sigma \left(\left\{ \theta \in S^{n-1} : \|\theta\|_D \leqslant \frac{t \cdot m(D)}{\tau \sqrt{n}} \right\} \right)$$

$$\leqslant 2\gamma_n \left(\frac{2t}{\tau} m(D) D \right),$$

6

where in the last step, we make use of (2.1). Therefore,

$$\nu_n\left(\left\{U \in O(n) : \left\|\sum_{i=1}^{n} \epsilon_i U(x_i)\right\|_D \geqslant t \cdot m(D)\,, \text{ for every } \epsilon \in S\right\}\right)$$

$$\geqslant 1 - \sum_{\epsilon \in S} \nu_n\left(\left\{U \in O(n) : \left\|\sum_{i=1}^{n} \epsilon_i U(x_i)\right\|_D \leqslant t \cdot m(D)\right\}\right)$$

$$> 1 - 2|S| \cdot \gamma_n\left(\frac{2t}{\tau}m(D)D\right).$$

This proves the claim of the proposition. □

Theorem 1.4 now is a direct corollary of Proposition 3.1.

Proof of Theorem 1.4. Let $t = \tau t_{D,\delta}$ and $S \subseteq \{-1,1\}^n$ with $|S| \leqslant 2^{\delta n}$. By the definition of $t_{D,\delta}$ we have that $|S|\gamma_n(\frac{2t}{\tau}m(D)D) < e^{-n}$. We can thus apply Proposition 3.1 to get $\mathcal{U} \subseteq O(n)$ such that $\nu_n(\mathcal{U}) \geqslant 1 - e^{-n}$ and

$$\left\|\sum_{i=1}^{n} \epsilon_i U x_i\right\|_D \geqslant t \cdot m(D) = \tau t_{D,\delta} m(D)$$

for every $U \in \mathcal{U}$ and all $(\epsilon_1, \ldots, \epsilon_n) \in S$, which is the statement of the theorem. □

Before we proceed, let us briefly explain at this point the general bounds on the parameter $t_{D,\delta}$, stated in the introduction.

Proof of (1.4). For the upper bound, it is straightforward by the definition of the median that

$$\gamma_n(m(D)D) = \gamma_n(\|Z\|_D \leqslant m(D)) \geqslant \frac{1}{2} > (2^\delta e)^{-n},$$

hence $t_{D,\delta} \leqslant \frac{1}{2}$. For the lower bound, since, integrating in polar coordinates, $\int_{S^{n-1}} \|x\|_D^{-n} \, d\sigma(x) = \frac{|D|}{|B_2^n|}$, Markov's inequality implies that

$$\sigma\left(\|x\|_D \leqslant e^{-\eta}\left(\frac{|B_2^n|}{|D|}\right)^{1/n}\right) \leqslant e^{-\eta n}$$

holds for every $\eta > 0$. We can relate the latter to the Gaussian measure; using the same argument as in the proof of [13, Lemma 2.1], one can prove that for any $a \geqslant 1$,

$$\gamma_n\left(\frac{1}{a}\sqrt{n}e^{-\eta}\left(\frac{|B_2^n|}{|D|}\right)^{1/n} D\right) \leqslant \sigma\left(\|x\|_D \leqslant e^{-\eta}\left(\frac{|B_2^n|}{|D|}\right)^{1/n}\right) + \gamma_n\left(\frac{1}{a}\sqrt{n}B_2^n\right) \leqslant e^{-\eta n} + \gamma_n\left(\frac{1}{a}\sqrt{n}B_2^n\right).$$

We can bound the second term using, e.g. [6, Proposition 2.2], to get $\gamma_n\left(\frac{1}{a}\sqrt{n}B_2^n\right) \leqslant a^{-n}\exp\left(\frac{n(a^2-1)}{2a^2}\right)$. Note that, for $\eta = \log(4e)$ we have $2e^{-\eta n} = 2^{-n+1}(2e)^{-n} \leqslant (2^\delta e)^{-n}$ for any $\delta \in (0,1)$, so it suffices to have

$$a^{-n}\exp\left(\frac{n(a^2-1)}{2a^2}\right) \leqslant (4e)^{-n} = e^{-\eta n},$$

which is satisfied if a is large enough ($a = 20$ will do). Now, since $\gamma_n\left(\frac{1}{80e}\sqrt{n}\left(\frac{|B_2^n|}{|D|}\right)^{1/n} D\right) \leqslant (2^\delta e)^{-n}$ for every $\delta \in (0,1)$, the definition of $t_{D,\delta}$ implies that

$$t_{D,\delta} \geqslant \frac{1}{160e}\frac{\sqrt{n}}{m(D)}\left(\frac{|B_2^n|}{|D|}\right)^{1/n} \geqslant c_1\frac{1}{|D|^{1/n}m(D)},$$

for some absolute constant $c_1 > 0$. □

3.2 Random points from convex bodies

We will see now that, suitably normalized, the hypothesis on the norm of the sum $\zeta_1 x_1 + \cdots + \zeta_n x_n$ in the statement of Proposition 3.1 is satisfied with overwhelming probability when the x_i's are chosen uniformly and independently at random from the interior of a general symmetric convex body K. The required lower bound, which actually holds for any norm in $\mathbb{R}^n$ in place of the Euclidean norm, will be deduced as a corollary of the following result of Gluskin and V. Milman [11]:

Proposition 3.2 (Gluskin-V. Milman)**.** *Let D be a star body in $\mathbb{R}^n$ with $0 \in \text{int}(D)$ and $V_1, \ldots, V_m$ be measurable subsets of $\mathbb{R}^n$ with $|D| = |V_1| = \cdots = |V_m|$. For any $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and any $0 < t < 1$ one has*

$$\mathbb{P}\left((x_i)_{i=1}^m, x_i \in V_i : \left\| \sum_{i=1}^m \lambda_i x_i \right\|_D \leqslant t \Big(\sum_{i=1}^m \lambda_i^2 \Big)^{1/2} \right) \leqslant \Big(t e^{\frac{1-t^2}{2}} \Big)^n.$$

Below, we will make use of the following special instance of Proposition 3.2, which also provides an alternative proof of Banaszczyk's general lower bound (1.1) for $\beta(K, D)$.

Corollary 3.3. *Let K and D be symmetric convex bodies in $\mathbb{R}^n$, and let $x_1, \ldots, x_n$ be random points chosen uniformly from K. The inequality*

$$\left\| \sum_{i=1}^n \epsilon_i x_i \right\|_D \geqslant \frac{1}{10} \left(\frac{|K|}{|D|} \right)^{1/n} \sqrt{n}$$

is then valid for any choice of $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$, with probability greater than $1 - e^{-n}$ with respect to $x_1, \ldots, x_n$.

Proof. Let $t > 0$ and assume first that $|K| = |D|$. We have

$$\mathbb{P}\left((x_i)_{i=1}^n \subseteq K : \left\| \sum_{i=1}^n \frac{1}{\sqrt{n}} \epsilon_i x_i \right\|_D \leqslant t, \text{ for some } \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\} \right)$$

$$\leqslant 2^n \mathbb{P}\left((x_i)_{i=1}^n \subseteq K : \left\| \sum_{i=1}^n \frac{1}{\sqrt{n}} \epsilon_i x_i \right\|_D \leqslant t \right)$$

Now if we apply Proposition 3.2 for $m := n$, $V_i := K$ and $\lambda_i := \frac{1}{\sqrt{n}} \epsilon_i$ for every i, and t such that $2 t e^{(1-t^2)/2} < e^{-1}$ (say $t = 1/10$) we get

$$\mathbb{P}\left((x_i)_{i=1}^n \subseteq K : \left\| \sum_{i=1}^n \frac{1}{\sqrt{n}} \epsilon_i x_i \right\|_D \leqslant t, \text{ for some } \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\} \right) \leqslant 2^n (t e^{\frac{1-t^2}{2}})^n < e^{-n}.$$

We deduce that, with probability greater than $1 - e^{-n}$ with respect to $(x_1, \ldots, x_n)$, we have

$$\left\| \sum_{i=1}^n \epsilon_i x_i \right\|_D > \frac{1}{10} \sqrt{n}$$

for every choice of $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

For a general pair of symmetric convex bodies K, D, set $a = (|D|/|K|)^{1/n}$ and $\tilde{K} = aK$. Then we can apply the above for the pair $\tilde{K}, D$ to deduce that for every choice of $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$,

$$\left\| \sum_{i=1}^n \epsilon_i a x_i \right\|_D > \frac{1}{10} \sqrt{n},$$

8

or, equivalently,

$$\left\|\sum_{i=1}^n \epsilon_i x_i\right\|_D > \frac{1}{10}\sqrt{n}\left(\frac{|K|}{|D|}\right)^{1/n}$$

holds with probability greater than $1 - e^{-n}$ with respect to $(x_1, \ldots, x_n)$. $\qquad\square$

Remark 3.4. Note that Corollary 3.3 immediately implies Banaszczyk's lower bound on $\beta(K, D)$:

$$\beta(K, D) \geqslant c\sqrt{n}(|K|/|D|)^{1/n}.$$

3.2.1 Points from the ball

First we deal with the situation where the vectors $x_1, \ldots, x_n$ are chosen independently and uniformly at random from B_2^n. The statement of Theorem 1.5 follows immediately from the definition of $t_{D,\delta}$ and the next consequence of Proposition 3.1. This can be viewed as a generalization of Hajela's result, in the spirit of Proposition 3.2.

Theorem 3.5. *Let D be a symmetric convex body in $\mathbb{R}^n$ and $S \subseteq \{-1, 1\}^n$. Then, for every $t > 0$,*

$$\mathbb{P}\left((x_i)_{i=1}^n \subseteq B_2^n : \left\|\sum_{i=1}^n \epsilon_i x_i\right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S\right) \leqslant 2|S| \cdot \gamma_n(20t \cdot m(D)D) + e^{-n}.$$

Proof. Let $A \subseteq (B_2^n)^n$ be the set

$$A := \left\{(x_i)_{i=1}^n \subseteq B_2^n : \left\|\sum \epsilon_i x_i\right\|_2 \geqslant \frac{1}{10}\sqrt{n}, \text{ for every } \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}\right\}.$$

By Corollary 3.3, applied for $K = D = B_2^n$, we have that $\mathbb{P}(A^c) < e^{-n}$. This fact, combined with the same reasoning as in the proof of Proposition 3.1 gives

$$\mathbb{P}\left((z_i)_{i=1}^n \subseteq B_2^n : \left\|\sum_{i=1}^n \epsilon_i z_i\right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S\right)$$

$$= \mathbb{P}\left(((x_i)_{i=1}^n, U) \in (B_2^n)^n \times O(n) : \left\|\sum_{i=1}^n \epsilon_i U(x_i)\right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S\right)$$

$$\leqslant \mathbb{P}\left(((x_i)_{i=1}^n, U) \in A \times O(n) : \left\|\sum_{i=1}^n \epsilon_i U(x_i)\right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S\right) + e^{-n}$$

$$= \int_A \left[\nu_n\left(\left\{U \in O(n) : \left\|\sum_{i=1}^n \epsilon_i U(x_i)\right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S\right\}\right)\right] d\mu(x_n) \cdots d\mu(x_1) + e^{-n}$$

$$< 2|S|\gamma_n(20t \cdot m(D)D) + e^{-n},$$

as claimed. $\qquad\square$

Application: the case of ℓ_p^n. As an application let us consider the case $D = B_p^n$, $1 \leqslant p \leqslant \infty$. We remark that, although the estimate $\beta(B_2^n, B_2^n) \leqslant \sqrt{n}$ seems to be well-known (a proof of this can be found in [5]), we could not locate in the literature any upper bound on the parameter $\beta(B_2^n, B_p^n)$, for $p \neq 2$. However, for $1 \leqslant p < 2$, one can use the previously mentioned estimate for $\beta(B_2^n, B_2^n)$: if we know that for every $x_1, \ldots, x_n \in B_2^n$ there exists $(\epsilon_i)_{i=1}^n \subseteq \{-1, 1\}$ such that $\|\epsilon_1 x_1 + \ldots + \epsilon_n x_n\|_2 \leqslant \sqrt{n}$, then one can write

$$\|\epsilon_1 x_1 + \ldots + \epsilon_n x_n\|_p \leqslant n^{\frac{1}{p} - \frac{1}{2}}\|\epsilon_1 x_1 + \ldots + \epsilon_n x_n\|_2 \leqslant n^{1/p}.$$

The above estimate is actually sharp, by the lower bound (1.1) and the fact that $(|B_2^n|/|B_p^n|)^{1/n}$ is of the order of $n^{\frac{1}{p}-\frac{1}{2}}$. On the other hand, for the case $p > 2$ one could use the estimate of Banaszczyk $\beta(B_2^n, B_\infty^n) \leqslant c\sqrt{\log n}$ in a similar fashion, to deduce that $\beta(B_2^n, B_p^n) \leqslant cn^{1/p}\sqrt{\log n}$.

Recall the definition of the parameter β in [19]:

$$\beta(D) = \frac{\operatorname{Var}_{\gamma_n}(\|\cdot\|_D)}{M^2(D)}.$$

For $D = B_p^n$, $\beta(D)$ has been computed in [19]: one has $\beta(B_p^n) \simeq \frac{2^p}{p^2 n}$ if $1 \leqslant p \leqslant c\log n$ and $\beta(B_p^n) \simeq (\log n)^{-2}$ if $C\log n \leqslant p \leqslant \infty$, for some absolute constants $c, C > 0$ (for a more detailed analysis, one may consult [16]). Moreover it is known that, in general, $m(D) \simeq \mathbb{E}\|Z\|_D$ for any symmetric convex body D in $\mathbb{R}^n$ (the inequality $\mathbb{E}\|Z\|_D \leqslant cm(D)$ can be deduced from [15, Lemma 3.1], while $m(D) \leqslant 2\mathbb{E}\|Z\|_D$ by the definition of $m(D)$ and Markov's inequality), and in particular

$$m(B_p^n) \simeq \mathbb{E}\|Z\|_p \simeq n^{1/p}\sqrt{p}, \qquad 1 \leqslant p \leqslant \log n,$$

while

$$m(B_p^n) \simeq \mathbb{E}\|Z\|_p \simeq \sqrt{\log n}, \qquad \log n \leqslant p \leqslant \infty.$$

Therefore, choosing $t = \frac{1}{4}$ in Theorem 3.5 and using the estimate $\gamma_n(\{x : \|x\|_p \leqslant t \cdot m(B_p^n)\}) \leqslant \frac{1}{2}t^{c/\beta(B_p^n)}$, we get:

Corollary 3.6. *For any $p \geqslant 1$ there exists a constant $c_p > 0$ such that for any $n \geqslant n_0(p)$ and any $S \subseteq \{-1, 1\}^n$ with $|S| \leqslant 2^{c_p n}$ we have that a random n-tuple of points in B_2^n satisfies, with probability greater than $1 - e^{-n}$,*

$$\left\|\sum_{i=1}^n \epsilon_i x_i\right\|_p \geqslant c\sqrt{p}n^{1/p}$$

for all $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$.

Remark 3.7. It is useful to compare the last result with (1.1). For every $p \leqslant \log n$, since $(|B_2^n|/|B_p^n|)^{1/n}$ is of the order of $n^{\frac{1}{p}-\frac{1}{2}}$ we see that for any $S \subseteq \{-1, 1\}^n$ with $|S| \leqslant 2^{c_p n}$ a random n-tuple of points in B_2^n satisfies, with probability greater than $1 - e^{-n}$,

$$\left\|\sum_{i=1}^n \epsilon_i x_i\right\|_p \geqslant c\sqrt{p}\sqrt{n}\left(\frac{|B_2^n|}{|B_p^n|}\right)^{1/n}$$

for all $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$. On the other hand, in the case $p > \log n$ one can use the fact that the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are equivalent to deduce from Theorem 1.2 (more precisely, Theorem 1.5 combined with the estimate given by Theorem 1.2) that for any $0 < \varrho < 1$ and $S \subseteq \{-1, 1\}^n$ with $|S| \leqslant 2^{n^{1-\varrho}}$, a random n-tuple of points in B_2^n satisfies, with probability greater than $1 - e^{-n}$,

$$\left\|\sum_{i=1}^n \epsilon_i x_i\right\|_p \geqslant c(\varrho)\sqrt{\log n}\sqrt{n}\left(\frac{|B_2^n|}{|B_p^n|}\right)^{1/n}$$

for all $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$.

In fact, the results of this section point out a general way to derive variants of (1.1) in this spirit. A meaningful lower bound on $\|\sum_{i=1}^n \epsilon_i x_i\|_D$ can be obtained with high probability for the random n-tuple $x_1, \ldots, x_n$, if one is willing to drop the requirement that this holds for any $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, but rather restrict themselves to a "big" subset $S \subseteq \{-1, 1\}^n$.

3.2.2 Points from a symmetric convex body

Finally, we study the case where $x_1, \ldots, x_n$ are chosen uniformly and independently from an arbitrary symmetric convex body K in $\mathbb{R}^n$. We shall prove the following generalization of Theorem 3.5, which in turn, for $t := ct_{D,\delta}$ will give the claim in Theorem 1.6.

Theorem 3.8. *There exists an absolute constant $c > 0$ such the following holds: Let D be a symmetric convex body in $\mathbb{R}^n$ and $S \subseteq \{-1, 1\}^n$. For any symmetric convex body K in $\mathbb{R}^n$ let $t > 0$ be such that*

$$|S|\gamma_n(ct(|B_2^n|/|K|)^{1/n}m(D)D) < e^{-n}.$$

Then, we may find $\mathcal{U} \subseteq O(n)$ with $\nu_n(\mathcal{U}) > 1 - 3e^{-n/2}$ such that, for all $U \in \mathcal{U}$,

$$\mathbb{P}\left((z_i)_{i=1}^n \subseteq U(K) \times \cdots \times U(K) : \left\| \sum_{i=1}^n \epsilon_i z_i \right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S \right) \leqslant e^{-n/2}.$$

Proof. Let

$$A = \left\{ (x_i)_{i=1}^n \subseteq K : \left\| \sum_{i=1}^n \epsilon_i x_i \right\|_2 > \frac{1}{10} \left(\frac{|K|}{|B_2^n|} \right)^{1/n} \sqrt{n}, \text{for every } \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\} \right\}.$$

By Corollary 3.3, applied for $D = B_2^n$, we have that $\mathbb{P}(A^c) < e^{-n}$. Using Proposition 3.1 we write

$$\int_{O(n)} \mathbb{P}\left((z_i)_{i=1}^n \in U(K)^n : \left\| \sum_{i=1}^n \epsilon_i z_i \right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S \right)$$

$$= \mathbb{P}\left(((x_i)_{i=1}^n, U) \in K^n \times O(n) : \left\| \sum_{i=1}^n \epsilon_i U x_i \right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S \right)$$

$$\leqslant \mathbb{P}\left(((x_i)_{i=1}^n, U) \in A \times O(n) : \left\| \sum_{i=1}^n \epsilon_i U x_i \right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S \right) + e^{-n}$$

$$= \int_A \left[\nu_n \left(\left\{ U \in O(n) : \left\| \sum_{i=1}^n \epsilon_i U x_i \right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S \right\} \right) \right] d\mu(x_n) \cdots d\mu(x_1) + e^{-n}$$

$$< 2|S|\gamma_n(20t(|B_2^n|/|K|)^{1/n}m(D)D) + e^{-n}.$$

Assume that t is chosen so that

$$|S|\gamma_n(20t(|B_2^n|/|K|)^{1/n}m(D)D) < e^{-n}.$$

Applying Markov's inequality we may find $\mathcal{U} \subseteq O(n)$ with $\nu_n(\mathcal{U}) > 1 - 3e^{-n/2}$ such that

$$\mathbb{P}\left((z_i)_{i=1}^n \subseteq U(K)^n : \left\| \sum_{i=1}^n \epsilon_i z_i \right\|_D \leqslant t \cdot m(D), \text{ for some } \epsilon \in S \right) \leqslant e^{-n/2}.$$

This proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 3.9. Theorem 3.8 illustrates once more the main idea behind the improvement upon Hajela's result, Theorem 1.2, as well as the rest of the results in this note; small ball probability estimates for the Gaussian measure of convex bodies can be linked to the deduction of lower bounds for variants of the quantity $\beta(K, D)$, where the D-norm of the signed sum of a random n-tuple of vectors can be lower bounded for every choice of signs in any appropriately large subset of $\{-1, 1\}^n$.

References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Part I*, Amer. Math. Soc., Mathematical Surveys and Monographs **202** (2015).

[2] W. Banaszczyk, *Balancing vectors and convex bodies*, Studia Math. **106** (1993), 93–100.

[3] W. Banaszczyk, *Balancing vectors and Gaussian measures of n-dimensional convex bodies*, Random Structures Algorithms **12** (1998), no. 4, 351–360.

[4] N. Bansal, D. Dadush, and S. Garg, *An algorithm for Komlós conjecture matching Banaszczyks bound*, In 57th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2016, New Brunswick, NJ, USA, October 9-11 2016, 788-799.

[5] I. Bárány, *On the power of linear dependencies*, Building Bridges, Bolyai Society Mathematical Studies, Vol. 19, Springer, Berlin (2008), pp.31–45.

[6] A. Barvinok, *Measure Concentration*, lecture notes, University of Michigan (2005), available at `http://www.math.lsa.umich.edu/~barvinok/total710.pdf`.

[7] W. Beckner,*Inequalities in Fourier analysis*, Ann. Math., 102, 159-182 (1975).

[8] H. J. Brascamp and E. H. Lieb, *Best constants in Youngs inequality, its converse, and its generalization to more than three functions*, Advances in Math., 20, no. 2, 151-173 (1976).

[9] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Amer. Math. Society, Mathematical Surveys and Monographs **196** (2014).

[10] E. D. Gluskin, *Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces*, Math. USSR Sbornik **64** (1989), 85–96.

[11] E. Gluskin and V. Milman, *Geometric probability and random cotype 2*, Geometric Aspects of Functional Analysis, volume 1850 of Lecture Notes in Math., pages 123-138. Springer, Berlin, 2004.

[12] D. Hajela, *On a Conjecture of Komlós about Signed Sums of Vectors inside the Sphere*, Eur. J. Comb. 9(1): 33–37, (1988).

[13] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky Theorem*, Israel J. Math. 157 (2007) , no. 1, 193-207.

[14] R. Latała and K. Oleszkiewicz, *Small ball probability estimates in terms of widths*, Studia Math. **169** (2005), 305–314.

[15] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer, Berlin, 1991.

[16] A. Lytova and K. E. Tikhomirov, *The variance of the ℓ_p^n-norm of the gaussian vector, and Dvoretzky's theorem*, St. Petersburg Math. J., to appear, `https://arxiv.org/abs/1705.05052`.

[17] V. D. Milman and G. Schechtman, *An "isomorphic" version of Dvoretzky's theorem*, C.R. Acad. Sci. Paris **321** (1995), 541–544.

[18] G. Paouris and P. Valettas, *A Gaussian small deviation inequality for convex functions*, Annals of Probability **46** (2018), 1441–1454.

[19] G. Paouris, P. Valettas and J. Zinn, *Random version of Dvoretzky's theorem in ℓ_p^n*, Stochastic Process. Appl. **127** (2017), no. 10, 3187–3227.

[20] J. Spencer, *Six standard deviations suffice*, Trans. Amer. Math. Soc. **289** (1985), 679–706.

[21] J. Spencer, *Balancing vectors in the max norm*, Combinatorica **6**(1) (1986), 55–65.

[22] J. Spencer, *Ten Lectures on the Probabilistic Method*, Society for Industrial and Applied Mathematics, Philadelphia, Penn. 1994.

[23] A. Szankowski, *On Dvoretzky's theorem on almost spherical sections of convex bodies*, Israel J. Math. **17** (1974), 325–338.

Giorgos Chasapis: Department of Mathematics, University of Athens, Panepistimioupolis 157-84, Athens, Greece.

Current Address: Department of Mathematical Sciences, Kent State University, Kent, OH USA.

E-mail: `gchasap1@kent.edu`

Nikos Skarmogiannis: Department of Mathematics, University of Athens, Panepistimioupolis 157-84, Athens, Greece.

E-mail: `nikskarm@math.uoa.gr`