Sub-Gaussian directions of isotropic convex bodies

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Abstract

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. A direction $\theta \in S^{n-1}$ is called sub-Gaussian for K with constant $b > 0$ if $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant b\|\langle \cdot, \theta\rangle\|_2$. We show that if K is isotropic then most directions are sub-Gaussian with a constant which is logarithmic in the dimension. More precisely, for any $a > 1$ we have

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^{3/2} \max\left\{\sqrt{\log n}, \sqrt{a}\right\} L_K$$

for all θ in a subset Θ_a of S^{n-1} with $\sigma(\Theta_a) \geqslant 1 - n^{-a}$, where $C > 0$ is an absolute constant.

1 Introduction

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$; we say that K is centered if it has its barycenter at the origin. A direction $\theta \in S^{n-1}$ is a ψ_α-direction (where $1 \leqslant \alpha \leqslant 2$) for K with constant $b > 0$ if

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_\alpha}(K)} \leqslant b\|\langle \cdot, \theta\rangle\|_2,$$

where

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_\alpha}(K)} := \inf\left\{t > 0 : \int_K \exp\left((|\langle x, \theta\rangle|/t)^\alpha\right) dx \leqslant 2\right\}.$$

From Markov's inequality it is clear that if K satisfies a ψ_α-estimate with constant b in the direction of θ then for all $t \geqslant 1$ we have $|\{x \in K : |\langle x, \theta\rangle| \geqslant t\|\langle \cdot, \theta\rangle\|_2\}| \leqslant 2e^{-t^a/b^\alpha}$. Conversely, it is a standard fact that tail estimates of this form imply that θ is a ψ_α-direction for K.

From the Brunn-Minkowski inequality it follows that every $\theta \in S^{n-1}$ is a ψ_1-direction for K with an absolute constant C. The starting point of this note is a question posed by V. Milman: is it true that there exists an absolute constant $C > 0$ such that every K has at least one sub-Gaussian direction (ψ_2-direction) with constant C? Klartag proved in [11] that for every centered convex body K of volume 1 in $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant ct\|\langle\cdot,\theta\rangle\|_2\}| \leqslant e^{-\frac{t^2}{[\log(t+1)]^{2a}}}$$

for all $t \geqslant 1$, where $a = 3$ (equivalently, $\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^a\|\langle\cdot,\theta\rangle\|_2$). This estimate was later improved by Giannopoulos, Paouris and Valettas in [8] and [9] (see also [7]). They considered the body $\Psi_2(K)$ with support function $y \mapsto \|\langle\cdot, y\rangle\|_{L_{\psi_2}(K)}$ and showed that for every centered convex body K of volume 1 in $\mathbb{R}^n$, one has

$$c_1 \leqslant \left(\frac{|\Psi_2(K)|}{|Z_2(K)|}\right)^{1/n} \leqslant c_2\sqrt{\log n}, \tag{1.1}$$

where $\{Z_q(K)\}_{q\geqslant 1}$ is the family of the L_q-centroid bodies of K, and $c_1, c_2 > 0$ are absolute constants (background information on isotropic convex bodies and their centroid bodies is provided in Section 2). An immediate consequence of (1.1) is the existence of at least one sub-Gaussian direction for K with constant $b \leqslant C\sqrt{\log n}$.

A natural question that arises is to consider a suitable position $T(K)$, $T \in SL(n)$, of the body K and to study the distribution of the ψ_2-norm $\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)}$ with respect to the rotationally invariant probability measure σ on the sphere. Klartag [11] offers a result of this type: if K has volume 1 and barycenter at the origin, then there exists $T \in SL(n)$ such that the body $K_1 = T(K)$ has the following property: there exists $\Theta \subseteq S^{n-1}$ with measure $\sigma(\Theta) \geqslant \frac{4}{5}$ such that, for every $\theta \in \Theta$ and every $t \geqslant 1$,

$$|\{x \in K_1 : |\langle x, \theta\rangle| \geqslant ct\|\langle\cdot,\theta\rangle\|_1\}| \leqslant \exp\left(-\frac{ct^2}{\log^2 n \log^5(t+1)}\right).$$

Exploiting Klartag's ideas we give a short proof of an analogous statement.

Proposition 1.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, with barycenter at the origin, such that $\Psi_2(K)$ has minimal mean width among all its linear images of the same volume. Then, for any $\delta \in (0,1)$ we may find $\Theta_\delta \subseteq S^{n-1}$ with measure $\sigma(\Theta_\delta) \geqslant 1-\delta$ such that every $\theta \in \Theta_\delta$ is a ψ_2-direction for K with constant $C\delta^{-1}(\log n)^{3/2}$.*

Note that $\Psi_2(T(K)) = T(\Psi_2(K))$ for all $T \in SL(n)$, and hence there exists a position $K_1 = T(K)$ of K such that Proposition 1.1 applies for K_1. A more natural and interesting case to consider is when K is in the isotropic position. Our main result provides logarithmic bounds for $\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)}$ with probability polynomially close to 1.

Theorem 1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for any $a > 1$ we have*

$$\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^{3/2}\max\left\{\sqrt{\log n}, \sqrt{a}\right\} L_K$$

for all θ in a subset Θ_a of S^{n-1} with $\sigma(\Theta_a) \geqslant 1 - n^{-a}$, where $C > 0$ is an absolute constant.

Theorem 1.2 shows that $\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K$ with probability greater than $1-\frac{1}{n}$. This allows us to estimate the expectation of $\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)}$ on S^{n-1}:

Theorem 1.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\int_{S^{n-1}} \|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)} d\sigma(\theta) \leqslant C(\log n)^2 L_K,$$

where $C>0$ is an absolute constant.

The previously known general estimate was $\mathbb{E}_\sigma(\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)}) \leqslant C\sqrt[4]{n}L_K$ (see [9]). Regarding the optimal expected result, it is useful to mention a number of sharp results for special classes of convex bodies. Bobkov and Nazarov (see [2] and [3]) have proved that if K is an isotropic unconditional convex body in $\mathbb{R}^n$ then, for every $\theta\in\mathbb{R}^n$,

$$\|f_\theta\|_{L_{\psi_2}(K)} \leqslant c\sqrt{n}\|\theta\|_\infty,$$

where $c>0$ is an absolute constant. It follows that

$$\int_{S^{n-1}} \|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)} d\sigma(\theta) \leqslant C\sqrt{\log n} \tag{1.2}$$

in the unconditional case. In particular, the upper bound of (1.2) holds true the normalized ℓ_p^n-balls $\overline{B}_p^n$ for all $1\leqslant p\leqslant\infty$. The estimate is sharp in the case of the normalized ℓ_1^n-ball $\overline{B}_1^n$: one has $\mathbb{E}_\sigma(\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(\overline{B}_1^n)}) \simeq \sqrt{\log n}$. Therefore, the estimate of Theorem 1.3 is best possible up to the power of $\log n$, and one cannot expect a general upper bound independent from the dimension. A very precise description of the behavior of linear functionals on the ℓ_p^n-balls, for all $1\leqslant p\leqslant\infty$, can be found in the article [1] of Barthe, Guédon, Mendelson and Naor; in particular, they show that in the case $2\leqslant p\leqslant\infty$ one has $\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(\overline{B}_p^n)} \leqslant C$ for every $\theta\in S^{n-1}$, where $C>0$ is a constant independent from p and n.

The main new tool for the proof of Theorem 1.2 and Theorem 1.3 is a recent result of E. Milman on the mean width $w(Z_q(K))$ of the L_q-centroid bodies $Z_q(K)$ of an isotropic convex body K in $\mathbb{R}^n$.

Theorem 1.4 (E. Milman [15])**.** *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for all $q\geqslant 1$ one has*

$$w(Z_q(K)) \leqslant C\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}},\sqrt{q}\right\}L_K$$

where $C>0$ is an absolute constant.

The proofs of the main results are given in Section 3.

2 Notation and background information

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, S^{n-1} for the unit sphere, and σ for the rotationally invariant probability measure on S^{n-1}. Volume is denoted by $|\cdot|$. The letters c_i, C_i denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. We will often use the fact that $|B_2^n|^{1/n} \simeq 1/\sqrt{n}$; to see this, recall that $|B_2^n| = \pi^{n/2}/\Gamma\left(\frac{n}{2}+1\right)$ and use Stirling's formula.

A convex body in $\mathbb{R}^n$ is a compact convex set $A \subset \mathbb{R}^n$ with non-empty interior. We say that A is symmetric if $x \in A$ implies that $-x \in A$. We say that A is unconditional with respect to an orthonormal basis of $\mathbb{R}^n$ if $x = (x_1, \dots, x_n) \in A$ implies that $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in A$ for every choice of signs $\varepsilon_i \in \{-1, 1\}$. The volume radius of A is the quantity $\mathrm{vrad}(A) = (|A|/|B_2^n|)^{1/n}$. Integration in polar coordinates shows that if the origin is an interior point of A then the volume radius of A can be expressed as

$$\mathrm{vrad}(A) = \left(\int_{S^{n-1}} \|\theta\|_A^{-n}\, d\sigma(\theta)\right)^{1/n},$$

where $\|\theta\|_A = \min\{t > 0 : \theta \in tA\}$. The support function of A is defined by $h_A(y) := \max\big\{\langle x, y\rangle : x \in A\big\}$, and the mean width of A is the average

$$w(A) := \int_{S^{n-1}} h_A(\theta)\, d\sigma(\theta) \tag{2.1}$$

of h_A on S^{n-1}. The radius $R(A)$ of A is the smallest $R > 0$ such that $A \subseteq RB_2^n$. For notational convenience we write $\overline{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\overline{A} := |A|^{-1/n} A$.

The polar body A° of a symmetric convex body A in $\mathbb{R}^n$ is defined by

$$A^\circ := \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in A\big\}. \tag{2.2}$$

The Blaschke-Santaló inequality states that $|A||A^\circ| \leqslant |B_2^n|^2$, with equality if and only if A is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [5] states that there exists an absolute constant $c > 0$ such that, conversely,

$$(|A||A^\circ|)^{1/n} \geqslant c/n. \tag{2.3}$$

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and if its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{2.4}$$

for every θ in the Euclidean unit sphere S^{n-1}. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C \tag{2.5}$$

for all $n \geqslant 1$. Bourgain proved in [4] that $L_n \leqslant c\sqrt[4]{n}\log n$, while Klartag [10] obtained the bound $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [12].

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ we consider the q-th moment of the Euclidean norm

$$I_q(K) = \left(\int_K \|x\|_2^q dx\right)^{1/q}.$$

Note that if K is isotropic then $I_2(K) = \sqrt{n}L_K$. For every $q \geqslant 1$ and every $y \in \mathbb{R}^n$ we set

$$h_{Z_q(K)}(y) := \|\langle \cdot, y\rangle\|_q = \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}. \tag{2.6}$$

The L_q-centroid body $Z_q(K)$ of K is the centrally symmetric convex body with support function $h_{Z_q(K)}$. Note that K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in SL(n)$ then $Z_q(T(K)) = T(Z_q(K))$. From Hölder's inequality it follows that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. Using Borell's lemma (see [6, Chapter 1]) one can check that

$$Z_q(K) \subseteq c_1 \frac{q}{p} Z_p(K) \tag{2.7}$$

for all $1 \leqslant p < q$. In particular, if K is isotropic, then $R(Z_q(K)) \leqslant c_1 q L_K$. One can also check that if K is centered, then $Z_q(K) \supseteq c_2 Z_\infty(K)$ for all $q \geqslant n$ (this was observed in [17]). An asymptotic approach to the family of centroid bodies was developed by Paouris in [19] and [20].

We refer the reader to the article of V. Milman and Pajor [16] and to the book [6] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

3 Proof of the results

Recall that $\Psi_2(K)$ is the symmetric convex body with support function $h_{\Psi_2(K)}(y) = \|\langle \cdot, y\rangle\|_{L_{\psi_2}(K)}$. One also has

$$h_{\Psi_2(K)}(y) \simeq \sup_{q \geqslant 2} \frac{h_{Z_q(K)}(y)}{\sqrt{q}} \simeq \sup_{2 \leqslant q \leqslant n} \frac{h_{Z_q(K)}(y)}{\sqrt{q}}$$

because $h_{Z_q(K)}(y) \simeq h_{Z_n(K)}(y)$ for all $q \geqslant n$. Taking into account the fact that if $2^s \leqslant q < 2^{s+1}$ then

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant \frac{h_{Z_{2^{s+1}}(K)}(y)}{2^{s/2}} \leqslant \sqrt{2}\frac{h_{Z_{2^{s+1}}(K)}(y)}{2^{(s+1)/2}},$$

we can further simplify and write

$$h_{\Psi_2(K)}(y) \simeq \max_{1\leqslant s\leqslant m} \frac{h_{Z_{2^s}(K)}(y)}{2^{s/2}} \tag{3.1}$$

where $m = \lfloor \log_2 n \rfloor$ (for all the above see [6, Chapter 5]).

Proof of Proposition 1.1. Since $\Psi_2(T(K)) = T(\Psi_2(K))$ for every $T \in SL(n)$, we may find $T \in SL(n)$ such that $K_1 = T(K)$ has the property that $\Psi_2(K_1)$ has minimal mean width among all its linear images of the same volume. It is well known (see [21] or [6, Chapter 1]) that in this case one has the estimate

$$w(\Psi_2(K_1)) \leqslant C_1(\log n)\,[\mathrm{vrad}(\Psi_2(K_1))],$$

where $C_1 > 0$ is an absolute constant. On the other hand, we may write

$$\begin{aligned}\int_{S^{n-1}} \frac{h_{\Psi_2(K_1)}(\theta)}{h_{Z_2(K_1)}(\theta)}\,d\sigma(\theta) &\leqslant \left(\int_{S^{n-1}} h^2_{\Psi_2(K_1)}(\theta)\,d\sigma(\theta)\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} h^{-2}_{Z_2(K_1)}(\theta)\,d\sigma(\theta)\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{S^{n-1}} h^2_{\Psi_2(K_1)}(\theta)\,d\sigma(\theta)\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} h^{-n}_{Z_2(K_1)}(\theta)\,d\sigma(\theta)\right)^{\frac{1}{n}} \\ &\leqslant C_2 w(\Psi_2(K_1))\mathrm{vrad}(Z_2^{\circ}(K_1)) = C_2\frac{w(\Psi_2(K_1))}{\mathrm{vrad}(Z_2(K_1))},\end{aligned}$$

where we have used Cauchy-Schwarz inequality, Hölder's inequality, the equality $\mathrm{vrad}(Z_2(K_1))\mathrm{vrad}(Z_2^{\circ}(K_1)) = 1$ which holds true because $Z_2(K_1)$ is an ellipsoid, and the equivalence of the L_1 and the L_2 norm of the function $h_{\Psi_2(K_1)}$ on S^{n-1} (this is a well-known Kahane-Khintchine type inequality; in fact, one can view it as a special case of the stronger inequality (3.2), due to Litvak, V. Milman and Schechtman [13]).

Combining the previous estimates we conclude that

$$\int_{S^{n-1}} \frac{h_{\Psi_2(K_1)}(\theta)}{h_{Z_2(K_1)}(\theta)}\,d\sigma(\theta) \leqslant C_3 \log n \left(\frac{|\Psi_2(K_1)|}{|Z_2(K_1)|}\right)^{1/n} \leqslant C_4(\log n)^{3/2},$$

where in the last step we have used (1.1). An application of Markov's inequality shows that for any $\delta \in (0,1)$ we may find $\Theta_\delta \subseteq S^{n-1}$ with measure $\sigma(\Theta_\delta) \geqslant 1-\delta$ such that every $\theta \in \Theta_\delta$ is a ψ_2-direction for K_1 with constant $C_4\delta^{-1}(\log n)^{3/2}$. $\square$

We proceed to the proof of Theorem 1.2 and Theorem 1.3. First, we give a simple argument that leads to the upper bound of Theorem 1.3 for

$$w(\Psi_2(K)) = \int_{S^{n-1}} \|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} d\sigma(\theta).$$

Let K be an isotropic convex body in $\mathbb{R}^n$. By (3.1), for any $y \in S^{n-1}$ we have

$$h_{\Psi_2(K)}(y) \leqslant C_1 \max_{1\leqslant s\leqslant m} \frac{h_{Z_{2^s}(K)}(y)}{2^{s/2}} \leqslant C_1 \sum_{s=1}^{m} \frac{h_{Z_{2^s}(K)}(y)}{2^{s/2}}$$

where $m = \lfloor \log_2 n \rfloor$. It trivially follows that

$$w(\Psi_2(K)) \leqslant C_1 \sum_{s=1}^{m} \frac{w(Z_{2^s}(K))}{2^{s/2}}.$$

From Theorem 1.4 we know that

$$w(Z_{2^s}(K)) \leqslant C_2 s 2^{s/2} \max\left\{\frac{s2^{s/2}}{\sqrt{n}}, 1\right\} L_K.$$

Therefore, denoting by k the largest integer for which $k^2 2^k \leqslant n$, and using summation by parts in the final step, we see that

$$\begin{aligned} w(\Psi_2(K)) &\leqslant C_3 \sum_{s=1}^{m} s \max\left\{\frac{s2^{s/2}}{\sqrt{n}}, 1\right\} L_K \leqslant C_3 \left(\sum_{s=1}^{k} s + \frac{1}{\sqrt{n}} \sum_{s=k+1}^{m} s^2 2^{s/2}\right) L_K \\ &\leqslant C_4 \left(k^2 + \frac{m^2 2^{m/2}}{\sqrt{n}}\right) L_K \leqslant C_5 m^2 L_K \leqslant C(\log n)^2 L_K, \end{aligned}$$

where $C > 0$ is an absolute constant.

A more careful use of the theory of centroid bodies, given below, leads to the probability estimate of Theorem 1.2 and to a second proof of Theorem 1.3.

Proof of Theorem 1.2 and Theorem 1.3. We will use the following observations that can be found e.g. in [6, Chapter 5]: given a symmetric convex body A in $\mathbb{R}^n$, if we set $k = k_*(A) = n\left(\frac{w(A)}{R(A)}\right)^2$ then

$$w_k(A) := \left(\int_{S^{n-1}} h_A^k(\theta)\, d\sigma(\theta)\right)^{1/k} \leqslant C_1 w(A) \tag{3.2}$$

where $C_1 > 0$ is an absolute constant; this is a result of Litvak, V. Milman and Schechtman from [13]. If $A = Z_q(K)$ then the results of Paouris in [19] (or, earlier, in [18]) show that, for all $2 \leqslant q \leqslant \sqrt{n}$,

$$w(Z_q(K)) \geqslant c_1 w_q(Z_q(K)) \geqslant c_2 \sqrt{q/n}\, I_q(K) \geqslant c_2 \sqrt{q/n}\, I_2(K) = c_2 \sqrt{q} L_K.$$

In fact, E. Milman and Klartag have obtained the stronger bound $\mathrm{vrad}(Z_q(K)) \geqslant c_3\sqrt{q} L_K$ for all $2 \leqslant q \leqslant q_H(K)$, where $q_H(K) \geqslant c_4\sqrt{n}$ is a hereditary parameter of K that was introduced and studied in [12] for this purpose. On the other hand, $R(Z_q(K)) \leqslant C_2 q L_K$, and hence $k_*(Z_q(K)) \geqslant c_5 n/q$ for all $2 \leqslant q \leqslant \sqrt{n}$. In the range $\sqrt{n} \leqslant q \leqslant n$ one has the weaker bound $w(Z_q(K)) \geqslant \mathrm{vrad}(Z_q(K)) \geqslant c_6\sqrt{q}$ which follows from Urysohn's inequality and a lower bound for $\mathrm{vrad}(Z_q(K))$ for the full range $2 \leqslant q \leqslant n$, which is due to Lutwak, Yang and Zhang [14]; this results in the estimate $k_*(Z_q(K)) \geqslant c_7 n/(qL_K^2)$.

Using Theorem 1.4 and (3.2) we get

$$\left(\int_{S^{n-1}} h_{Z_q(K)}^{k_*}(\theta)\,d\sigma(\theta)\right)^{1/k_*} \leqslant C_3 \log(1+q) \max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\} L_K$$

where $k_* := k_*(Z_q(K))$, and using Markov's inequality we conclude that, for every $q \leqslant n$ there exists a subset Θ_q of S^{n-1} such that $\sigma(S^{n-1}\setminus\Theta_q) \leqslant \exp(-c_8 n/(qL_K^2))$ and

$$h_{Z_q(K)}(\theta) \leqslant C_4\sqrt{q}\log(1+q)\max\left\{\frac{\sqrt{q}\log(1+q)}{\sqrt{n}}, 1\right\} L_K$$

for all $\theta \in \Theta_q$.

We fix $a > 1$ and define $q_0 = \frac{c_8 n}{2aL_K^2 \log n}$. Then, for every $q \leqslant q_0$ we have $\sigma(S^{n-1}\setminus\Theta_q) \leqslant \frac{1}{n^{2a}}$. It follows that

$$\sigma\Big(S^{n-1} \setminus \bigcap_{s=1}^{\lfloor \log_2 q_0 \rfloor} \Theta_{2^s}\Big) \leqslant \frac{c_9 \log n}{n^{2a}} \leqslant \frac{1}{n^a}.$$

If $\Theta := \bigcap_{s=1}^{\lfloor \log_2 q_0\rfloor} \Theta_{2^s}$ then, for every $\theta \in \Theta$ and every $q \leqslant q_0$ we have

$$\begin{aligned} \frac{h_{Z_q(K)}(\theta)}{\sqrt{q}} &\leqslant C_4 \log(1+q)\max\left\{\frac{\sqrt{q}\log(1+q)}{\sqrt{n}}, 1\right\} L_K \\ &\leqslant C_5(\log n)\max\left\{\frac{\log(1+q_0)}{\sqrt{a}L_K\sqrt{\log n}}, 1\right\} L_K, \end{aligned} \tag{3.3}$$

while for $q_0 \leqslant q \leqslant n$ we use (2.7) to write

$$\begin{aligned} \frac{h_{Z_q(K)}(\theta)}{\sqrt{q}} &\leqslant \frac{C_6 q}{q_0}\frac{h_{Z_{q_0}(K)}(\theta)}{\sqrt{q}} = C_6\sqrt{q/q_0}\frac{h_{Z_{q_0}(K)}(\theta)}{\sqrt{q_0}} \\ &\leqslant C_6\sqrt{n/q_0}\log(1+q_0)\max\left\{\frac{\log(1+q_0)}{\sqrt{a}L_K\sqrt{\log n}}, 1\right\} L_K \\ &\leqslant C_7\sqrt{a}(\log n)^{3/2}\max\left\{\frac{\sqrt{\log n}}{\sqrt{a}L_K}, 1\right\} L_K^2. \end{aligned}$$

Combining this estimate with (3.3) we see that

$$\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)} \leqslant C_8\sqrt{a}(\log n)^{3/2}\max\left\{\frac{\sqrt{\log n}}{\sqrt{a}L_K}, 1\right\} L_K^2$$

with probability greater that $1 - n^{-a}$.

Now, we use Theorem 3.1 from [8]: If K is an isotropic convex body in $\mathbb{R}^n$, there exists an isotropic convex body K_1 such that $L_{K_1} \leqslant C_0$, where $C_0 > 0$ is an absolute constant, and

$$\Psi_2(K) \subseteq c_{10} L_K \Psi_2(K_1).$$

Our previous reasoning, applied to K_1, shows that, with probability greater than $1 - n^{-a}$,

$$\begin{aligned} \|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} &\leqslant c_{10} L_K \|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K_1)} \leqslant C_8 c_{10} L_K \sqrt{a} (\log n)^{3/2} \max\left\{\frac{\sqrt{\log n}}{\sqrt{a}}, 1\right\} C_0^2 \\ &\leqslant C_9 L_K \sqrt{a} (\log n)^{3/2} \max\left\{\frac{\sqrt{\log n}}{\sqrt{a}}, 1\right\}. \end{aligned}$$

This proves Theorem 1.2. In particular, we have

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C_{10} (\log n)^2 L_K$$

with probability greater than $1 - \frac{1}{n}$. Since $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C_{11}\sqrt{n} L_K$ for all $\theta \in S^{n-1}$ (see e.g. [6]) this gives one more proof of Theorem 1.3. $\square$

Remark. Let K be an isotropic convex body in $\mathbb{R}^n$. The function $\psi_K : [1, \infty) \to \mathbb{R}$ with

$$\psi_K(t) := \sigma\left(\{\theta \in S^{n-1} : \|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant ct\sqrt{\log n} L_K\}\right)$$

was introduced in [9], where it was shown that for every $t \geqslant 1$ one has

$$\psi_K(t) \geqslant \exp(-cn/t^2),$$

where $c > 0$ is an absolute constant. Theorem 1.3 provides much stronger information; it implies that $\psi_K(t) \geqslant 1/2$ for $t \simeq (\log n)^{3/2}$.

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