

Generalized Sudakov via Dimension Reduction - A Program

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Abstract

We propose a program for establishing a conjectural extension to the class of (origin-symmetric) log-concave probability measures μ , of the classical (dual) Sudakov Minoration on the expectation of the supremum of a Gaussian process:

$$M(Z_p(\mu), C \int \|x\|_K d\mu \cdot K) \leq \exp(Cp) \quad \forall p \geq 1. \quad (0.1)$$

Here K is an origin-symmetric convex body, $Z_p(\mu)$ is the L_p -centroid body associated to μ , $M(A, B)$ is the packing-number of B in A , and $C > 0$ is a universal constant. The Program consists of first establishing a Weak Generalized Sudakov Minoration, involving the dimension n of the ambient space, which is then self-improved to a dimension-free estimate after applying a dimension-reduction step. The latter step may be thought of as a conjectural “small-ball one-sided” variant of the Johnson–Lindenstrauss dimension-reduction lemma. We establish the Weak Generalized Sudakov Minoration for a variety of log-concave probability measures and convex bodies (for instance, this step is fully resolved assuming a positive answer to the Slicing Problem). The Separation Dimension-Reduction step is fully established for ellipsoids and, up to logarithmic factors in the dimension, for cubes, resulting in a corresponding Generalized (regular) Sudakov Minoration estimate for these bodies and arbitrary log-concave measures, which are shown to be (essentially) best-possible. Along the way, we establish a regular version of (0.1) for all $p \geq n$ and provide a new direct proof of Sudakov’s Minoration via The Program.

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1 Introduction

Let γ_n denote the standard Gaussian measure on \mathbb{R}^n , and let $K \subset \mathbb{R}^n$ denote a convex body, that is a convex compact set with non-empty interior. For simplicity, we assume that K is origin-symmetric, $K = -K$, and denote by $\|\cdot\|_K$ the associated norm whose unit-ball is K . The dual norm is denoted $\|\cdot\|_K^*$. Given two compact sets $A, B \subset \mathbb{R}^n$, we denote by $M(A, B)$ the packing number of B in A , i.e. the maximal integer M so that there exist $\{x_i\}_{i=1, \dots, M} \subset A$ with $x_i + B$ mutually disjoint (“ $\{x_i\}$ are B -separated”).

This paper is dedicated to the study of a conjectural generalized version of the classical Sudakov Minoration estimate [55] and its dual version (due to Pajor–Tomczak-Jaegermann [44], see also [34]):

Theorem (Sudakov and Dual Sudakov Minoration).

(1) *Sudakov Minoration for $\ell^*(K) := \int \|x\|_K^* d\gamma_n(x)$:*

$$M(K, tB_2^n) \leq \exp(C\ell^*(K)/t^2) \quad \forall t > 0.$$

(2) *Dual Sudakov Minoration for $\ell(K) := \int \|x\|_K d\gamma_n(x)$:*

$$M(B_2^n, tK) \leq \exp(C\ell(K)/t^2) \quad \forall t > 0.$$

Here and throughout this work, C, C', C'', c , etc... denote positive universal constants, independent of any other parameter (and in particular the dimension n), whose value may change from one occurrence to the next. We use $A \simeq B$ to signify that $c \leq A/B \leq C$. The Euclidean unit-ball is denoted by B_2^n .

Sudakov’s Minoration plays a key-role in the proof of M. Talagrand’s “Majorizing Measures” theorem [56, 57, 58], which gives two-sided bounds on the expected supremum of a Gaussian process in terms of a certain geometric parameter associated with the indexing set. Subsequently (see [58] and the references therein), Talagrand extended his characterization to more general processes sampled from i.i.d. Bernoulli random variables and measures of the form $\exp(-\sum_{i=1}^n |x_i|^p) dx$, $p \in [1, \infty]$, and the case of general log-concave *product* measures (with moderate tail decay) was obtained by R. Latała [31]. Recall that a probability measure μ on \mathbb{R}^n is called log-concave if $\mu = \exp(-V(x)) dx$ with $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ convex. Motivated by an attempt to extend Talagrand’s characterization to more general log-concave measures, Latała [32] and independently the authors (unpublished) conjectured the following *Generalized Sudakov Minoration* bounds:

Conjecture. *For any origin-symmetric log-concave probability measure μ on \mathbb{R}^n and origin-symmetric convex body $K \subset \mathbb{R}^n$:*

(1) *Generalized Sudakov Minoration for $I_1^*(\mu, K) := \int \|x\|_K^* d\mu(x)$:*

$$M(K, CI_1^*(\mu, K)B_p(\mu)) \leq \exp(Cp) \quad \forall p \geq 1.$$

(2) *Generalized Dual Sudakov Minoration for $I_1(\mu, K) := \int \|x\|_K d\mu(x)$:*

$$M(Z_p(\mu), CI_1(\mu, K)K) \leq \exp(Cp) \quad \forall p \geq 1.$$

Here $B_p(\mu)$ denotes the unit-ball of the norm given by:

$$\|x\|_{B_p(\mu)} := \left(\int |\langle y, x \rangle|^p d\mu(y) \right)^{1/p},$$

and $Z_p(\mu)$ denotes the polar-body $B_p(\mu)^\circ$, defined by $\|\cdot\|_{Z_p(\mu)} = \|\cdot\|_{B_p(\mu)}^*$. Up to normalization, the $Z_p(\mu)$ bodies coincide with the L_p centroid-bodies introduced by E. Lutwak and G. Zhang [36], and have played a pivotal role in the development of our understanding of the volumetric properties of log-concave measures in the last decade (e.g. [10]). Note that when $\mu = \gamma_n$, since $Z_p(\gamma_n) \simeq \sqrt{p}B_2^n$, the above conjecture precisely coincides with the classical Sudakov Minoration and its dual version. We refer to [57, 31, 32, 37] for further partial results confirming this conjecture in particular cases and additional applications. Let us presently only mention that the Generalized Sudakov Minoration conjecture has been confirmed by Latała in [31] for all log-concave (and in-fact, only regular) *product* measures, for all log-concave measures when the extremal points of K are antipodal pairs with disjoint supports, when $\mu = \exp(-\varphi(\|x\|_p))dx$, $p \in [1, \infty]$, with $\varphi : [0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\}$ non-decreasing and convex, and for all $p \geq 2n \log(e+n)$.

1.1 Johnson–Lindenstrauss Lemma

In this work, we choose to concentrate on the conjectured Generalized *Dual* Sudakov estimate, and propose a novel program for establishing it. The Program is based on a conjectural dimension reduction step, which may be thought of as a “*one-sided* Johnson–Lindenstrauss lemma”. Recall that the classical lemma of W. B. Johnson and J. Lindenstrauss [22] asserts that if $\{x_i\}_{i=1, \dots, e^k}$ is a collection of (say distinct) points in Euclidean space $X = (\mathbb{R}^n, |\cdot|)$ and $\epsilon \in (0, 1)$, then there exists a map $T : X \rightarrow Y$, $Y = (\mathbb{R}^m, |\cdot|)$ Euclidean, so that:

$$1 - \epsilon \leq \frac{\|Tx_i - Tx_j\|_Y}{\|x_i - x_j\|_X} \leq 1 + \epsilon \quad \forall i \neq j \tag{1.1}$$

with $m \leq Ck/\epsilon^2$. Moreover, T may be chosen to be linear, and a random (appropriately rescaled) orthogonal projection does the job with very high-probability (see Lemma 3.4).

For The Program, we will require an extension of this classical result to more general normed spaces. Such a question was studied by Johnson and A. Naor [23], who showed

that this is essentially impossible - even for a fixed $\epsilon \in (0, 1)$, if the normed space Y is an m -dimensional subspace of X with $m \leq C_\epsilon k$, the distance-preservation property (1.1) for a linear map T implies that X must be *essentially* Hilbertian (see [23] for the precise formulation, and also [23, Section 4, Remark 7] for the case that Y is not assumed to be a subspace of X). It follows that when $X = (\mathbb{R}^n, \|\cdot\|_K)$, we cannot in general hope for a *two-sided* estimate (1.1).

However, for our purposes, we will only need to satisfy the *left-hand-side* inequality in (1.1): if the points $\{x_i\}$ are well-separated in X , so should their images $\{Tx_i\}$ be in Y (“Separation Dimension Reduction”). Of course, without some additional requirement, this is always possible, simply by scaling the norm of Y or the map T in the numerator above. The additional requirement which replaces the right-hand-side inequality in (1.1) is that the unit-ball of Y be “massive enough”, as measured with respect to $T_*\mu := \mu \circ T^{-1}$, the push-forward of the measure μ by T , thereby precluding trivial rescaling attempts. In a sense, this is an averaged variant (with respect to the given measure μ) of the pointwise right-hand-side requirement in (1.1). This conjectural “one-sided Johnson–Lindenstrauss” separation dimension-reduction is in our opinion a fascinating question, which will be explored more in depth elsewhere; it constitutes Part 1 (or more precisely, Part 1’) of our proposed program. We are now ready to describe it and the remaining parts of The Program in more detail; for simplicity, we describe here a simplified version, postponing a description of the full version to Section 5.

1.2 The Program - Simplified Version

Our proposed program consists of three parts, which we first describe in general terms. Part 1 is a conjectural dimension-reduction step, already eluded to above: if $Z_p(\mu)$ is separated by e^k translates of K , then there should be a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m \simeq k$ so that $TZ_p(\mu) = Z_p(T_*\mu)$ is separated by e^k translates of another star-body $L \subset \mathbb{R}^m$, which we may choose at our discretion from a family of candidates \mathcal{L}_m , with the only requirement being that it should be massive enough with respect to $T_*\mu$. Part 2 consists of establishing a weak version of the Generalized Dual Sudakov estimate for the pair $Z_p(T_*\mu)$ and L , which is allowed to depend (in an appropriate manner) on the dimension m of the ambient space. Part 3 consists of establishing the Generalized Dual Sudakov estimate for the latter pair when p is larger than m . By means of a bootstrap argument, we will shortly show that confirmation of these three parts would imply the Generalized Dual Sudakov Minoration Conjecture. We begin with describing (a simplified version of) The Program in greater detail.

A compact set $L \subset \mathbb{R}^n$ having the origin in its interior is called a star-body if $tL \subset L$ for all $t \in [0, 1]$. Given an absolutely continuous probability measure μ on \mathbb{R}^n , denote $m_q(\mu, L) := \sup \{s > 0; \mu(sL) \leq e^{-q}\}$ so that $\mu(m_q(\mu, L)L) = e^{-q}$.

Fix an origin-symmetric convex body $K \subset \mathbb{R}^n$, origin-symmetric log-concave measure μ on \mathbb{R}^n and $p \geq 1$. It is known (see Lemma 2.1) that in such a case $I_1(\mu, K) \simeq m_1(\mu, K)$,

and so up to universal constants we need not distinguish between these two parameters. The following simplified version of The Program for establishing the Generalized Dual Sudakov estimate:

$$\mu(K) \geq \frac{1}{e} \quad \Rightarrow \quad M(Z_p(\mu), RK) \leq \exp(C_{A,B,\varphi} p), \quad (1.2)$$

consists of establishing the following three parts for some constants $R, A, B \geq 1$ and a certain function φ , described below. For all $m = 1, \dots, n$, set $\mathcal{M}_m := \{T_*\mu; T: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ linear}\}$, which is a family of log-concave measures on \mathbb{R}^m by the Prékopa–Leindler theorem (e.g. [13]). In addition, let \mathcal{L}_m denote some family of origin-symmetric star-bodies in \mathbb{R}^m , with $\mathcal{L}_n = \{K\}$. Below k is a positive real number.

(1) **Part 1 (Massive Separation Dimension Reduction).**

If $M(Z_p(\mu), RK) = e^k$ with $\mu(K) \geq \frac{1}{e}$ and $2B \leq k \leq n/A$, show that there exists a linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L \in \mathcal{L}_m$, with $m \leq Ak$, so that:

- (a) $M(TZ_p(\mu), L) \geq e^k$ (“**Separation Dimension Reduction**”).
- (b) $T_*\mu(L) \geq \exp(-q_m)$, $1 \leq q_m \leq k/2$ (“ **L is sufficiently massive**”).

(2) **Part 2 (Weak Generalized Dual Sudakov).**

For all $m = 1, \dots, n$, $L \in \mathcal{L}_m$ and $\nu \in \mathcal{M}_m$, show that:

$$1 \leq p \leq m \quad , \quad \nu(L) \geq \exp(-q_m) \quad \Rightarrow \quad M(Z_p(\nu), L) \leq \exp(q_m + m\varphi(p/m)),$$

where $\varphi: [0, 1] \rightarrow \mathbb{R}_+$ is an increasing function with $\varphi(0) = 0$ and $x \mapsto \varphi(x)/x$ non-increasing (which is independent of all other parameters).

(3) **Part 3 (Large p).**

For all $m = 1, \dots, n$, $L \in \mathcal{L}_m$ and $\nu \in \mathcal{M}_m$, show that:

$$p \geq m \quad , \quad \nu(L) \geq \exp(-q_m) \quad \Rightarrow \quad M(Z_p(\nu), L) \leq \exp(q_m + Bp).$$

Remark 1.1. The following *linear* version of Part 1 should be kept in mind:

(1') **Part 1' - Linear Version**

If $\{x_i\}_{i=1, \dots, e^k} \subset \mathbb{R}^n$ is a collection of K -separated points with $\mu(K) \geq \frac{1}{e}$ and $2B \leq k \leq n/A$, show that there exists a linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L \in \mathcal{L}_m$, with $m \leq Ak$, so that:

- (a) $\{T(x_i)\}_{i=1, \dots, e^k} \subset \mathbb{R}^m$ are $\frac{1}{R}L$ -separated (“**One-sided Johnson–Lindenstrauss**”).
- (b) $T_*\mu(L) \geq \exp(-q_m)$, $1 \leq q_m \leq k/2$ (“ **L is sufficiently massive**”).

By applying this linear version of Part 1 to the maximal collection of K -separated points $\{x_i\}$ in $\frac{1}{R}Z_p(\mu)$, it is evident that establishing Part 1' is sufficient for establishing Part 1 of The Program. However, this is not an equivalent reformulation, and we will also see in Section 10 an example where a non-linear combinatorial argument is required for establishing Part 1.

Remark 1.2. Note that using $\varphi(t) = t$ in Part 2 with $m = n$ precisely corresponds to establishing the Generalized Dual Sudakov Minoration conjecture. Part 2 provides the added flexibility of using a weaker function φ . For instance, using $\varphi(t) = t^q$ for some $q \in (0, 1)$ corresponds to establishing $M(Z_p(\nu), L) \leq \exp(qm + m^{1-q}p^q)$, i.e. a weak dimension-dependent confirmation of the conjecture for $\nu \in \mathcal{M}_m$ and $L \in \mathcal{L}_m$.

Theorem 1.3 (The Program Yields Generalized Dual Sudakov). *Establishing The Program above yields the Generalized Dual Sudakov Estimate (1.2).*

Proof. Assume that $\mu(K) \geq 1/e$. We will show that:

$$e^k := M(Z_p(\mu), RK) \leq \exp(C_{A,B,\varphi}p), \quad C_{A,B,\varphi} := \max\left(2B, \frac{1}{A\varphi^{-1}(1/(2A))}\right). \quad (1.3)$$

Since $p \geq 1$, we may assume that $k \geq C_{A,B,\varphi} \geq 2B$, otherwise there is nothing to prove. We now claim there exists a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L \in \mathcal{L}_m$ for some $m \leq \min(n, Ak)$, so that $M(TZ_p(\mu), L) \geq e^k$ and $T_*\mu(L) \geq \exp(-q_m)$, $1 \leq q_m \leq k/2$. Indeed, if $k < n/A$ this follows from Part 1, whereas if $k \geq n/A$ this is actually trivial by using $m = n$, $T = Id$, $L = K$ and $q_m = 1$. Denoting $\nu = T_*\mu \in \mathcal{M}_m$, note that $TZ_p(\mu) = Z_p(\nu)$. Consequently, if $p \geq m$ then by Part 3:

$$\exp(k) \leq M(Z_p(\nu), L) \leq \exp(q_m + Bp) \leq \exp(k/2 + Bp),$$

implying that $k \leq 2Bp$, as required. Alternatively, if $p \leq m$ then by Part 2 and the assumption that $x \mapsto \varphi(x)/x$ is non-increasing:

$$\exp(k) \leq M(Z_p(\nu), L) \leq \exp(q_m + m\varphi(p/m)) \leq \exp(k/2 + Ak\varphi(p/(Ak))).$$

It follows since φ is increasing from 0 that:

$$\frac{p}{Ak} \geq \varphi^{-1}\left(\frac{1}{2A}\right) > 0,$$

implying that $k \leq C_{A,B,\varphi}p$, and concluding the proof. \square

1.3 Results

Besides introducing The Program, our main results in this work are as follows:

- (1) As a warm-up, we demonstrate in Section 3 the usefulness of The Program by running an analogous version which yields a new proof of the classical Sudakov Minoration (in fact, an improved version, known to experts). To take care of the Separation Dimension-Reduction step (Part 1), we simply employ the usual Johnson–Lindenstrauss Lemma, while for Parts 2 and 3 we invoke an elementary weak volumetric estimate based on Urysohn’s inequality.
- (2) In Section 4, we establish Part 3 of The Program in full generality, for all (origin-symmetric) log-concave measures ν and star-bodies L in \mathbb{R}^m . In fact, we obtain the following *regular version* thereof:

$$p \geq m \quad \Rightarrow \quad M(Z_p(\nu), Ctm_q(\nu, L)L) \leq \exp\left(1 + q + \frac{p}{t}\right) \quad \forall t, q > 0. \quad (1.4)$$

- (3) In Section 5, we formulate the full version of The Program, which extends the simplified one presented above in two aspects: first, in Part 1, we allow the packing number after dimension reduction to drop from e^k down to e^l for $l \geq k/D$ (where $D \geq 1$ is an additional parameter); and second, we introduce a regularity parameter $t > 0$ whose role is to scale the bodies K and L . We prove an analogue of Theorem 1.3, stating that establishing Parts 1 and 2 of The (full) Program, together with the regular version of Part 3 established in (1.4), yields a Generalized Regular Dual Sudakov upper bound on $M(Z_p(\mu), tm_1(\mu, K)K)$ for all $t > 0$.
- (4) In Section 6, we establish Part 2 of The Program for the case that L is an (origin-symmetric) ellipsoid, by invoking a weak volumetric estimate involving all intrinsic volumes of $Z_p(\nu)$.
- (5) In Section 7, we establish the remaining Part 1 of The Program for the case that K is an ellipsoid, by decoupling the separation dimension-reduction and massiveness requirements using a general probabilistic argument, and applying a small-ball one-sided variant of the Johnson–Lindenstrauss Lemma. Running The Program, we obtain the following estimate:

$$M(Z_p(\mu), tm_1(\mu, \mathcal{E})\mathcal{E}) \leq \exp\left(C\left(\frac{p}{t^2} + \frac{p}{t}\right)\right) \quad \forall t > 0, \quad (1.5)$$

for any (origin-symmetric) ellipsoid $\mathcal{E} \subset \mathbb{R}^n$. We verify in Section 11 that for general log-concave measures μ and ellipsoids \mathcal{E} , this estimate is best-possible (up to numeric constants) for all $p \in [1, n]$ and $t \geq \sqrt{p/n}$. When μ has identity covariance matrix (“ μ is isotropic”) and $\mathcal{E} = B_2^n$, we have $m_1(\mu, \mathcal{E}) \simeq \sqrt{n}$, and so the estimate (1.5) precisely coincides with the one obtained in [16] for *isotropic* log-concave measures and Euclidean balls. In the isotropic case, further improved packing estimates (for an appropriate range of p, t) were obtained in [40, Subsection 3.3].

- (6) In Section 8, we introduce the class of h -pure log-concave probability measures μ , which includes several important sub-families, such as unconditional, sub-Gaussian and super-Gaussian log-concave measures. In particular, a log-concave measure is called 1-pure if all of its lower-dimensional marginals have uniformly bounded isotropic constant (see Section 8). A regular packing estimate for $M(B_2^n, tZ_n(\mu))$ when μ is an isotropic 1-pure log-concave probability measure was obtained by Giannopoulos–Milman in [14], and we extend it here to general h -pure measures, as it plays an important role in the subsequent section.
- (7) In Section 9, we use the previous results to establish Part 2 of The Program in a variety of scenarios, such as when the log-concave measure ν is assumed h -pure, or when $Z_m(\nu)$ or $L \in \mathcal{L}_m$ are assumed to have regular small-diameter, such as for type-2 convex bodies, sub-Gaussian convex bodies or unconditional convex bodies with small-diameter, and in particular for ℓ_q^m unit-balls, $q \in [2, \infty]$. In particular, in view of Remark 1.2, this confirms the Generalized Dual Sudakov Minoration conjecture for such ν and L up to non-trivial, but unfortunately dimension-dependent, constants.
- (8) In Section 10 we establish Part 1 of (the full version of) The Program for $K = B_\infty^n$, the n -dimensional cube, with additional logarithmic terms in the dimension. Running The Program, this yields for all $p \geq 1$ and $t > 0$:

$$M(Z_p(\mu), tC \log \log(e + n)m_1(\mu, B_\infty^n)B_\infty^n) \leq \exp\left(C \log(e + n) \left(\frac{p}{t^2} + \frac{p}{t}\right)\right). \quad (1.6)$$

In Section 11, we verify that for general log-concave measures μ and up to the above logarithmic terms, this estimate is best-possible (up to numeric constants) for all $p \in [1, n]$ and $t \geq \min(1, \sqrt{p/n^\alpha})$, for any fixed $\alpha \in (0, 1)$. Removing these logarithmic terms would establish the Generalized Dual Sudakov conjecture in full generality, since any origin-symmetric convex-body $K \subset \mathbb{R}^n$ may be approximated by an n -dimensional section of B_∞^N as $N \rightarrow \infty$ (in fact, using $N = e^n$ would be enough). So from an optimistic perspective, we are only $\log n$ far from establishing the conjecture. A similar argument verifies that (1.6) also holds with B_∞^n replaced by any origin-symmetric polytope with n^β facets, for any fixed $\beta \geq 1$.

In Section 11 we present some further concluding remarks.

2 Notation

We work in Euclidean space $(\mathbb{R}^n, |\cdot|)$, where $|\cdot|$ denotes the standard Euclidean norm. The Euclidean unit-ball is denoted by B_2^n and the Euclidean unit-sphere by S^{n-1} . We also use $|A|$ to denote the volume (or Lebesgue measure) of a Borel set $A \subset \mathbb{R}^n$ in its m -dimensional affine hull (there will be no ambiguity with this standard double role of $|\cdot|$);

the volume-radius of A is then defined as $\text{vrad}(A) := (|A|/|B_2^m|)^{1/m}$. It is well-known that $|B_2^m|^{1/m} \simeq 1/\sqrt{m}$.

The Grassmannian of all m -dimensional linear subspaces of \mathbb{R}^n is denoted by $G_{n,m}$, $m = 1, \dots, n$. All homogeneous spaces G of the group of rotations $SO(n)$ are equipped with their Haar probability measures σ_G , and in particular $\sigma = \sigma_{S^{n-1}}$ denotes the corresponding Haar probability measure on S^{n-1} . Given $F \in G_{n,m}$, we denote by P_F the orthogonal projection onto F , and set $B_2(F) := B_2^n \cap F$ and $S(F) := S^{n-1} \cap F$. Given a Borel measure μ on \mathbb{R}^n , its marginal $\pi_F \mu$ is defined as the push-forward $(P_F)_*(\mu) = \mu \circ P_F^{-1}$. A consequence of the Prékopa–Leindler celebrated extension of the Brunn–Minkowski inequality (e.g. [13]), is that the marginal $\pi_F \mu$ of a log-concave measure μ is itself log-concave on F .

The support function of a compact set L is defined as $h_L(\theta) := \max\{\langle x, \theta \rangle; x \in L\}$, $\theta \in S^{n-1}$. Recall that a star-body $L \subset \mathbb{R}^n$ is a compact set containing the origin in its interior so that $tL \subset L$ for all $t \in [0, 1]$. We denote $\|x\|_L := \min\{t > 0; x \in tL\}$. The radial function $\rho_L(\theta)$ is defined as $1/\|\theta\|_L$ for $\theta \in S^{n-1}$. When K is an origin-symmetric convex body, $\|\cdot\|_K$ is a genuine norm whose unit-ball is precisely K , and its support function coincides with the dual-norm $h_K(\theta) = \|\theta\|_K^*$. The Minkowski sum of two compact sets $A, B \subset \mathbb{R}^n$ is defined as the compact set $A+B := \{a+b; a \in A, b \in B\}$, and satisfies $h_{A+B} = h_A + h_B$. We will write $L_1 \simeq L_2$ if $cL_2 \subset L_1 \subset CL_2$ for some universal constants $c, C > 0$.

2.1 Quantiles of log-concave probability measures

Given an absolutely continuous probability measure μ on \mathbb{R}^n , a star-body $L \subset \mathbb{R}^n$ and $q > 0$, recall that:

$$m_q(\mu, L) := \sup\{s > 0; \mu(sL) \leq e^{-q}\},$$

so that $\mu(m_q(\mu, L)L) = e^{-q}$. In addition, given $q > -1$, we define:

$$I_q(\mu, L) := \left(\int \|x\|_L^q d\mu(x)\right)^{1/q}.$$

Lemma 2.1. *Let K be an origin-symmetric convex body and let μ denote a log-concave probability measure on \mathbb{R}^n . Then for all $q \geq 1$:*

$$ce^{-q}m_1(\mu, K) \leq m_q(\mu, K) \leq m_1(\mu, K) \simeq I_1(\mu, K) \leq I_q(\mu, K) \leq CqI_1(\mu, K).$$

Proof. The first inequality follows by a Kahane–Khintchine-type inequality for negative moments due to O. Guédon [19], which asserts that under our assumptions:

$$\mu(\epsilon m_1(\mu, K)K) \leq 2 \ln\left(\frac{e}{e-1}\right)\epsilon \quad \forall \epsilon \in [0, 1].$$

The second inequality is trivial. The inequality $m_1(\mu, K) \leq \frac{e}{e-1}I_1(\mu, K)$ follows directly by the Markov-Chebyshev inequality. The reverse inequality $I_1(\mu, K) \leq Cm_1(\mu, K)$

follows again by Markov-Chebyshev in conjunction with the negative moment comparison $I_1(\mu, K) \leq C_q I_q(\mu, K)$ for all $q \in (-1, 0]$ established in [19]. The inequality $I_1(\mu, K) \leq I_q(\mu, K)$ is immediate by Jensen's inequality. Finally, the Kahane–Khintchine-type inequality $I_q(\mu, K) \leq C_q I_1(\mu, K)$ is a known consequence of Borell's lemma [6] (e.g. [41], [42, Appendix III] or [10]). \square

2.2 Centroid Bodies

Recall that the L_p ($p \geq 1$) centroid-bodies $Z_p(\mu)$ associated to a log-concave probability measure μ on \mathbb{R}^n are defined by:

$$h_{Z_p(\mu)}(\theta) = \left(\int |\langle x, \theta \rangle|^p d\mu(x) \right)^{1/p}, \quad \theta \in S^{n-1}.$$

Note that $T(Z_p(\mu)) = Z_p(T_*\mu)$ for any linear mapping T , and in particular $P_F Z_p(\mu) = Z_p(\pi_F \mu)$ for all $F \in G_{n,m}$. It is well-known that:

$$1 \leq p \leq q \Rightarrow Z_p(\mu) \subset Z_q(\mu) \subset C \frac{q}{p} Z_p(\mu), \quad (2.1)$$

for some constant $C \geq 1$. The first inequality is simply Jensen's inequality, whereas the second one is due to Berwald [5], or may be deduced as in Lemma 2.1 as a consequence of Borell's Lemma [6]. In fact, it was noted by Latała and Wojtaszczyk [33, Proposition 3.8] that when μ is origin-symmetric, one may use $C = 1$ above (note that the argument in [33] applies to the entire range $1 \leq p \leq q$).

Lemma 2.2. *For any probability measure μ , origin-symmetric convex body K and $p \geq 1$:*

$$Z_p(\mu) \subset I_p(\mu, K)K.$$

Proof. For all $\theta \in S^{n-1}$:

$$h_{Z_p(\mu)}^p(\theta) = \int |\langle x, \theta \rangle|^p d\mu(x) \leq \int \|x\|_K^p d\mu(x) h_K^p(\theta).$$

\square

We denote by $\text{Cov}(\mu)$ the covariance matrix of μ , defined as $\text{Cov}(\mu) := \int x \otimes x d\mu(x) - \int x d\mu(x) \otimes \int x d\mu(x)$. We will say that μ is isotropic if its barycenter is at the origin and $\text{Cov}(\mu)$ is the identity matrix Id . It is easy to see that by applying an affine transformation, any absolutely continuous probability measure may be brought to isotropic “position”, which is unique up to orthogonal transformations. We will always assume that μ has barycenter at the origin, so that μ is isotropic if and only if $Z_2(\mu) = B_2^n$; more generally, we always have $Z_2(\mu) = \text{Cov}(\mu)^{1/2}(B_2^n)$, so that $|Z_2(\mu)| = |B_2^n| (\det \text{Cov}(\mu))^{1/2}$ (where we identified between a matrix and its associated linear operator).

2.3 Packing and Covering Numbers

Recall that given two compact sets $A, B \subset \mathbb{R}^n$, the packing number $M(A, B)$ of B in A is defined as the maximal integer M so that there exist $\{x_i\}_{i=1, \dots, M} \subset A$ with $x_i + B$ mutually disjoint (“ $\{x_i\}$ are B -separated”) - note that a more standard definition in the literature is to assume that $x_i - x_j \notin B$, differing from our preferred definition by a factor of 2. The covering number $N(A, B)$ of A by B is defined as the minimal integer N so that there exist $\{x_i\}_{i=1, \dots, N}$ with $A \subset \cup_{i=1}^N (x_i + B)$. The following relation between packing-numbers and covering-numbers is well-known (see e.g. [50, 1]):

$$N(A, B - B) \leq M(A, B) \leq N(A, -B).$$

When B is an origin-symmetric convex body K , it follows that:

$$N(A, 2K) \leq M(A, K) \leq N(A, K), \quad (2.2)$$

and so up to this immaterial factor of 2, we need not distinguish between packing and covering numbers.

Note that Lemma 2.2 implies that $N(Z_p(\mu), I_p(\mu, K)K) = 1$. The Dual Generalized Sudakov Conjecture asserts that for log-concave measures, it is possible to replace $I_p(\mu, K)$ by $CI_1(\mu, K)$, and still cover $Z_p(\mu)$ with $\exp(Cp)$ copies of $CI_1(\mu, K)K$.

Clearly $M(A, B)$ and $N(A, B)$ are both invariant under simultaneously applying a non-singular linear transformation to both A and B , and under translation of A or B . We will frequently use the following obvious volumetric estimates:

$$\frac{|A|}{|B|} \leq N(A, B) \quad , \quad M(A, B) \leq \frac{|A + B|}{|B|}. \quad (2.3)$$

In particular, when $K \subset \mathbb{R}^n$ is an origin-symmetric convex body, we have the standard volumetric estimate:

$$\left(\frac{1}{t}\right)^n \leq N(K, tK) \leq M(K, (t/2)K) \leq \left(\frac{1+t/2}{t/2}\right)^n \leq \left(1 + \frac{2}{t}\right)^n \quad \forall t \in (0, 1]. \quad (2.4)$$

Lemma 2.3. *Assume that for some compact A and convex body K in \mathbb{R}^n :*

$$N(A, tK) \leq \exp(n\varphi(t)) \quad \forall t \geq t_0,$$

for some function $\varphi : [t_0, \infty) \rightarrow \mathbb{R}_+$. Then the same estimate holds for all $t > 0$ after defining:

$$\varphi(t) := \varphi(t_0) + \log(1 + (2t_0)/t) \quad , \quad t \in (0, t_0).$$

Proof.

$$N(A, tK) \leq N(A, t_0K)N(t_0K, tK) \leq \exp(n\varphi(t_0))(1 + (2t_0)/t)^n.$$

□

The following triangle inequality for covering numbers is obvious for all compact A, B, D :

$$N(A, B) \leq N(A, D)N(D, B).$$

Note that the following variant also holds for packing numbers:

Lemma 2.4.

$$M(A, B) \leq N(A, D)M(D, B).$$

Proof. Let Z denote a B -separated set in A of cardinality M , and assume that $A \subset \cup_{i=1}^N (x_i + D)$. Clearly $(Z - x_i) \cap D$ is a B -separated subset of D , and hence:

$$M = \#Z \leq \sum_{i=1}^N \#(Z \cap (x_i + D)) \leq NM(D, B).$$

□

3 Warm Up - Sudakov's Minoration via The Program

In this section, we demonstrate the usefulness of The Program by applying an analogous program which yields a new proof of Sudakov's Minoration Inequality. Given a compact set $K \subset \mathbb{R}^n$, denote (half) the mean-width of K by:

$$M_{\mathbb{R}^n}^*(K) = M^*(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

Theorem 3.1 (Improved Sudakov Minoration). *For any compact $K \subset \mathbb{R}^n$, one has:*

$$M(K, CtM^*(K)B_2^n) \leq \exp\left(\frac{n}{\max(t, t^2)}\right) \quad \forall t > 0,$$

for some universal constant $C \geq 1$.

Remark 3.2. Note that $M^*(K)$ is invariant under taking the convex hull of K , and so we may as well assume that K is a convex body above. In that case, it is well-known (e.g. [42]) and easy to check by polar-integration that $I_1^*(\gamma_n, K) \simeq \sqrt{n}M^*(K)$. Translating the classical Sudakov Minoration stated in the Introduction using the present notation, it asserts that the left-hand-side is majorized by $\exp(n/t^2)$. The improved version above for $t \in (0, 1)$ is known to experts, and follows from an elementary volumetric argument, reproduced below. The term “minoration” refers to the resulting lower bound on $M^*(K)$ as a function of the packing numbers.

The simplest text-book proof of Sudakov's Minoration we are aware of is obtained by first establishing a dual version using a covering estimate of Talagrand, and then applying a duality argument due to Tomczak-Jaegermann (see [34, 1]). The proof we

provide below is very different: we work with the primal version directly; we first establish the easy “weak” covering estimate $\exp(n/t)$ from elementary volumetric considerations (taking care of the analogues of Part 2 and Part 3 of The Program); and finally self-improve this estimate when $t \geq 1$ by employing dimension reduction (Part 1 of The Program) via the usual Johnson–Lindenstrauss lemma.

Lemma 3.3 (Weak Sudakov Inequality, folklore). *For any convex body $K \subset \mathbb{R}^n$:*

$$M(K, tM^*(K)B_2^n) \leq \exp\left(\frac{n}{t}\right) \quad \forall t > 0.$$

Proof. By linearity of the support functions we have $M^*(K+tL) = M^*(K) + tM^*(L)$ for all $t \geq 0$. We invoke Urysohn’s inequality [18, 53], which states that $\text{vrad}(K) \leq M^*(K)$. Coupled with the standard volumetric covering estimate (2.3), we obtain:

$$M(K, tM^*(K)B_2^n) \leq \frac{|K + tM^*(K)B_2^n|}{|tM^*(K)B_2^n|} \leq \left(\frac{M^*(K + tM^*(K)B_2^n)}{tM^*(K)}\right)^n = \left(\frac{1+t}{t}\right)^n \leq \exp\left(\frac{n}{t}\right).$$

□

Lemma 3.4 (Johnson–Lindenstrauss Lemma [22]). *Let $F \in G_{n,m}$ be a random m -dimensional subspace of Euclidean space $(\mathbb{R}^n, |\cdot|)$ distributed according to the Haar probability measure on $G_{n,m}$, $m = 1, \dots, n$, and let P_F denote the orthogonal projection onto F . Then for all $x \in S^{n-1}$:*

(1)

$$\mathbb{P}\left(\left|\sqrt{n/m}|P_F x| - 1\right| \geq \epsilon\right) \leq C \exp(-c m \epsilon^2) \quad \forall \epsilon > 0.$$

(2) *Let $\{x_i\}_{i=1, \dots, M} \subset \mathbb{R}^n$ be a collection of (say distinct) points. Then:*

$$\mathbb{P}\left(1 - \epsilon \leq \frac{\sqrt{n/m}|P_F x_i - P_F x_j|}{|x_i - x_j|} \leq 1 + \epsilon \quad \forall i \neq j\right) \geq 1 - \binom{M}{2} C \exp(-c m \epsilon^2).$$

Proof Sketch. The first assertion follows from concentration on the sphere and the fact that for a fixed $F_0 \in G_{n,m}$, $S^{n-1} \ni x \mapsto |P_{F_0} x|$ is a 1-Lipschitz function. The second part follows immediately from the first part, linearity, and the union-bound. □

Proof of Theorem 3.1. When $t \in (0, C_0]$, where $C_0 \geq 1$ is a large-enough constant to be determined, the assertion follows from Lemma 3.3. When $t \geq C_0$, we proceed as follows. Set:

$$e^k := M(K, tM^*(K)B_2^n),$$

Lemma 3.3 ensures that $k \leq n/C_0$, and since the packing number is an integer, we may assume that $k \geq \log 2$ (otherwise $k = 0$ and there is nothing to prove). Let $\{x_i\}_{i=1, \dots, e^k}$ denote the maximal collection of points in K which are $tM^*(K)B_2^n$ -separated. By the

Johnson–Lindenstrauss Lemma 3.4, we may choose C_0 large enough so that setting $m := \lceil C_0 k \rceil \in [C_0 \log 2, n]$, an orthogonal projection P_F onto a randomly selected $F \in G_{n,m}$ with respect to its Haar probability measure $\sigma_{G_{n,m}}$, will satisfy with probability at least $1 - \binom{e^k}{2} C \exp(-cm(1/2)^2) > 1/2$ that:

$$\{P_F(x_i)\} \text{ are } \frac{1}{2}tM^*(K)\sqrt{\frac{m}{n}}B_2(F)\text{-separated.} \quad (3.1)$$

In addition, since $h_{P_F K} = h_K|_F$, note that:

$$\begin{aligned} M^*(K) &= \int_{S^{n-1}} h_K(\theta) d\sigma_{S^{n-1}}(\theta) = \int_{G_{n,m}} \int_{S(F)} h_K(\theta) d\sigma_{S(F)}(\theta) d\sigma_{G_{n,m}}(F) \\ &= \int_{G_{n,m}} M_F^*(P_F K) d\sigma_{G_{n,m}}(F), \end{aligned}$$

and so by the Markov–Chebyshev inequality,

$$M_F^*(P_F K) \leq 2M^*(K) \quad (3.2)$$

with probability at least $1/2$ (in fact, it follows by a result of Klartag–Vershynin [30, Section 3] that this holds with much higher probability, but this is not required here). By the union bound, it follows that there exists a subspace $F \in G_{n,m}$ for which both (3.1) and (3.2) hold. Hence, applying Lemma 3.3 to $P_F K$ in $F \in G_{n,m}$:

$$e^k \leq M(K, tM^*(K)B_2^n) \leq M\left(P_F K, \frac{1}{4}tM_F^*(P_F K)\sqrt{\frac{m}{n}}B_2(F)\right) \leq \exp\left(4\frac{m}{t\sqrt{m/n}}\right).$$

Using that $m \leq C_0 k + 1 \leq (C_0 + 1/\log(2))k$ and solving for k , we obtain:

$$k \leq C' \frac{n}{t^2},$$

and hence:

$$M(K, tM^*(K)B_2^n) = e^k \leq \exp\left(C' \frac{n}{t^2}\right) \quad \forall t \geq C_0,$$

concluding the proof. □

4 Part 3 - the case $p \geq n$

In this section, we establish a regular version of Part 3 of The Program. Specifically:

Theorem 4.1. *Let μ denote an origin-symmetric log-concave probability measure on \mathbb{R}^n , and let $L \subset \mathbb{R}^n$ denote a star-body. Then for any $p \geq n$, we have:*

$$M(Z_p(\mu), Ctm_q(\mu, L)L) \leq \exp\left(1 + q + \frac{p}{t}\right) \quad \forall t, q > 0$$

In particular, if $\mu(L) \geq e^{-p}$ with $p \geq n$ then $M(Z_p(\mu), CL) \leq \exp(3p)$.

For the proof, we require a bit of preparation, emphasizing that L need not be convex but only be star-shaped, which might prove useful for establishing Part 1 of The Program. We start with the following variation on Talagrand's proof of the dual Sudakov Minoration (e.g. [34, 1]), which was already used by Hartzoulaki in her PhD Thesis [21] and subsequently employed by other authors as well (cf. [35, 16]).

Recall that λ_K denotes the uniform probability measure on the convex body K .

Proposition 4.2. *Let K denote a convex body and let L denote a star-body in \mathbb{R}^n . Then:*

$$M(K, 2tm_q(\lambda_K, L)) \leq \exp(1 + q + \frac{n}{t}) \quad \forall q, t > 0.$$

For the proof, we will utilize the following auxiliary probability measure on \mathbb{R}^n , which may be associated to any star-body $K \subset \mathbb{R}^n$:

$$\mu_K := \frac{1}{n! |K|} e^{-\|x\|_K} dx.$$

Lemma 4.3. *With the same assumptions as in Proposition 4.2:*

$$M(K, tm_q(\mu_K, L)L) \leq \exp(q + \frac{1}{t}) \quad \forall q, t > 0.$$

Proof. Let us show the following equivalent formulation:

$$M(K, rL) \leq \frac{\exp(s/r)}{\mu_K(sL)} \quad \forall r, s > 0. \tag{4.1}$$

By definition, there exists $M := M(K, rL)$ points $z_1, \dots, z_M \in K$ so that the sets $\{z_i + rL\}$ are mutually disjoint. Hence, for all $s > 0$, the sets $\{\frac{s}{r}z_i + sL\}$ are also mutually disjoint. In addition, by convexity of K :

$$\mu_K(\frac{s}{r}z_i + sL) = \frac{1}{n! |K|} \int_{sL} e^{-\|\frac{s}{r}z_i + x\|_K} dx \geq \frac{1}{n! |K|} e^{-\frac{s}{r}\|z_i\|_K} \int_{sL} e^{-\|x\|_K} dx \geq e^{-\frac{s}{r}} \mu_K(sL).$$

Consequently:

$$1 \geq \sum_{i=1}^M \mu_K(\frac{s}{r}z_i + sL) \geq M e^{-\frac{s}{r}} \mu_K(sL),$$

establishing (4.1), as required. □

Lemma 4.4. *For all star-bodies $K, L \subset \mathbb{R}^n$ and $q > 1$, we have:*

$$m_{q-1}(\lambda_K, L) \geq \frac{1}{2n} m_q(\mu_K, L).$$

Proof. For all $s > 0$:

$$\mu_K(sL) = \frac{1}{n!|K|} \int_{sL} e^{-\|x\|_K} dx = \frac{1}{n!|K|} \int_0^\infty |tK \cap sL| e^{-t} dt = \frac{1}{n!} \int_0^\infty t^n e^{-t} \lambda_K\left(\frac{s}{t}L\right) dt.$$

Applying this to $s := m_q(\mu_K, L)$, we obtain:

$$e^{-q} = \mu_K(sL) = \frac{1}{n!} \int_0^\infty t^n e^{-t} \lambda_K\left(\frac{s}{t}L\right) dt \geq \lambda_K\left(\frac{s}{2n}L\right) \frac{1}{n!} \int_0^{2n} t^n e^{-t} dt \geq \frac{1}{e} \lambda_K\left(\frac{s}{2n}L\right),$$

where the very rough estimate $\int_0^{2n} t^n e^{-t} dt \geq \frac{1}{e} \int_0^\infty t^n e^{-t} dt$ is standard and may be easily verified by direct calculation (or e.g. by Markov's inequality when $n \geq 4$). It follows that $\frac{s}{2n} \leq m_{q-1}(\lambda_K, L)$, as asserted. \square

Proof of Proposition 4.2. Applying the previous two lemmas, the proof is immediate:

$$M(K, 2tm_q(\lambda_K, L)) \leq M(K, \frac{t}{n}m_{q+1}(\mu_K, L)) \leq \exp(1 + q + \frac{n}{t}).$$

\square

One final ingredient we require for the proof of Theorem 4.1 involves the following star-body, introduced by K. Ball [3]. Given a probability measure μ on \mathbb{R}^n with continuous and exponentially-decaying density f_μ with $f_\mu(0) > 0$ (“non-degenerate measure”), and $p \geq 1$, denote by $K_p(\mu) \subset \mathbb{R}^n$ the star-body with radial function:

$$\rho_{K_p(\mu)}(\theta) = \left(\frac{p}{\max f_\mu} \int_0^\infty r^{p-1} f_\mu(r\theta) dr \right)^{\frac{1}{p}}, \quad \theta \in S^{n-1}.$$

Note our slightly non-standard normalization involving $\max f_\mu$ instead of $f_\mu(0)$, which seems more convenient. Integration in polar coordinates immediately verifies that $|K_n(\mu)| = \frac{1}{\max f_\mu}$, and that (cf. [47]):

$$Z_p(\lambda_{K_{n+p}(\mu)}) = \left(\frac{|K_n(\mu)|}{|K_{n+p}(\mu)|} \right)^{1/p} Z_p(\mu). \quad (4.2)$$

Lemma 4.5. *For any star-body $L \subset \mathbb{R}^n$:*

$$\mu(L) \leq \lambda_{K_n(\mu)}(L).$$

In particular, for all $q > 0$:

$$m_q(\mu, L) \geq m_q(\lambda_{K_n(\mu)}, L).$$

Proof. Simply note that for all $\theta \in S^{n-1}$:

$$\begin{aligned} \int_0^{\rho_L(\theta)} r^{n-1} f_\mu(r\theta) dr &\leq \min\left(\left(\max f_\mu\right) \frac{\rho_L^n(\theta)}{n}, \int_0^\infty r^{n-1} f_\mu(r\theta) dr\right) \\ &= \frac{\max f_\mu}{n} \min(\rho_L^n(\theta), \rho_{K_n(\mu)}^n(\theta)) = \frac{\max f_\mu}{n} \rho_{K_n(\mu) \cap L}^n(\theta). \end{aligned}$$

Integrating the above ray-wise inequality on S^{n-1} , we obtain:

$$\begin{aligned} \mu(L) &= \int_{S^{n-1}} \int_0^{\rho_L(\theta)} r^{n-1} f_\mu(r\theta) dr d\theta \leq \max f_\mu \int_{S^{n-1}} \int_0^{\rho_{K_n(\mu) \cap L}(\theta)} r^{n-1} dr d\theta \\ &= \max f_\mu |K_n(\mu) \cap L| = \lambda_{K_n(\mu)}(L), \end{aligned}$$

as required. \square

Remarkably, it was observed by K. Ball [3] that when μ is an origin-symmetric log-concave probability measure, $K_p(\mu)$ is in fact a convex body for all $p \geq 1$; this was extended in [24] to the non-symmetric case (assuming that $f_\mu(0) > 0$). Consequently, Proposition 4.2 immediately yields the following:

Corollary 4.6. *For any log-concave probability measure μ on \mathbb{R}^n so that $f_\mu(0) > 0$, star-body $L \subset \mathbb{R}^n$, and $q, t > 0$:*

$$M(K_n(\mu), 2tm_q(\mu, L)L) \leq M(K_n(\mu), 2tm_q(\lambda_{K_n(\mu)}, L)L) \leq \exp\left(1 + q + \frac{n}{t}\right).$$

It remains to pass from $K_n(\mu)$ to $Z_n(\mu)$ in the packing estimate above. This is standard, but for completeness, and in order to prove an additional estimate we will require later on, we provide a proof. First, it is known [45] that:

$$Z_n(\lambda_K) \simeq \text{conv}(K \cup -K), \quad (4.3)$$

for any convex body $K \subset \mathbb{R}^n$. It is also known (see [4, 3, 41] for the even case and [24, Lemmas 2.5, 2.6] or [49, Lemma 3.2 and (3.12)] for the general one, noting our non-standard normalization) that for any log-concave measure μ on \mathbb{R}^n whose barycenter is at the origin, we have:

$$1 \leq p \leq q \Rightarrow K_p(\mu) \subset K_q(\mu) \subset \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} e^{n(\frac{1}{p}-\frac{1}{q})} K_p(\mu). \quad (4.4)$$

In particular, $K_n(\mu) \simeq K_{2n}(\mu)$. Combining this with (4.2) and (4.3), we obtain for an *origin-symmetric* log-concave measure μ (for which $f_\mu(0) = \max f_\mu > 0$):

$$Z_n(\mu) = \left(\frac{|K_{2n}(\mu)|}{|K_n(\mu)|}\right)^{1/n} Z_n(\lambda_{K_{2n}(\mu)}) \simeq Z_n(\lambda_{K_{2n}(\mu)}) \simeq K_{2n}(\mu) \simeq K_n(\mu). \quad (4.5)$$

Note that the origin-symmetry of μ was crucially used to ensure $Z_n(\lambda_{K_{2n}(\mu)}) \simeq K_{2n}(\mu)$. It is possible to dispose of this restriction by employing the one-sided variants $Z_n^+(\mu)$ introduced in [20], but we do not pursue this here.

Summarizing, we deduce from Corollary 4.6 and (4.5) that for an appropriate constant $C > 0$, we have under the assumptions of Theorem 4.1:

$$M(Z_n(\mu), Ctm_q(\mu, L)L) \leq M(K_n(\mu), 2tm_q(\mu, L)L) \leq \exp(1 + q + \frac{n}{t}),$$

concluding the theorem for the case $p = n$. When $p \geq n$, simply use (2.1):

$$Z_p(\mu) \subset \frac{p}{n}Z_n(\mu),$$

and conclude:

$$M(Z_p(\mu), Ctm_q(\mu, L)L) \leq M(Z_n(\mu), Ct\frac{n}{p}m_q(\mu, L)L) \leq \exp(1 + q + \frac{p}{t}).$$

The proof of Theorem 4.1 is complete.

Before concluding this section, we also record for future use the following well-known fact (cf. [29, 26, 25, 49]); as we did not find a precise reference, we provide a proof for completeness.

Lemma 4.7. *For any log-concave probability measure μ on \mathbb{R}^n with barycenter at the origin we have:*

$$I_1(\mu, Z_n(\mu)) \leq I_n(\mu, Z_n(\mu)) \leq C.$$

Proof. As in the proof of (4.2), it is immediate to verify by polar-integration that for any non-degenerate measure ν and star-body L in \mathbb{R}^n :

$$I_p(\nu, L) = \left(\frac{|K_{n+p}(\nu)|}{|K_n(\nu)|} \right)^{1/p} I_p(\lambda_{K_{n+p}(\nu)}, L).$$

Applying this to $\nu = \mu$, $L = Z_n(\mu)$ and $p = n$, and using that $K_{2n}(\mu) \simeq K_n(\mu)$ by (4.4), we obtain:

$$I_n(\mu, Z_n(\mu)) \leq C' I_n(\lambda_{K_{2n}(\mu)}, Z_n(\mu)).$$

It remains to use as in (4.5) that (without any symmetry assumptions):

$$Z_n(\mu) = \left(\frac{|K_{2n}(\mu)|}{|K_n(\mu)|} \right)^{1/n} Z_n(\lambda_{K_{2n}(\mu)}) \supset Z_n(\lambda_{K_{2n}(\mu)}) \supset c \text{Conv}(K_{2n}(\mu) \cup -K_{2n}(\mu)) \supset cK_{2n}(\mu).$$

Consequently:

$$I_n(\mu, Z_n(\mu)) \leq \frac{C'}{c} I_n(\lambda_{K_{2n}(\mu)}, K_{2n}(\mu)) \leq \frac{C'}{c},$$

concluding the proof. □

5 The Program - Full Version

We are now ready to state the full version of The Program. The full version extends the simplified one presented in the Introduction in two aspects. First, in Part 1, we allow the packing number after dimension reduction to *drop* by a D -th root, where $D \geq 1$ is an additional parameter we introduce; this additional flexibility will be crucial for applying The Program to the case of the cube $K = B_\infty^n$, analyzed in Section 10. Second, we also introduce a *regularity* parameter $t > 0$, whose role is to scale the bodies K and L ; this is important for obtaining a regular version of the Generalized Dual Sudakov estimate for ellipsoids in Section 7, which is later used for establishing a Weak Generalized Dual Sudakov estimate (Part 2 of The Program) for more general convex bodies in Section 9.

As usual, we fix an origin-symmetric convex body $K \subset \mathbb{R}^n$, origin-symmetric log-concave measure μ on \mathbb{R}^n and $p \geq 1$, and fix an additional scaling parameter $t > 0$. The Program for establishing the Generalized Regular Dual Sudakov estimate:

$$\mu(K) \geq \frac{1}{e} \quad \Rightarrow \quad M(Z_p(\mu), tRK) \leq \exp(C_{A,B,D,\varphi_t,t} p), \quad (5.1)$$

consists of establishing the first 2 parts below for some constants $R, A, B, D \geq 1$ and a certain function φ_t , described below. For all $m = 1, \dots, n$, set as usual $\mathcal{M}_m := \{T_*\mu; T: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ linear}\}$, which is a family of log-concave measures on \mathbb{R}^m by the Prékopa–Leindler theorem. In addition, let \mathcal{L}_m denote some family of origin-symmetric star-bodies in \mathbb{R}^m , with $\mathcal{L}_n = \{K\}$.

(1) **Part 1 (Massive Partial Separation Dimension Reduction).**

If $M(Z_p(\mu), tRK) = e^k$ with $\mu(K) \geq \frac{1}{e}$ and $4BD \leq k \leq n/A$, show that there exists $l \in [k/D, k]$ and a linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L \in \mathcal{L}_m$, with $m \leq Al$, so that:

- (a) $M(TZ_p(\mu), tL) \geq e^l$ (“**Partial Separation Dimension Reduction**”).
- (b) $T_*\mu(L) \geq \exp(-q_m)$, $1 \leq q_m \leq l/2$ (“ **L is sufficiently massive**”).

(2) **Part 2 (Weak Generalized Regular Dual Sudakov).**

For all $m = 1, \dots, n$, $L \in \mathcal{L}_m$ and $\nu \in \mathcal{M}_m$, show that:

$$1 \leq p \leq m, \quad \nu(L) \geq \exp(-q_m) \quad \Rightarrow \quad M(Z_p(\nu), tL) \leq \exp(B + q_m + m\varphi_t(p/m)),$$

where $\varphi_t: [0, 1] \rightarrow \mathbb{R}_+$ is an increasing function with $\varphi_t(0) = 0$ and $x \mapsto \varphi_t(x)/x$ non-increasing (depending only on t and independent of all other parameters).

(3) **Part 3 (Large p).**

For all $m = 1, \dots, n$, $L \in \mathcal{L}_m$ and $\nu \in \mathcal{M}_m$, Theorem 4.1 verifies that:

$$p \geq m, \quad \nu(L) \geq \exp(-q_m) \quad \Rightarrow \quad M(Z_p(\nu), tL) \leq \exp\left(1 + q_m + C\frac{p}{t}\right),$$

for some universal constant $C \geq 1$.

Remark 5.1. As in the Introduction, we state the following *linear* version of Part 1:

(1') **Part 1' - Linear Version**

If $\{x_i\}_{i=1,\dots,e^k} \subset \mathbb{R}^n$ is a collection of K -separated points with $\mu(K) \geq \frac{1}{e}$ and $4BD \leq k \leq n/A$, show that there exists $l \in [k/D, k]$ and a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L \in \mathcal{L}_m$, with $m \leq Al$, so that:

- (a) There exists $I \subset \{1, \dots, e^k\}$ with $\#I \geq e^l$ so that $\{T(x_i)\}_{i \in I} \subset \mathbb{R}^m$ are $\frac{1}{R}L$ -separated (“**Partial One-sided Johnson–Lindenstrauss**”).
- (b) $T_*\mu(L) \geq \exp(-q_m)$, $1 \leq q_m \leq l/2$ (“ **L is sufficiently massive**”).

By applying this linear version of Part 1 to the maximal collection of K -separated points $\{x_i\}$ in $\frac{1}{tR}Z_p(\mu)$, it is evident that establishing Part 1' is sufficient (but not necessary) for establishing Part 1 of The (full) Program.

Theorem 5.2. *Establishing The (full) Program above yields the Generalized Regular Dual Sudakov Estimate (5.1) with:*

$$C_{A,B,D,\varphi_t,t} := D \max \left(\frac{4 \max(C, C' \frac{B}{R})}{t}, \frac{1}{A\varphi_t^{-1}(\frac{1}{4A})} \right). \quad (5.2)$$

Proof. We assume that $\mu(K) \geq 1/e$. We will first show that:

$$e^k := M(Z_p(\mu), tRK) \leq \exp \left(D \max \left(\frac{4C}{t}, \frac{1}{A\varphi_t^{-1}(\frac{1}{4A})} \right) p \right), \quad (5.3)$$

under the assumption that $k \geq 4BD$.

Under this assumption, there exists $l \in [k/D, k]$ and a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L \in \mathcal{L}_m$ for some $m \leq \min(n, Al)$, so that $M(TZ_p(\mu), tL) \geq e^l$ and $T_*\mu(L) \geq \exp(-q_m)$, $1 \leq q_m \leq l/2$. Indeed, if $k < n/A$ this follows from Part 1, whereas if $k \geq n/A$ this is actually trivial by using $m = n$, $l = k$, $T = Id$, $L = K$ and $q_m = 1$. Denoting $\nu = T_*\mu \in \mathcal{M}_m$, note that $TZ_p(\mu) = Z_p(\nu)$. Also note that $l/4 \geq B \geq 1$. Consequently, if $p \geq m$ then by Part 3:

$$\exp(l) \leq M(Z_p(\nu), tL) \leq \exp(q_m + Bp) \leq \exp(1 + l/2 + C \frac{p}{t}) \leq \exp(l/4 + l/2 + C \frac{p}{t}),$$

implying that $k \leq Dl \leq 4DC \frac{p}{t}$, as required. Alternatively, if $p \leq m$ then by Part 2 and the assumption that $x \mapsto \varphi_t(x)/x$ is non-increasing:

$$\exp(l) \leq M(Z_p(\nu), tL) \leq \exp(B + q_m + m\varphi_t(p/m)) \leq \exp(l/4 + l/2 + Al\varphi_t(p/(Al))).$$

It follows since φ_t is increasing from 0 that:

$$\frac{p}{Al} \geq \varphi^{-1} \left(\frac{1}{4A} \right) > 0,$$

implying that $k \leq Dl \leq D \frac{1}{A\varphi_t^{-1}(\frac{1}{4A})}p$, and establishing (5.3) under the assumption that $k \geq 4BD$.

To complete the proof, recall that $Z_p(\mu) \subset I_p(\mu, K)K$ by Lemma 2.2. Since $I_p(\mu, K) \leq C'p m_1(\mu, K) \leq C'p$ by Lemma 2.1, it follows that the left-hand-side of (5.3) is actually 1 (equivalently, $k = 0$) for $t \geq C'p/R$, in which case there is nothing to prove. On the other hand, in the non-trivial range $t \in (0, C'p/R)$, we have $4DC' \frac{B}{R} \frac{1}{t} p \geq 4BD$, which leads to the definition of $C_{A,B,D,\varphi_t,t}$ in (5.2) and confirms (5.1) for all $t > 0$. \square

6 Part 2 - Weak Generalized Dual Sudakov: Ellipsoids

Recall that:

$$I_q(\mu, K) := \left(\int_{\mathbb{R}^n} \|x\|_K^q d\mu(x) \right)^{1/q}.$$

When $K = B_2^n$, we simply denote $I_q(\mu) = I_q(\mu, B_2^n)$.

Theorem 6.1. *Let μ denote an origin-symmetric log-concave probability measure on \mathbb{R}^n and let $\mathcal{E} \subset \mathbb{R}^n$ denote an (origin-symmetric) ellipsoid. Then for any $p \in [1, n]$:*

$$M(Z_p(\mu), tI_1(\mu, \mathcal{E})\mathcal{E}) \leq \exp \left(C \frac{p^{2/3} n^{1/3}}{t^{2/3}} + C \frac{\sqrt{p}\sqrt{n}}{t} \right) \quad \forall t > 0.$$

Since $N(Z_p(\mu), tI_1(\mu, \mathcal{E})\mathcal{E})$ is invariant under simultaneously applying a linear transformation to μ and \mathcal{E} , we may and will reduce to the case $\mathcal{E} = B_2^n$. For the proof, our strategy will be to invoke the standard volumetric estimate on the covering numbers (see Section 2):

$$M(Z_p(\mu), tB_2^n) \leq \frac{|Z_p(\mu) + tB_2^n|}{|tB_2^n|} \quad \forall t > 0.$$

To handle the numerator, we use Steiner's classical formula [53], stating that for any convex body $K \subset \mathbb{R}^n$:

$$|K + tB_2^n| = \sum_{k=0}^n \binom{n}{k} W_k(K) t^{n-k},$$

where $W_k(K)$ denotes the k -th quermassintegral (or mixed-volume) of K . Recall that by Kubota's formula (e.g. [53]), we have:

$$W_k(K) = \frac{|B_2^n|}{|B_2^k|} \int_{G_{n,k}} |P_F K| d\sigma_{G_{n,k}}(F) = |B_2^n| \int_{G_{n,k}} \text{vrad}(P_F K)^k d\sigma_{G_{n,k}}(F), \quad (6.1)$$

where $\sigma_{G_{n,k}}$ denotes the Haar probability measure on $G_{n,k}$ (with the interpretation when $k = 0$ that $W_0(K) = |B_2^n|$).

To bound $W_k(Z_p(\nu))$, we will need the following averaged version of [15, Theorem 2.4],[40, Proposition 3.1]:

Proposition 6.2. *Let μ denote an origin-symmetric log-concave probability measure on \mathbb{R}^n . Then for all $p \geq 1$ and $k = 1, \dots, n$:*

$$W_k(Z_p(\mu))^{\frac{1}{k}} \leq C \max(\sqrt{p}, p/\sqrt{k}) W_k(Z_2(\mu))^{\frac{1}{k}}.$$

Proof. It was shown in [47, Theorem 6.2] (see also [40, Corollary 2.2]) that for any (say origin-symmetric) log-concave probability measure η on \mathbb{R}^k , one has:

$$\text{vrad}(Z_p(\eta)) \leq C \sqrt{p} \det \text{Cov}(\eta)^{\frac{1}{2k}} \quad \forall 1 \leq p \leq k.$$

When $p \geq k$, since $Z_p(\eta) \subset \frac{p}{k} Z_k(\eta)$, it follows that:

$$\text{vrad}(Z_p(\eta)) \leq \frac{p}{k} \text{vrad}(Z_k(\eta)) \leq C \frac{p}{\sqrt{k}} \det \text{Cov}(\eta)^{\frac{1}{2k}} \quad \forall p \geq k.$$

Finally, noting that $\det \text{Cov}(\eta)^{\frac{1}{2k}} = \text{vrad}(Z_2(\eta))$ as $Z_2(\eta)$ is an ellipsoid, we obtain:

$$\text{vrad}(Z_p(\eta)) \leq C \max(\sqrt{p}, p/\sqrt{k}) \text{vrad}(Z_2(\eta)) \quad \forall p \geq 1.$$

Applying the above to $\eta = \pi_F \mu$, using that $Z_q(\pi_F \mu) = P_F Z_q(\mu)$, integrating over $F \in G_{n,k}$ and applying Kubota's formula (6.1), the assertion readily follows. \square

The quermassintegrals of the ellipsoid $Z_2(\mu)$ are particularly easy to compute using elementary linear algebra, and one may show that $(W_k(Z_2(\mu))/|B_2^n|)^{2/k}$ is closely related to the k -th root of the k -th symmetric elementary polynomial in the eigenvalues of $\text{Cov}(\mu)$ (appropriately normalized). However, we will only require:

Lemma 6.3. *For all log-concave probability measures μ on \mathbb{R}^n :*

$$\det \text{Cov}(\mu)^{\frac{1}{2n}} \leq \left(\frac{W_k(Z_2(\mu))}{|B_2^n|} \right)^{\frac{1}{k}} \leq \left(\frac{1}{n} \text{tr} \text{Cov}(\mu) \right)^{\frac{1}{2}} \quad \forall k = 1, \dots, n.$$

Proof. By the Alexandrov-Fenchel inequalities for the quermassintegrals [53], we have:

$$\left(\frac{W_n(Z_2(\mu))}{|B_2^n|} \right)^{\frac{1}{n}} \leq \left(\frac{W_k(Z_2(\mu))}{|B_2^n|} \right)^{\frac{1}{k}} \leq \frac{W_1(Z_2(\mu))}{|B_2^n|},$$

so it is enough to calculate the expressions on either side. Indeed:

$$\begin{aligned} \frac{W_1(Z_2(\mu))}{|B_2^n|} &= \int_{S^{n-1}} h_{Z_2(\mu)}(\theta) d\sigma(\theta) \leq \left(\int_{S^{n-1}} h_{Z_2(\mu)}^2(\theta) d\sigma(\theta) \right)^{\frac{1}{2}} \\ &= \left(\int_{S^{n-1}} \langle \text{Cov}(\mu)\theta, \theta \rangle d\sigma(\theta) \right)^{\frac{1}{2}} = \left(\frac{1}{n} \text{tr} \text{Cov}(\mu) \right)^{\frac{1}{2}}, \end{aligned}$$

while:

$$\left(\frac{W_n(Z_2(\mu))}{|B_2^n|} \right)^{\frac{1}{n}} = \left(\frac{|Z_2(\mu)|}{|B_2^n|} \right)^{\frac{1}{n}} = \det \text{Cov}(\mu)^{\frac{1}{2n}}.$$

\square

Finally, it is useful to note that by Lemma 2.1:

$$(\operatorname{tr} \operatorname{Cov}(\mu))^{\frac{1}{2}} = \left(\int_{\mathbb{R}^n} |x|^2 d\mu(x) \right)^{\frac{1}{2}} = I_2(\mu) \simeq I_1(\mu).$$

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1.

$$M(Z_p(\mu), tI_1(\mu)B_2^n) \leq \frac{|Z_p(\mu) + tI_1(\mu)B_2^n|}{|tI_1(\mu)B_2^n|} = \sum_{k=0}^n \binom{n}{k} \frac{W_k(Z_p(\mu))}{(tI_1(\mu))^k |B_2^n|}.$$

Employing Proposition 6.2 and Lemma 6.3, we know that:

$$\left(\frac{W_k(Z_p(\mu))}{|B_2^n|} \right)^{1/k} \leq C \max(\sqrt{p}, p/\sqrt{k}) \frac{I_1(\mu)}{\sqrt{n}} \quad \forall k = 1, \dots, n.$$

Using the standard estimate $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, we obtain:

$$\begin{aligned} M(Z_p(\mu), tI_1(\mu)B_2^n) &\leq 1 + \sum_{k=1}^n \left(C \frac{e\sqrt{n}}{tk} \max(\sqrt{p}, p/\sqrt{k}) \right)^k \\ &= 1 + \sum_{k=1}^{\lfloor p \rfloor} \left(C \frac{ep\sqrt{n}}{tk^{3/2}} \right)^k + \sum_{k=\lfloor p \rfloor+1}^n \left(C \frac{e\sqrt{p}\sqrt{n}}{tk} \right)^k \\ &= 1 + \sum_{k=1}^{\lfloor p \rfloor} \left(C' \frac{p^{2/3}n^{1/3}}{t^{2/3}k} \right)^{\frac{3}{2}k} + \sum_{k=\lfloor p \rfloor+1}^n \left(C \frac{e\sqrt{p}\sqrt{n}}{tk} \right)^k \\ &\leq \sum_{m=0}^{\infty} \frac{1}{m!} \left(C'' \frac{p^{2/3}n^{1/3}}{t^{2/3}} \right)^m + \sum_{k=0}^{\infty} \frac{1}{k!} \left(C'' \frac{\sqrt{p}\sqrt{n}}{t} \right)^k \\ &= \exp \left(C'' \frac{p^{2/3}n^{1/3}}{t^{2/3}} + C'' \frac{\sqrt{p}\sqrt{n}}{t} \right). \end{aligned}$$

The assertion is thus established for $\mathcal{E} = B_2^n$, and hence for arbitrary ellipsoids, as explained above. The proof is complete. \square

7 Part 1 - Separation Dimension Reduction: Ellipsoids

7.1 The Probabilistic Approach

The following proposition decouples the question of separation dimension reduction and the massiveness requirement of Part 1 of The Program. Not surprisingly, this is achieved by introducing some randomness.

Proposition 7.1. *Let $K \subset \mathbb{R}^n$ denote a star-body, and assume that $\{x_i\}_{i=1,\dots,M} \subset \mathbb{R}^n$ is a collection of K -separated points. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote a random linear map and $L, S \subset \mathbb{R}^m$ denote two random star-bodies defined on a common probability space, so that $L = L_T$ and $S = S_T$ are measurable functions of T (when equipping the family of star-bodies with the Hausdorff metric). Assume that:*

- (1) *If $x \notin K$ then $\mathbb{P}(Tx \in L_T) \leq p_{out}$.*
- (2) *If $x \in K$ then $\mathbb{P}(Tx \in S_T) \geq p_{in}$.*

Then for any Borel probability measure μ on \mathbb{R}^n , if:

$$M^2 p_{out} \leq \mu(K) p_{in}, \tag{7.1}$$

then there exists a linear map $T^0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and star-bodies $L^0, S^0 \subset \mathbb{R}^m$ so that $\{T^0(x_i)\}_{i=1,\dots,M}$ are L^0 -separated and $T_^0(\mu)(S^0) \geq \frac{1}{2}\mu(K)p_{in}$.*

Proof. We may assume that $p_{in}, p_{out}, \mu(K) > 0$. By linearity and the union-bound, the random set $\{T(x_i)\}_{i=1,\dots,M}$ is clearly L_T -separated with probability at least:

$$1 - \binom{M}{2} p_{out} > 1 - \frac{M^2}{2} p_{out},$$

so it remains to verify the second requirement. Denoting $G_T := T_*(\mu)(S_T)$, note that:

$$\mathbb{E}(G_T) = \mathbb{E} \left(\int_{\mathbb{R}^n} 1_{\{Tx \in S_T\}} d\mu(x) \right) = \int_{\mathbb{R}^n} \mathbb{P}(Tx \in S_T) d\mu(x) \geq \int_K \mathbb{P}(Tx \in S_T) d\mu(x) \geq \mu(K) p_{in} =: q.$$

Since $0 \leq G_T \leq 1$ and $\mathbb{E}(G_T) \geq q$, it follows that $\mathbb{P}(G_T \geq q/2) \geq q/2$. Consequently, the assumption (7.1) guarantees that the event that $\{T(x_i)\}_{i=1,\dots,M}$ are L_T -separated and the one that $G_T \geq q/2$ have non-empty intersection, yielding the claim. \square

7.2 Part 1' For Ellipsoids

In view of Proposition 7.1, Part 1' of The Program will follow from the following one-sided variant of the Johnson–Lindenstrauss lemma, which pertains to small-ball probability, see e.g. [43, Fact 3.2] or [10, Lemma 8.1.15]:

Lemma 7.2. *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote a random orthogonal projection, that is $T = P \circ U$ where U is uniformly distributed on $SO(n)$ and P is the canonical projection on the first m coordinates, $m = 1, \dots, n$. Then for all $x \in S^{n-1}$:*

$$\mathbb{P} \left(\sqrt{n/m} |Tx| \leq \epsilon \right) \leq (C'\epsilon)^m \quad \forall \epsilon \in [0, 1].$$

Corollary 7.3 (Part 1' for Euclidean Ball). *Let $\{x_i\}_{i=1,\dots,e^k} \subset \mathbb{R}^n$ be a collection of B_2^n -separated points, $k \in [1, n]$, and let μ denote a Borel probability measure with $\mu(B_2^n) \geq e^{-q} \geq e^{-k}$. Set $m = \lceil k \rceil$, and denote $L := \sqrt{2}\sqrt{m/n}B_2^m$. Then there exists an orthogonal projection $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (as above) so that:*

- (1) $\{T(x_i)\}_{i=1,\dots,e^k} \subset \mathbb{R}^m$ are L/C -separated.
- (2) $T_*(\mu)(L) \geq \frac{1}{4}e^{-q}$.

Proof. By appropriately choosing $c > 0$, we may ensure that:

- (1) $\mathbb{P}(|Tx| \leq c\sqrt{m/n}) \leq p_{\text{out}} := \frac{1}{2}e^{-3m}$ for any $x \notin B_2^n$.
- (2) $\mathbb{P}(|Tx| \leq \sqrt{2}\sqrt{m/n}) \geq p_{\text{in}} := \frac{1}{2}$ for any $x \in B_2^n$.

Indeed, the first estimate is ensured by Lemma 7.2, while the second one follows by simply noting that $\mathbb{E}|Tx|^2 = \frac{m}{n}|x|^2$ and applying the Markov–Chebyshev inequality (or by invoking the Johnson–Lindenstrauss Lemma 3.4, but this is actually unnecessary). The assertion then follows by Proposition 7.1 with $C = \sqrt{2}/c$, $L_T \equiv L/C$ and $S_T \equiv L$. \square

7.3 Running The Program for Ellipsoids

Running (the regular version of) The Program of Section 5, we can finally obtain:

Theorem 7.4 (Generalized Regular Dual Sudakov For Ellipsoids). *For any origin-symmetric log-concave measure μ on \mathbb{R}^n and any (origin-symmetric) ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, we have:*

$$M(Z_p(\mu), tm_1(\mu, \mathcal{E})\mathcal{E}) \leq \exp\left(C\left(\frac{p}{t^2} + \frac{p}{t}\right)\right) \quad \forall p \geq 1 \quad \forall t > 0.$$

Proof. Since the expression on the left-hand-side is invariant under simultaneously applying a linear transformation to μ and \mathcal{E} , it is enough to establish it for the case that $\mathcal{E} = B_2^n$. Since this expression is also invariant under scaling μ , we may assume that $m_1(\mu, B_2^n) = 1$.

Given $p \geq 1$ and $t > 0$, we run The Program for $K = B_2^n$, with \mathcal{L}_m consisting of (centered) Euclidean balls in \mathbb{R}^m . Corollary 7.3 applied with $q = 1$ verifies Part 1' of The Program regarding Massive Separation Dimension Reduction (with say $D = 1$, $B = 1$, $q_m = 3$, $A = 5/4$ and $R \leq C'$). Part 2 of The Program regarding Weak Generalized Regular Dual Sudakov, with parameters $q_m = 3$ and $\varphi_t(x) = C'' \max((x/t)^{2/3}, \sqrt{x}/t)$, is established by Theorem 6.1 in conjunction with Lemma 2.1 (which implies that $m_3(\nu, L) \simeq I_1(\nu, L)$ for all $\nu \in \mathcal{M}_m$ and $L \in \mathcal{L}_m$). Since $\varphi_t^{-1}(y) \simeq \min(y^{3/2}t, y^2t^2)$, we have $\varphi_t^{-1}(1/(4A)) \simeq \min(t, t^2)$, and Theorem 5.2 yields the asserted estimate. \square

8 Pure Measures

In this section, we introduce the class of pure log-concave probability measures, and study their properties.

8.1 Definitions

Recall that the isotropic constant of a probability measure μ on \mathbb{R}^n having log-concave density f_μ is defined as the following affine-invariant quantity:

$$L_\mu := (\max f_\mu)^{\frac{1}{n}} (\det \text{Cov}(\mu))^{\frac{1}{2n}} . \quad (8.1)$$

It is well-known (e.g. [41, 26]) that $L_\mu \geq c$ for some universal constant $c > 0$. See Bourgain [7, 8], Milman–Pajor [41], Ball [2] and Brazitikos–Giannopoulos–Valettas–Vritsiou [10] for background on the yet unresolved Slicing Problem, which is concerned with obtaining a dimension independent upper-bound on L_μ . The current best-known estimate $L_\mu \leq Cn^{1/4}$ is due to B. Klartag [24], who improved the previous estimate $L_\mu \leq Cn^{1/4} \log(1+n)$ of J. Bourgain [8] (proved when μ is the uniform measure on an origin-symmetric convex body, but applies to general log-concave probability measures, see [2, 24]); see also Klartag–Milman [28] and Vritsiou [59] for subsequent refinements.

The following key estimate, which plays a fundamental role in previous groundbreaking works of Paouris [47, 49] and Klartag [26], relates between $|Z_n(\mu)|$ and L_μ (see e.g. the proof of [40, Theorem 2.1]):

Theorem 8.1 (Paouris, Klartag). *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Then:*

$$\text{vrad}(Z_n(\mu)) \simeq \frac{\sqrt{n}}{L_\mu} \text{vrad}(Z_2(\mu)) = \sqrt{n} \frac{\det \text{Cov}(\mu)^{\frac{1}{2n}}}{L_\mu} = \frac{\sqrt{n}}{\max f_\mu^{1/n}} .$$

In other words, if μ is isotropic then $|Z_n(\mu)|^{1/n} \simeq \frac{1}{L_\mu}$.

Definition 8.2 (*h-pure measure*). *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. We will say that μ is h-pure ($h = 1, \dots, n$), with constants (A, B) , if the following two conditions hold:*

- (1) $Z_n(\mu) \supset \frac{1}{A} \sqrt{h} Z_2(\mu)$.
- (2) For all $E \in G_{n,m}$ with $m = h, \dots, n$, we have that $L_{\pi_E \mu} \leq B$.

When μ is h-pure with some universally bounded constants $A, B \leq C < \infty$, we will simply say that μ is h-pure (with implicitly bounded constants).

Note that in the second condition, only the marginals of μ of dimension not smaller than h are taken into account. For example, if μ is isotropic and $Z_n(\mu) \supset \frac{1}{A} \sqrt{n} B_2^n$, it follows from Theorem 8.1 that $L_\mu \leq CA$, and hence μ is n-pure (with constants (A, CA)). On the other, if all marginals of μ (of arbitrary dimension) have isotropic constant bounded by a universal constant $C > 0$, since $Z_n(\mu) \supset Z_2(\mu)$ (if $n \geq 2$, and up to a constant otherwise), we see that μ is 1-pure (with constants $(1, C)$). The Slicing Problem

may be equivalently restated as asking whether all log-concave probability measures on \mathbb{R}^n ($n \geq 2$) are 1-pure with constants $(1, C)$, for some universal constant $C > 0$ independent of n .

The following is immediate from the definition:

Lemma 8.3. *If μ is both h_1 -pure and h_2 -pure with $1 \leq h_1 < h_2 \leq n$, then it is also h -pure for all $h = h_1, \dots, h_2$.*

8.2 Families of Pure Measures

We now provide several useful examples of families of log-concave measures which are pure. For simplicity, we restrict our attention to origin-symmetric measures. Recall that a measure is called unconditional if it is invariant under reflections with respect to all coordinate hyperplanes. A probability measure μ on \mathbb{R}^n is called Ψ_2 or sub-Gaussian if $Z_p(\mu) \subset C\sqrt{p}Z_2(\mu)$ for some universal constant $C \geq 1$ and all $p \geq 2$. It is called super-Gaussian if $Z_p(\mu) \supset c\sqrt{p}Z_2(\mu)$ for some universal constant $c > 0$ and all $p \in [2, n]$. It is immediate to verify (see e.g. [20]) that if $\tilde{\mu}$ is an isotropic origin-symmetric log-concave probability measure and γ_n is the standard Gaussian measure, then the convolved measure $\mu = \tilde{\mu} * \gamma_n$ is log-concave and super-Gaussian. We refer to [10] for missing additional definitions, which will not be required for the rest of this work.

Proposition 8.4. *The following families of log-concave measures are n -pure:*

- (1) *Super-Gaussian measures.*
- (2) *Uniform measures on 2-convex bodies.*

Proof. If μ is super-Gaussian we have by definition that $Z_n(\mu) \supset c\sqrt{n}Z_2(\mu)$. If μ is the uniform measure on a 2-convex body K (see [27] for missing definitions), it was shown in [27] that $Z_n(\mu) \supset c\sqrt{n}Z_2(\mu)$ for some universal constant $c > 0$ (depending only on the 2-convexity constant of K). In either case, we deduce by Theorem 8.1 that $L_\mu \leq C$, establishing both properties of an n -pure measure. \square

Proposition 8.5. *The following families of log-concave measures are 1-pure:*

- (1) *Super-Gaussian measures.*
- (2) *Sub-Gaussian (Ψ_2) measures.*
- (3) *Unconditional measures.*

Furthermore, the class of 1-pure measures is closed under taking marginals.

Proof. We may assume that all measures in question are isotropic.

(1) If μ is super-Gaussian, we have for all $E \in G_{n,m}$:

$$Z_m(\pi_E\mu) = P_E Z_m(\mu) \supset c\sqrt{m}B_2(E),$$

so that $\frac{1}{L_{\pi_E\mu}} \simeq |Z_m(\pi_E\mu)|^{1/m} \geq c' > 0$, as asserted.

(2) Similarly, if μ is Ψ_2 then for all $E \in G_{n,m}$ and $p \geq 2$:

$$Z_p(\pi_E\mu) = P_E Z_p(\mu) \subset C\sqrt{p}B_2(E),$$

confirming that $\pi_E\mu$ is also Ψ_2 (with the same universal bound C on its Ψ_2 constant). It is well-known [9, 28] that a Ψ_2 measure has bounded isotropic constant, confirming the assertion in this case.

(3) If μ is an isotropic unconditional measure and χ is the uniform measure on the cube $[-1, 1]^n$, it was noted in [11, p. 2829] following Latała that $Z_p(\mu) \supset cZ_p(\chi)$ for all $p \geq 1$. It follows as before that if $E \in G_{n,m}$ then:

$$Z_m(\pi_E\mu) = P_E Z_m(\mu) \supset cP_E Z_m(\chi) = cZ_m(\pi_E\chi).$$

Taking volumes, we deduce:

$$\frac{1}{L_{\pi_E\mu}} \simeq |Z_m(\pi_E\mu)|^{1/m} \geq c|Z_m(\pi_E\chi)|^{1/m} \simeq \frac{c}{L_{\pi_E\chi}}.$$

But since χ is a Ψ_2 measure, we know that $L_{\pi_E\chi}$ is universally bounded above, establishing (3).

The closure under marginals is immediate from the definition and the fact that $Z_m(\mu) \supset cZ_2(\mu)$ for all $m \geq 1$ by (2.1). \square

Remark 8.6. It was shown by Paouris in [48] that product measures (having arbitrarily many factors) of sub-Gaussian or super-Gaussian log-concave measures are 1-pure.

Remark 8.7. A well-known argument due to V. Milman involving the M-position [1, 51], in combination with K. Ball's observation that the isotropic position is an M-position if the isotropic constant is bounded [2], shows that if μ is an origin-symmetric isotropic log-concave probability measure with $L_\mu \leq C$, then with high-probability, a random marginal $\pi_F\mu$ with $F \in G_{n,n/2}$ is $n/2$ -pure with universal constants (A, B) depending solely on C . Moreover, it may be shown that with high-probability, a random marginal as above is super-Gaussian, and therefore h -pure for all $h = 1, \dots, n/2$ with universal constants (A, B) depending solely on C .

8.3 Properties of Pure Measures

Given a convex body $K \subset \mathbb{R}^n$ and $m = 1, \dots, n$, we use the following notation:

$$\begin{aligned} v_m^-(K) &:= \inf \{ \text{vrad}(P_E(K)); E \in G_{n,m} \}, \\ e_m(B_2^n, K) &:= \inf \{ t > 0 ; N(B_2^n, tK) \leq 2^m \}. \end{aligned}$$

We will need the following crucial estimate on the regularity of dual covering numbers of pure isotropic log-concave measures, essentially established by Giannopoulos and Milman in [14]:

Theorem 8.8. *Let μ denote a h -pure isotropic log-concave probability measure (with constants (A, B)). Then for all $k = 1, \dots, n$:*

$$v_k^-(Z_n(\mu)) \geq \max \left(\frac{1}{A} \sqrt{h}, \frac{c}{B} \sqrt{k} \right). \quad (8.2)$$

Furthermore, for all $k = 1, \dots, n$ we have:

$$e_k(B_2^n, Z_n(\mu)) \leq \min \left(\frac{A}{\sqrt{h}}, C_{A,B} \frac{1}{\sqrt{k}} \frac{n}{k} \log \left(e + \frac{n}{k} \right) \right), \quad (8.3)$$

or equivalently, we have for all $t > 0$:

$$N(\sqrt{n}B_2^n, tZ_n(\mu)) \leq \begin{cases} \exp \left(C'_{A,B} n \left(\frac{\log(e+t)}{t} \right)^{\frac{2}{3}} \right) & t \leq A\sqrt{n/h} \\ 1 & \text{otherwise} \end{cases}. \quad (8.4)$$

Proof Sketch. Since μ is assumed isotropic we have $Z_2(\mu) = B_2^n$. The case $h = 1$ appears explicitly in [14, Lemma 12 and Theorem 16]. For the general case, an inspection of the proof of [14, Theorem 16] reveals that the only ingredient required to obtain an estimate on $e_k(B_2^n, Z_n(\mu))$ is a lower bound on $v_m^-(Z_n(\mu))$ for $m = 1, \dots, k$.

When $m \leq h$, we may simply use $Z_n(\mu) \supset \frac{1}{A} \sqrt{h} B_2^n$ and conclude that:

$$v_m^-(Z_n(\mu)) \geq \frac{1}{A} \sqrt{h}.$$

When $m = h, \dots, n$, [14, Lemma 12] ensures that:

$$v_m^-(Z_n(\mu)) \geq \frac{c}{\sup \{ L_{\pi_E \mu}; E \in G_{n,m} \}} \sqrt{m},$$

for some universal constant $c > 0$, and so we see that one only needs to control the isotropic constants of marginals of μ of dimension not smaller than h . Combining these two estimates, (8.2) follows for a h -pure isotropic measure with constants (A, B) .

Now, according to [14, Corollary 9 and Remark 6], one has for any $\alpha > 0$:

$$e_k(B_2^n, K) \leq C_\alpha \sup_{m=1, \dots, k} \left(\frac{m}{k}\right)^\alpha \frac{n}{m} \log\left(e + \frac{n}{m}\right) \frac{1}{v_m^-(K)}.$$

Applying this to $K = Z_n(\mu)$ (with, say, $\alpha = 2$), and plugging the estimate (8.2) on $v_m^-(K)$, we obtain:

$$e_k(B_2^n, Z_n(\mu)) \leq C_{A,B} \frac{1}{\sqrt{\max(h, k)}} \frac{n}{k} \log\left(e + \frac{n}{k}\right)$$

Combining this with the trivial estimate:

$$e_k(B_2^n, Z_n(\mu)) \leq \frac{A}{\sqrt{h}}$$

(since $Z_n(\mu) \supset \frac{1}{A}\sqrt{h}B_2^n$), the asserted (8.3) follows (note that we replaced $\max(h, k)$ by the looser k since we do not care here about the dependence of $C_{A,B}$ on (A, B)). The equivalent (8.4) is obtained in the range $t \geq 1$ by direct inspection of (8.3), and extended to all $t > 0$ by Lemma 2.3 after adjusting the constant $C'_{A,B}$. \square

9 Part 2 - Weak Generalized Dual Sudakov: Pure Measures and Regular Small-Diameter Bodies

It is naturally of interest to establish the Weak Generalized Dual Sudakov estimate for general (say origin-symmetric) log-concave measures μ and convex bodies K . Unfortunately, we have not been able to accomplish this in that generality. In this section, we establish a Weak Generalized Dual Sudakov estimate when the log-concave measure μ is assumed to be h -pure, or when K is assumed to have α -regular small-diameter (defined below).

9.1 Part 2 for Pure Measures

Theorem 9.1. *Let μ be a h -pure log-concave measure on \mathbb{R}^n (with constants (A, B)), and let $p \in [1, n]$. Then for any $q > 0$ and star-body $L \subset \mathbb{R}^n$:*

$$M\left(Z_p(\mu), t\sqrt{\frac{p}{n}}m_q(\mu, L)L\right) \leq \exp\left(1 + q + C_{A,B}n\left(\frac{\log(e+t)}{t}\right)^{\frac{1}{3}}\right) \quad \forall t > 0.$$

In particular, if $\mu(L) \geq \exp(-p)$ with $p \in [1, n]$ then:

$$M(Z_p(\mu), L) \leq \exp(C'_{A,B}n^{5/6}p^{1/6}\log^{1/3}(1+n/p)).$$

This will follow from Part 3 of The Program together with the following:

Theorem 9.2. *Let μ be a h -pure log-concave measure (with constants (A, B)) and let $p \in [1, n]$. Then:*

$$N\left(Z_p(\mu), t\sqrt{\frac{p}{n}}Z_n(\mu)\right) \leq \exp\left(C_{A,B}n\left(\frac{\log(e+t)}{t}\right)^{\frac{1}{2}}\right) \quad \forall t > 0.$$

Proof. Since the statement is invariant under linear transformations, we may assume that μ is isotropic. By the triangle inequality for covering numbers, we have for all $s > 0$:

$$N\left(Z_p(\mu), t\sqrt{\frac{p}{n}}Z_n(\mu)\right) \leq N\left(Z_p(\mu), \frac{t}{s}\sqrt{p}B_2^n\right) N\left(\sqrt{n}B_2^n, sZ_n(\mu)\right).$$

The best known (essentially optimal) estimates on $N(Z_p(\mu), Cr\sqrt{p}B_2^n)$ for general isotropic origin-symmetric log-concave probability measures were obtained by Giannopoulos–Paouris–Valettas [16] (for the range $r \in [1, \sqrt{n/p}\log^2(1+n)]$) and Milman [40, Subsection 3.3] (for $r \geq \sqrt{n/p}\log^2(1+n)$). Note that since $Z_p(\mu) \subset CpB_2^n$, $N(Z_p(\mu), Cr\sqrt{p}B_2^n) = 1$ for $r \geq \sqrt{p}$. To avoid a lengthy multiple case analysis, we will simply use:

$$N(Z_p(\mu), r\sqrt{p}B_2^n) \leq \exp\left(C_2\frac{n}{r^2} + C_3\frac{\sqrt{n}\sqrt{p}}{r}\right) \quad \forall r > 0.$$

For $r \geq C_1 > 0$, this estimate was proved in [16], and after adjustment of constants it continues to hold for all $r > 0$ by Lemma 2.3. Note that since $I_1(\mu, B_2^n) \simeq \sqrt{n}$ for isotropic μ , this estimate is a particular case of the Generalized Dual Sudakov estimate for ellipsoids we obtained in Theorem 7.4 for general (not necessarily isotropic) origin-symmetric log-concave measures.

Next, by Theorem 8.8 on the regularity of dual covering numbers for pure measures, we have:

$$N(\sqrt{n}B_2^n, sZ_n(\mu)) \leq \exp\left(C_{A,B}n\left(\frac{\log(e+s)}{s}\right)^{\frac{2}{3}}\right) \quad \forall s > 0.$$

Setting $r = t/s$ above and combining both estimates, we obtain for all $s, t > 0$:

$$N(Z_p(\mu), t\sqrt{p/n}Z_n(\mu)) \leq \exp\left(C_2\frac{n}{t^2}s^2 + C_3\frac{\sqrt{n}\sqrt{p}}{t}s + C_{A,B}n\left(\frac{\log(e+s)}{s}\right)^{\frac{2}{3}}\right).$$

Optimizing on s , we set $s = t^{3/4}\log^{1/4}(e+t)$, yielding:

$$N(Z_p(\mu), t\sqrt{p/n}Z_n(\mu)) \leq \exp\left(C'_{A,B}n\left(\frac{\log(e+t)}{t}\right)^{1/2} + C_3\sqrt{n}\sqrt{p}\left(\frac{\log(e+t)}{t}\right)^{1/4}\right) \quad \forall t > 0.$$

However, note that since $Z_p(\mu) \subset Z_n(\mu)$, the left-hand-side is exactly 1 for all $t \geq \sqrt{n/p}$, and that in the non-trivial range $t \in (0, \sqrt{n/p}]$, the first term on the right-hand-side always dominates the second one. Adjusting constants, the assertion is thus established. \square

Proof of Theorem 9.1. By the triangle inequality for packing numbers, we have for every $s > 0$:

$$M\left(Z_p(\mu), t\sqrt{\frac{p}{n}}m_q(\mu, L)L\right) \leq N\left(Z_p(\mu), s\sqrt{\frac{p}{n}}Z_n(\mu)\right) M\left(Z_n(\mu), \frac{t}{s}m_q(\mu, L)L\right).$$

Invoking Theorems 9.2 and 4.1 to estimate the terms on the right-hand-side, we obtain:

$$M\left(Z_p(\mu), t\sqrt{\frac{p}{n}}m_q(\mu, L)L\right) \leq \exp\left(C_{A,B}n\left(\frac{\log(e+s)}{s}\right)^{1/2} + 1 + q + C'n\frac{s}{t}\right).$$

Optimizing on $s > 0$, we set $s := t^{2/3}\log^{1/3}(e+t)$. Adjusting constants, the assertion is established. \square

9.2 Part 2 for Regular Small-Diameter Bodies

The only property we will require for the ensuing proof is encapsulated in the following:

Definition (Regular Small-Diameter). *An origin-symmetric convex body $K \subset \mathbb{R}^n$ is called α -regular ($\alpha \in (0, 2]$) small-diameter (with constant $R \geq 1$) if there exists $T \in GL_n$ so that, denoting $K_0 = T(K)$:*

- (1) $K_0 \subset RB_2^n$.
- (2) $N(B_2^n, tK_0) \leq \exp(n/t^\alpha)$ for all $t > 0$.

9.2.1 Examples of Regular Small-Diameter Bodies

Proposition 9.3. *Assume that $K \subset \mathbb{R}^n$ is a origin-symmetric convex body so that λ_K , the uniform measure on K , is h -pure (with constants (A, B)). Assume in addition that $K \subset D\sqrt{n}Z_2(\lambda_K)$. Then K is α -regular small-diameter with constant $R_{A,B,D,\alpha}$ for all $\alpha \in (0, 2/3)$.*

In particular, the following families are $\frac{1}{2}$ -regular small-diameter:

- (1) *Sub-Gaussian (Ψ_2) Convex-Bodies are $\frac{1}{2}$ -regular small-diameter with constant $R \leq C$.*
- (2) *Unconditional Convex Bodies K satisfying $K \subset D\sqrt{n}Z_2(\lambda_K)$ are $\frac{1}{2}$ -regular small-diameter with constant $R \leq CD$.*

Proof. We may assume that λ_K is isotropic, so that $Z_2(\lambda_K) = B_2^n$. Since $Z_n(\lambda_K) \simeq K$ by origin-symmetry, if we define $K_1 = K/\sqrt{n}$, Theorem 8.8 ensures that:

$$N(B_2^n, tK_1) \leq \exp(C_{A,B,\alpha}n/t^\alpha) \quad \forall t > 0,$$

for any $\alpha \in (0, 2/3)$, while we are given that $K_1 \subset DB_2^n$. It follows that K is α -regular with constant $R = C_{A,B,\alpha}^{1/\alpha} D$.

In particular, if K is sub-Gaussian, Proposition 8.5 ensures that λ_K is 1-pure, while it is also well-known (e.g. [46]) that $K \simeq Z_n(\lambda_K) \subset D\sqrt{n}Z_2(\lambda_K)$ where D is the Ψ_2 constant of K , which is assumed to be bounded by a universal constant. In addition, Proposition 8.5 ensures that λ_K is 1-pure if K is unconditional, and so if in addition $K \subset D\sqrt{n}Z_2(\lambda_K)$, then it is $\frac{1}{2}$ -regular with constant $R = CD$. \square

To describe another important class of regular small-diameter bodies, recall that the (Gaussian) type-2 constant of a normed space $(X, \|\cdot\|)$ over \mathbb{R} , denoted $T_2(X)$, is the minimal $T > 0$ for which:

$$\left(\mathbb{E} \left\| \sum_{i=1}^m G_i x_i \right\|^2 \right)^{\frac{1}{2}} \leq T \left(\sum_{i=1}^m \|x_i\|^2 \right)^{\frac{1}{2}}$$

for any $m \geq 1$ and any $x_1, \dots, x_m \in X$, where G_1, \dots, G_m denote independent real-valued standard Gaussian random variables. We will often identify between a normed space and its unit-ball, and given an origin-symmetric convex body $K \subset \mathbb{R}^n$, refer to the type-2 constant $T_2(X_K)$ of the normed space X_K whose unit-ball is K . We will not distinguish between the Gaussian and the Rademacher type-2 constants, since it is well known that the former constant is always majorized by the latter one (e.g. [42]), and all our results will involve upper bounds in terms of the Gaussian type-2 constant.

Note that a Hilbert-space has type-2 constant exactly 1. It is also well-known (e.g. [42]) that subspaces of L_p for $p \geq 2$ have type-2 constant of the order of \sqrt{p} . In a finite dimensional setting, it is clear by John's theorem that $T_2(X_K) \leq \sqrt{n}$ for all origin-symmetric $K \subset \mathbb{R}^n$. Since ℓ_∞^n is isomorphic to a subspace of $L_{\log n}$, it similarly follows that $T_2(\ell_\infty^n) \leq C\sqrt{\log n}$, and in fact this is the correct order.

Proposition 9.4. *Every origin-symmetric convex body $K \subset \mathbb{R}^n$ is 2-regular small-diameter with constant $CT_2(X_K)$, for some universal constant $C \geq 1$.*

Proof. It was shown by W. J. Davis, V. Milman and N. Tomczak-Jaegermann [12] using operator-theoretic notation, and in [39, Corollary 3.5] using a geometric argument, that when B_2^n is the minimal volume ellipsoid containing K (the Lowner position), then:

$$M(K) \leq M_2(K) := \left(\int_{S^{n-1}} \|\theta\|_K^2 d\sigma(\theta) \right)^{1/2} \leq T_2(X_K).$$

Setting $K_0 := RK$ with $R = CT_2(X_K)$ for an appropriate constant $C \geq 1$, we have that $M(K_0) \leq \frac{1}{C}$, and hence by the Dual Sudakov Minorant:

$$N(B_2^n, tK_0) \leq \exp(n/t^2) \quad \forall t > 0.$$

Since $K_0 \subset RB_2^n$, the assertion is established. \square

Remark 9.5. Applying Proposition 9.3, we may conclude that the unit-balls B_q^n of ℓ_q^n , which for all $q \in [2, \infty]$ are both Ψ_2 and small-diameter unconditional, are $1/2$ -regular with uniformly bounded universal constant $C > 0$. Note that by the previous remarks, we cannot get a uniform estimate for the type-2 constant of ℓ_q^n in the range $q \in [2, \infty]$, and in particular ℓ_∞^n is not 2-regular small-diameter with dimension-independent constant. However, using precise covering estimates due to C. Schutt [54], we know that in fact B_∞^n is $(2 - \epsilon)$ -regular small-diameter with constant $C_\epsilon > 0$ for all $\epsilon > 0$, and the same applies to all B_q^n for $q \in [2, \infty]$. For simplicity, as this is not crucial for any of our ensuing estimates, we will only use below that they are all 1-regular small-diameter with a uniformly bounded universal constant $C > 0$ for all $q \in [2, \infty]$.

9.2.2 Weak Generalized Regular Dual Sudakov

Theorem 9.6. *Let μ denote an origin-symmetric log-concave probability measure on \mathbb{R}^n , and let $K \subset \mathbb{R}^n$ denote an α -regular small-diameter convex body (with constants $\alpha \in (0, 2]$ and $R \geq 1$). Then for any $p \in [1, n]$:*

$$N(Z_p(\mu), tm_1(\mu, K)K) \leq \exp\left(C(n(R/t)^\alpha)^{\frac{2}{2+\alpha}} p^{\frac{\alpha}{2+\alpha}} + C(n(R/t)^\alpha)^{\frac{1}{1+\alpha}} p^{\frac{\alpha}{1+\alpha}}\right) \quad \forall t > 0. \quad (9.1)$$

In particular:

$$N(Z_p(\mu), T_2(X_K)m_1(\mu, K)K) \leq \exp(C'\sqrt{n}\sqrt{p}),$$

and for all $q \in [2, \infty]$:

$$N(Z_p(\mu), tm_1(\mu, B_q^n)B_q^n) \leq \exp\left(C''\frac{p^{1/3}n^{2/3}}{t^{2/3}} + C''\frac{\sqrt{p}\sqrt{n}}{\sqrt{t}}\right) \quad \forall t > 0.$$

Proof. Since the statement is clearly invariant under applying (non-degenerate) linear transformations on both μ and K , we may and will assume that $K \subset RB_2^n$ and:

$$N(B_2^n, tK) \leq \exp(n/t^\alpha) \quad \forall t > 0.$$

The small-diameter property ensures that:

$$I_1(\mu) = \int |x| d\mu(x) \leq R \int \|x\|_K d\mu(x) = RI_1(\mu, K).$$

Together with the Generalized Regular Dual Sudakov estimate for ellipsoids (Theorem 7.4), we obtain for any $s > 0$:

$$\begin{aligned} N(Z_p(\mu), tI_1(\mu, K)K) &\leq N(Z_p(\mu), sI_1(\mu)B_2^n)N(I_1(\mu)B_2^n, (t/s)I_1(\mu, K)K) \\ &\leq N(Z_p(\mu), sI_1(\mu)B_2^n)N(B_2^n, t/(Rs)K) \\ &\leq \exp\left(C\left(\frac{p}{s^2} + \frac{p}{s}\right) + n\frac{R^\alpha s^\alpha}{t^\alpha}\right). \end{aligned}$$

Optimizing on $s > 0$, we set $s = C' \max(s_{1+\alpha}, s_{2+\alpha})$ where $s_\beta := \left(\frac{p}{n} \left(\frac{t}{R}\right)^\alpha\right)^{1/\beta}$. Recalling that $I_1(\mu, K) \simeq m_1(\mu, K)$ by Lemma 2.1, we obtain the first assertion. Applying the first part with $\alpha = 2$, $t = C'' T_2(X_K)$ and invoking Proposition 9.4, the second assertion follows after an adjustment of constants. The last assertion follows in view of Remark 9.5. \square

As a corollary, since we always have:

$$m_1(\mu, Z_n(\mu)) \simeq I_1(\mu, Z_n(\mu)) \leq C,$$

by Lemma 4.7, and in addition $N(Z_p(\mu), tZ_n(\mu)) = 1$ for all $p \in [1, n]$ and $t \geq 1$, so that only the first term in the right-hand-side of (9.1) is relevant, we obtain:

Corollary 9.7. *Let μ denote an origin-symmetric log-concave probability measure on \mathbb{R}^n , so that $Z_n(\mu)$ has α -regular small-diameter (with constants $\alpha \in (0, 2]$ and $R \geq 1$). Then for $p \in [1, n]$:*

$$N\left(Z_p(\mu), t\sqrt{\frac{p}{n}}Z_n(\mu)\right) \leq \exp\left(Cn\frac{R^{\frac{2\alpha}{2+\alpha}}}{t^{\frac{2\alpha}{2+\alpha}}}\right) \quad \forall t > 0.$$

In particular:

$$N\left(Z_p(\mu), t\sqrt{\frac{p}{n}}Z_n(\mu)\right) \leq \exp\left(Cn\frac{T_2(X_{Z_n(\mu)})}{t}\right) \quad \forall t > 0.$$

In analogy with the previous subsection, we deduce:

Theorem 9.8. *Let μ denote an origin-symmetric log-concave probability measure on \mathbb{R}^n , so that $Z_n(\mu)$ has α -regular small-diameter (with constants $\alpha \in (0, 2]$ and $R \geq 1$). Then for $p \in [1, n]$, $q > 0$ and star-body $L \subset \mathbb{R}^n$:*

$$M\left(Z_p(\mu), t\sqrt{\frac{p}{n}}m_q(\mu, L)L\right) \leq \exp\left(1 + q + Cn\left(\frac{R}{t}\right)^{\frac{2\alpha}{3\alpha+2}}\right) \quad \forall t > 0.$$

In particular:

$$M(Z_p(\mu), m_p(\mu, L)L) \leq \exp\left(C'\sqrt{T_2(X_{Z_n(\mu)})}\sqrt{n}\sqrt{p}\right).$$

Proof. By the triangle inequality for packing numbers, we have for every $s > 0$:

$$M\left(Z_p(\mu), t\sqrt{\frac{p}{n}}m_q(\mu, L)L\right) \leq N\left(Z_p(\mu), s\sqrt{\frac{p}{n}}Z_n(\mu)\right) M\left(Z_n(\mu), \frac{t}{s}m_q(\mu, L)L\right).$$

Invoking Corollary 9.7 and Theorem 4.1 to estimate the terms on the right-hand-side, we obtain:

$$M \left(Z_p(\mu), t \sqrt{\frac{p}{n}} m_q(\mu, L) L \right) \leq \exp \left(C n \frac{R^{\frac{2\alpha}{2+\alpha}}}{s^{\frac{2\alpha}{2+\alpha}}} + 1 + q + C' n \frac{s}{t} \right).$$

Optimizing on $s > 0$, we set $s := t^{\frac{\alpha+2}{3\alpha+2}} R^{\frac{2\alpha}{3\alpha+2}}$, establishing the assertion after adjustment of constants. The last part follows by Proposition 9.4. \square

10 Part 1 - Combinatorial Dimension Reduction: Cube

In this section, we establish Part 1 of The (full) Program for the case that $K = B_\infty^n$, the n -dimensional cube, albeit with $D = C \log(e+n)$ and $R = C \log \log(e+n)$. Contrary to the linear “One-Sided Johnson–Lindenstrauss” approach that worked well for $K = B_2^n$, we employ a non-linear combinatorial dimension reduction, based on the fundamental work of M. Rudelson and R. Vershynin [52] on the combinatorial dimension, extending the work of Mendelson and Vershynin from [38].

10.1 Part 1 via Cell Content and Combinatorial Dimension

Denote by G_{crd} the collection of all 2^n coordinate subspaces of \mathbb{R}^n (of arbitrary dimension $m = 0, 1, \dots, n$). Given a convex body $K \subset \mathbb{R}^n$, its cell content $\Sigma(K)$ is defined as:

$$\Sigma(K) := \sum_{F \in G_{\text{crd}}} \text{number of integer cells contained in } P_F K,$$

where an integer cell is defined as a unit-cube with integer coordinates, i.e. $x + [0, 1]^m$ with $x \in \mathbb{Z}^m$. When $F = \{0\}$, the number of integer cells contained in $P_F K$ is defined to be 1. The combinatorial dimension $v(K)$ is defined to be:

$$v(K) := \max \{ \dim(F) ; F \in G_{\text{crd}} \text{ and } P_F K \text{ contains at least one integer cell} \}.$$

Recall that $B_\infty^n := [-1, 1]^n$. The combinatorial information we will require is summarized in the following theorem, which is a particular case of [52, Theorem 4.2]:

Theorem 10.1 (Rudelson–Vershynin). *Let $K \subset \mathbb{R}^n$ denote a convex body so that $N(K, B_\infty^n) \geq \exp(an)$, $a > 0$. Then for all $\epsilon > 0$:*

$$N(K, B_\infty^n) \leq \Sigma \left(\frac{C}{\epsilon} K \right)^{M_\epsilon}, \quad M_\epsilon := 4 \log^\epsilon(e + 1/a).$$

We will also require an additional standard combinatorial lemma (see [52, Lemma 4.6]), which may be seen as an integer-valued extension of the Sauer–Shelah lemma:

Lemma 10.2. *If $K \subset aB_\infty^n$ then:*

$$\Sigma(K) \leq \left(\frac{C a n}{v(K)} \right)^{v(K)}.$$

We can now state:

Theorem 10.3 (Part 1 for $K = B_\infty^n$ with logarithmic factors). *Let μ be an origin-symmetric log-concave measure on \mathbb{R}^n , let $p \in [1, n]$ and $t \geq 1/n$. Set:*

$$D := C_1 \log(e + n), \quad R := C_2 \log \log(e + n),$$

for appropriate universal constants $C_1, C_2 \geq 1$. Assume that $M(Z_p(\mu), tR B_\infty^n) = e^k$ with $\mu(B_\infty^n) \geq \frac{1}{e}$ and $1 \leq k \leq n$. Then there exists $F \in G_{\text{crd}}$ of $\dim(F) = m \in [k/D, k]$, so that:

- (1) $M(P_F Z_p(\mu), tP_F B_\infty^n) \geq e^m$ (“**Partial Separation Dimension Reduction**”).
- (2) $\pi_F \mu(P_F B_\infty^n) \geq \mu(B_\infty^n) \geq \frac{1}{e}$ (“ **$P_F B_\infty^n$ is sufficiently massive**”).

Proof. We know that:

$$N\left(\frac{1}{tR} Z_p(\mu), B_\infty^n\right) \geq M(Z_p(\mu), tR B_\infty^n) = e^k,$$

and so by Theorem 10.1, we have for any $\epsilon > 0$:

$$k \leq 4 \log^\epsilon \left(e + \frac{n}{k} \right) \log \Sigma \left(\frac{C}{tR\epsilon} Z_p(\mu) \right). \quad (10.1)$$

By Lemma 2.2 and 2.1, $\mu(B_\infty^n) \geq 1/e$ implies that:

$$Z_p(\mu) \subset I_p(\mu, B_\infty^n) B_\infty^n \subset C' p m_1(\mu, B_\infty^n) B_\infty^n \subset C' p B_\infty^n,$$

and so $\frac{C}{tR\epsilon} Z_p(\mu) \subset \frac{C'' p}{tR\epsilon} B_\infty^n$. Applying Lemma 10.2, we deduce that:

$$\log \Sigma \left(\frac{C}{tR\epsilon} Z_p(\mu) \right) \leq m_\epsilon \log \left(\frac{C_3 p n}{tR\epsilon m_\epsilon} \right), \quad m_\epsilon := v \left(\frac{C}{tR\epsilon} Z_p(\mu) \right). \quad (10.2)$$

Setting $\epsilon = 1/\log \log(e + n)$ and $C_2 = 8C$, we ensure by (10.1) and (10.2) that $m := v(\frac{1}{8t} Z_p(\mu))$ satisfies:

$$k \leq 4em \log \left(\frac{C_3 p n}{tC_2 m} \right).$$

Since $p \in [1, n]$ and $t \geq 1/n$, by appropriately selecting C_1 we may ensure that:

$$m \geq k/D.$$

This means that there exists $F \in G_{\text{crd}}$ of $\dim(F) = m \geq k/D$ so that $\frac{1}{8t}P_F Z_p(\mu)$ contains an integer cell. In particular:

$$M(P_F Z_p(\mu), tP_F B_\infty^n) = M\left(\frac{1}{8t}P_F Z_p(\mu), \frac{1}{8}P_F B_\infty^n\right) \geq M\left(\frac{1}{2}P_F B_\infty^n, \frac{1}{8}P_F B_\infty^n\right) \geq e^m.$$

Of course, by decreasing m if necessary, we may also always ensure that $m \leq k$. This concludes the proof of the first assertion. The second assertion is obvious since $\pi_F \mu(P_F B_\infty^n) = \mu(P_F^{-1} P_F B_\infty^n) \geq \mu(B_\infty^n)$. \square

10.2 Running The Program For Cubes

Running The (full) Program, we finally obtain:

Theorem 10.4 (Generalized Regular Dual Sudakov For Cubes with Logarithmic Terms). *For any origin-symmetric log-concave measure μ on \mathbb{R}^n , we have:*

$$M(Z_p(\mu), tC \log \log(e+n)m_1(\mu, B_\infty^n)B_\infty^n) \leq \exp\left(C \log(e+n) \left(\frac{p}{t^2} + \frac{p}{t}\right)\right) \quad \forall p \geq 1 \quad \forall t > 0.$$

Proof. Since the expression on the left-hand-side is invariant under scaling of μ , we may assume that $m_1(\mu, B_\infty^n) = 1$.

Given $p \geq 1$ and $t \geq 1/n$, we run The Program for $K = B_\infty^n$, with $\mathcal{L}_m = \{B_\infty^m\}$. Theorem 10.3 applied with $q = 1$ verifies Part 1 of The Program with $D = C_1 \log(e+n)$, $R = C_2 \log \log(e+n)$, $A = 1$, $B = 1$ and $q_m = 1$. Part 2 of The Program regarding Weak Generalized Regular Dual Sudakov, with parameters $q_m = 1$ and $\varphi_t(x) = C' \max(x^{1/3}/t^{2/3}, \sqrt{x}/\sqrt{t})$, is established in Theorem 9.6 (recalling (2.2)). Since $\varphi_t^{-1}(y) \simeq \min(y^3 t^2, y^2 t)$, we have $\varphi_t^{-1}(1/(4A)) \simeq \min(t, t^2)$, and Theorem 5.2 yields the asserted estimate in the range $t \geq 1/n$. The estimate remains valid after adjustment of constants (and in fact can be significantly improved) in the remaining non-interesting range $t \in (0, 1/n)$ by Lemma 2.3 and (2.2). This concludes the proof. \square

Since every origin-symmetric polytope K in \mathbb{R}^n with $2N$ facets is the unit-ball of an n -dimensional subspace E of ℓ_∞^N (in an appropriate basis), by applying Theorem 10.4 in \mathbb{R}^N to log-concave measures μ_k which weakly converge to a given log-concave measure μ on E , we immediately obtain:

Corollary 10.5 (Generalized Regular Dual Sudakov For Polytopes with Few Facets and Logarithmic Terms). *For any origin-symmetric log-concave measure μ on \mathbb{R}^n , and any origin-symmetric polytope K with $2N$ facets, we have:*

$$M(Z_p(\mu), tC \log \log(e+N)m_1(\mu, K)K) \leq \exp\left(C \log(e+N) \left(\frac{p}{t^2} + \frac{p}{t}\right)\right) \quad \forall p \geq 1 \quad \forall t > 0.$$

11 Concluding Remarks

11.1 Generalized Regular Sudakov Minoration: Ellipsoids

Recall that the following estimate was established in Theorem 7.4:

$$M(Z_p(\mu), tm_1(\mu, \mathcal{E})\mathcal{E}) \leq \exp\left(C\left(\frac{p}{t^2} + \frac{p}{t}\right)\right) \quad \forall p \geq 1 \quad \forall t > 0, \quad (11.1)$$

for any origin-symmetric log-concave measure μ on \mathbb{R}^n and any (origin-symmetric) ellipsoid $\mathcal{E} \subset \mathbb{R}^n$. Let us expand on some of the comments regarding this estimate given in the Introduction.

In terms of sharpness, first recall that $Z_p(\mu) \subset pI_1(\mu, \mathcal{E})\mathcal{E}$, and so the left-hand-side is 1 for $t \geq C'p$ and the estimate is of the correct order (up to the value of $C > 0$) in that range. Moreover, our estimate yields the correct worst-case behavior in the range $t \in [1, C'p]$ as well. This is easily seen by degenerating μ to a one-dimensional (two-sided) exponential measure, in which case $Z_p(\mu)$ approximates an interval of length of order p , and $m_1(\mu, B_2^n)$ is of the order of 1.

We now claim that (11.1) yields the correct worst-case behavior for all $t \in [\sqrt{p/n}, 1]$. To see this, set μ to be the standard Gaussian measure γ_n so that $Z_p(\gamma_n) \simeq \sqrt{p}B_2^n$, and let \mathcal{E} denote the cylinder $\sqrt{k}B_2^k \times \mathbb{R}^{n-k}$ (which we think of as a degenerate ellipsoid, as it can obviously be approximated by proper ones). Clearly $m_1(\gamma_n, \mathcal{E}) \simeq I_2(\gamma_n, \mathcal{E}) = 1$, and we have by the volumetric estimate (2.4):

$$M(\sqrt{p}B_2^n, t_0\mathcal{E}) = M(\sqrt{p}B_2^n, t_0\sqrt{k}B_2^k) \geq e^k$$

for $t_0 = \frac{1}{2e}\sqrt{p/k}$. Consequently, we confirm that for an appropriate constant $c > 0$:

$$M(Z_p(\gamma_n), c\sqrt{p/k} m_1(\gamma_n, \mathcal{E})\mathcal{E}) \geq e^k,$$

and letting k range from $\lceil p \rceil$ to n , the sharpness of (11.1) for all $t \in [\sqrt{p/n}, 1]$ is established. When $t \in (0, \sqrt{p/n})$ the estimate is definitely loose, as simply seen by a volumetric argument (as explained in Lemma 2.3); however, we do not try to improve the estimate in this non-interesting range.

As already mentioned in the Introduction, when μ is isotropic and $\mathcal{E} = B_2^n$ (and hence $m_1(\mu, B_2^n) \simeq \sqrt{n}$), the estimate (11.1) was already obtained by Giannopoulos–Paouris–Valettas in [16] using a delicate refinement of Talagrand’s approach for proving the (dual) Sudakov Minoration. An alternative proof of this particular case was obtained in [17] using a very similar approach to the one we employ in this work, namely self-improving a weak Sudakov Minoration estimate via dimension-reduction. For $p \geq \sqrt{n} \log^2(1+n)$, improved covering estimates in the range $t \in [\log^2(1+n), p/\sqrt{n}]$ have been obtained for the isotropic case in [40, Subsection 3.3]. In the non-isotropic case, a general formula in terms of the eigenvalues of $\text{Cov}(\mu)$ may also be obtained by employing Sudakov Minoration and the estimate on $M^*(Z_p(\mu))$ from [40, Theorem 1.3].

11.2 Generalized Regular Sudakov Minoration: Cubes

We now turn to the estimate established in Theorem 10.4:

$$M(Z_p(\mu), tC \log \log(e+n)m_1(\mu, B_\infty^n)B_\infty^n) \leq \exp\left(C \log(e+n) \left(\frac{p}{t^2} + \frac{p}{t}\right)\right) \quad \forall p \geq 1 \quad \forall t > 0. \quad (11.2)$$

for any origin-symmetric log-concave measure μ on \mathbb{R}^n . Up to the logarithmic terms above, this estimate is again seen to be sharp in the range $t \geq 1$, exactly as in the preceding analysis for ellipsoids.

In the range $t \in [\sqrt{p/n}, 1]$, the estimate (11.2) remains sharp up to these and perhaps additional logarithmic terms in the dimension. To see this, set again μ to be the Gaussian measure γ_n , for which it is well-known that $m_1(\gamma_n, B_\infty^n) \simeq \sqrt{\log(1+n)}$. Applying the precise covering estimates of Schutt [54], we have:

$$M\left(Z_p(\gamma_n), C \frac{\sqrt{\log(1+n/k)}}{\sqrt{\log(1+n)}} \frac{\sqrt{p}}{\sqrt{k}} m_1(\gamma_n, B_\infty^n) B_\infty^n\right) \geq M\left(B_2^n, C' \frac{\sqrt{\log(1+n/k)}}{\sqrt{k}} B_\infty^n\right) \geq e^k,$$

for all $\log(1+n) \leq k \leq n$. This confirms the sharpness of (11.2) up to the logarithmic terms for all $t \in [\sqrt{p/n^\alpha}, 1]$ for any fixed $\alpha \in (0, 1)$. Curiously, it seems that this example exhibits an additional advantageous $\log(1+n)$ term in the range $t \in [\sqrt{p/n}, \sqrt{p/n^\alpha}]$, and we do not know whether this is indeed the worst-possible expected behaviour. As in the case of ellipsoids, the estimate is definitely loose in the range $t \in (0, \sqrt{p/n})$ by a simple volumetric estimate.

11.3 Completing The Program

The results we obtain in this work completely resolve Part 3 of The Program, and almost entirely Part 2 as well. For instance, if the initial log-concave probability measure μ is assumed 1-pure (e.g. super-Gaussian, sub-Gaussian or unconditional), then by Proposition 8.5, so will be all of its marginals $\nu \in \mathcal{M}_m$, for which we have a Weak Sudakov Minoration result by Theorem 9.1. In particular, up to the Slicing Problem, Part 2 is completely established.

Consequently, it is clear that the main remaining challenge in completing The Program lies in establishing Part 1 of The Program. This is a significant challenge even for some specific convex bodies K besides ellipsoids, such as for $K = B_1^n$. To carry out this Separation Dimension-Reduction step, it seems that we would need to employ other measures on the Grassmannian $G_{n,m}$ besides the uniform Haar measure, upon which most of the (Euclidean) Asymptotic Geometric Analysis theory is built. In our opinion, this is a fascinating challenge, which will be explored in a separate work.

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