On the existence of supergaussian directions on convex bodies

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Abstract

We study the question whether every centered convex body K of volume 1 in $\mathbb{R}^n$ has "supergaussian directions", which means $\theta \in S^{n-1}$ such that

$$\left| \left\{ x \in K : |\langle x, \theta \rangle| \geqslant t \int_K |\langle x, \theta \rangle| dx \right\} \right| \geqslant e^{-ct^2},$$

for all $1 \leqslant t \leqslant \sqrt{n}$, where $c > 0$ is an absolute constant. We verify that a "random" direction is indeed supergaussian for isotropic convex bodies that satisfy the hyperplane conjecture. On the other hand, we show that if, for all isotropic convex bodies, a random direction is supergaussian then the hyperplane conjecture follows.

1 Introduction

A well known conjecture in the theory of convex bodies is the hyperplane conjecture: there exists $c > 0$ such that for any $n \geqslant 1$ and any convex body K of volume 1 in $\mathbb{R}^n$ with centre of mass at the origin, there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^{\perp}| \geqslant c. \tag{1.1}$$

This question was posed explicitly in [9]. A classical reference is the paper of Milman and Pajor [45] (see also [22]). In the late 80's, J. Bourgain verified the conjecture in the case of "ψ_2" bodies. These are the convex bodies for which every direction is subgaussian: if μ is the uniform measure on K, a direction $\theta \in S^{n-1}$ is called subgaussian for μ with constant $r > 0$, if

$$\mu\left(\{x : |\langle x, \theta \rangle| \geqslant t m_\theta\}\right) \leqslant e^{-\frac{t^2}{r^2}}, \tag{1.2}$$

for all $1 \leqslant t \leqslant r\sqrt{n}$, where m_θ is the median of $|\langle \cdot, \theta \rangle|$ with respect to μ. The best possible value of $r > 0$ for which (1.2) holds true is the subgaussian constant $b_\mu(\theta)$ of μ in the direction of θ. Bourgain provided a lower bound in (1.1), depending on the

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parameter $\sup_\theta b_\mu(\theta)$ (see [10], [11], [16]). The main idea that the uniform measure μ on K may "resemble" the Gaussian measure became a central and fruitful line of research: a central limit theorem for convex bodies (proposed in [2], [14]) was proved (see [29], [30], [21], [18], [20]), various concentration results ([49], [50]) have appeared, and several strong conjectures have been proposed ([27], [6], [39], [19]) and have been verified in special cases ([54], [32], [43]).

In this setting, the following question was posed by V. Milman: is it true that every convex body has at least one "subgaussian" direction? An affirmative answer was given by Bobkov and Nazarov ([7], [8]) for the class of 1-unconditional convex bodies; the same is true for the class of zonoids (see [48]). For a general convex body, B. Klartag ([31]) established the existence of a "subgaussian" direction up to a logarithmic in the dimension factor (see also [23]). The best known estimate is $\inf_\theta b_\mu(\theta) = O(\sqrt{\log n})$ [24].

In this paper we consider the following "dual" question: is it possible to find at least one "supergaussian" direction on every convex body? We say that a direction $\theta \in S^{n-1}$ is supergaussian for μ with constant $r > 0$ if, for all $1 \leqslant t \leqslant \frac{\sqrt{n}}{r}$,

$$\mu\left(\{x : |\langle x, \theta\rangle| \geqslant t m_\theta\}\right) \geqslant e^{-r^2 t^2}. \tag{1.3}$$

The best possible value of $r > 0$ for which (1.3) holds true is called the supergaussian constant of μ in the direction of θ and will be denoted by $\overline{sg}_\mu(\theta)$. The question about supergaussian directions was considered in a recent paper of P. Pivovarov [52], who gave an affirmative answer (up to a logarithmic in the dimension factor) for the class of 1-unconditional bodies.

The first main result of this paper provides an affirmative answer to the question for all convex bodies that satisfy the hyperplane conjecture. In fact, we show that, for isotropic convex bodies which satisfy the hyperplane conjecture, a random direction is supergaussian. Here, the randomness is with respect to the rotation invariant probability measure σ on S^{n-1}. In order to give the precise formulation, we recall that an isotropic convex body K in $\mathbb{R}^n$ is a convex body of volume 1 which has centre of mass at the origin and satisfies

$$\int_K |\langle x, \theta\rangle|^2 dx = L_K^2 \tag{1.4}$$

for all $\theta \in S^{n-1}$ and some constant $L_K > 0$. The question if the isotropic constant L_K of K is uniformly bounded from above is an equivalent formulation of the hyperplane conjecture. This is based on the fact, proved by Hensley [25], that if K has volume 1 and centre of mass at the origin, then

$$\int_K |\langle x, \theta\rangle|^2 dx \simeq |K \cap \theta^\perp|^{-2}. \tag{1.5}$$

for all $\theta \in S^{n-1}$. We write $\mathbb{E}(\overline{sg}_\mu)$ for the expectation of $\overline{sg}_\mu(\theta)$ with respect to σ. Then, we have the following:

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$ with centre of mass at the origin and volume 1. Assume that $L_K = O(1)$. Then, there exists $\theta \in S^{n-1}$ such that*

$$\left| \left\{ x \in K : |\langle x, \theta \rangle| \geqslant t \int_K |\langle x, \theta \rangle| dx \right\} \right| \geqslant e^{-c_1 t^2} \tag{1.6}$$

for all $1 \leqslant t \leqslant \sqrt{n}$. More generally, for any isotropic convex body K in $\mathbb{R}^n$, one has

$$\mathbb{E}(\overline{sg}_K) \leqslant c_2 L_K, \tag{1.7}$$

where $c_1, c_2 > 0$ are absolute constants.

The best known upper bound for L_K is due to B. Klartag: $L_K \leqslant O(\sqrt[4]{n})$ (see [28]), although the hyperplane conjecture has been verified for a large variety of classes of convex bodies: 1-unconditional bodies [45], projection bodies and polars of projections bodies[26], [4], intersection bodies [36], unit balls of the Schatten classes [37], polytopes with few vertices [1], various random polytopes [34], [17]. Therefore, Theorem 1.1 shows that, at least in all these cases, there exists a supergaussian direction.

We give two different proofs of Theorem 1.1. The first one is based on inequalities for the volume of L_p–centroid bodies, proved by Lutwak, Zhang and Yang [42]. The second proof is based on the techniques developed in [49], [50]. Actually, the proof can be carried out in a setting much broader than the one of log-concave measures. This is somehow expected: a recent result of B. Klartag [33] shows that any non-degenerate n-dimensional measure has at least one direction which behaves in a "supergaussian way" on a non-trivial interval.

The second main result of this paper shows that if, for all isotropic convex bodies, a random direction is supergaussian, then the hyperplane conjecture follows:

Theorem 1.2. *There exists an absolute constant $c > 0$ such that for every $n \geqslant 1$,*

$$\sup_{K \text{ isotropic in } \mathbb{R}^n} L_K \leqslant c \sup_{K \text{ isotropic in } \mathbb{R}^n} [\mathbb{E}(\overline{sg}_K)]^3 \sqrt{\log [\mathbb{E}(\overline{sg}_K)]^2}. \tag{1.8}$$

The proof of Theorem 1.2 is based on the techniques developed in [16]. The paper is organized as follows: In §2 we introduce notation, terminology and some background material which is needed for the rest of the paper. In §3 we show some basic properties of the supergaussian constant. In §4 we give a first proof of Theorem 1.1. In §5 we give a second proof of Theorem 1.1 and the proof of Theorem 1.2.

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2 Preliminaries

2.1 Basic notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the

Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. We also write $\widetilde{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$, i.e. $\widetilde{A} := \frac{A}{|A|^{1/n}}$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

2.2 Probability measures. We denote by $\mathcal{P}_{[n]}$ the class of all probability measures on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_{[n]}$ is denoted by f_μ. A probability measure $\mu \in \mathcal{P}_{[n]}$ is called symmetric if f_μ is an even function on $\mathbb{R}^n$. We say that $\mu \in \mathcal{P}_{[n]}$ is centered if for all $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = 0. \tag{2.1}$$

A measure μ on $\mathbb{R}^n$ is called log-concave if for any Borel sets A, B and any $\lambda \in (0,1)$,

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}. \tag{2.2}$$

A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if $\log f$ is concave on its support $\{f > 0\}$. It is known that if μ is log-concave and if $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave (see [13]).

2.3 Convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if it has centre of mass at the origin: $\int_C \langle x, \theta \rangle\, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. The mean width of C is defined by

$$W(C) = \int_{S^{n-1}} h_C(\theta)\sigma(d\theta). \tag{2.3}$$

For each $-\infty < p < \infty$, $p \neq 0$, we define the p-mean width of C by

$$W_p(C) = \left(\int_{S^{n-1}} h_C^p(\theta)\sigma(d\theta) \right)^{1/p}. \tag{2.4}$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } x \in C\}. \tag{2.5}$$

Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_{\widetilde{K}}$ is the density of a log-concave measure. We refer to the books [53], [46] and

[51] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

2.4 L_q***–centroid bodies***. Let $\mu \in \mathcal{P}_{[n]}$. For every $q \geqslant 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(\mu)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^q f(x)\, dx \right)^{1/q}, \tag{2.6}$$

where f is the density of μ. If μ is log-concave then $h_{Z_q(\mu)}(\theta) < \infty$ for every $q \geqslant 1$ and every $\theta \in S^{n-1}$. We define the L_q-centroid body $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$.

L_q–centroid bodies were introduced, with a different normalization, in [41] (see also [42] where an L_q affine isoperimetric inequality was proved). Here we follow the normalization (and notation) that appeared in [49]. The original definition concerned the class of densities $\mathbf{1}_K$ where K is a convex body of volume 1. In this case, we also write $Z_q(K)$ instead of $Z_q(\mathbf{1}_K)$.

If K is a compact set in $\mathbb{R}^n$ and $|K| = 1$, it is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}(\{K, -K\})$. Note that if $T \in SL_n$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, if K is convex body, as a consequence of the Brunn–Minkowski inequality (see, for example, [49]), one can check that

$$Z_q(K) \subseteq \overline{c}_0\, q\, Z_2(K) \tag{2.7}$$

for every $q \geqslant 2$ and, more generally,

$$Z_q(K) \subseteq \overline{c}_0 \frac{q}{p} Z_p(K) \tag{2.8}$$

for all $1 \leqslant p < q$, where $\overline{c}_0 \geqslant 1$ is an absolute constant. Also, if K has its centre of mass at the origin, then

$$Z_q(K) \supseteq \overline{c}\, K \tag{2.9}$$

for all $q \geqslant n$, where $\overline{c} > 0$ is an absolute constant. For a proof of this fact and additional information on L_q–centroid bodies, we refer to [48] and [50].

2.5 Isotropic probability measures. Let μ be a centered measure in $\mathcal{P}_{[n]}$. We say that μ is isotropic if $Z_2(\mu) = B_2^n$. We say that a centered convex body K is isotropic if $Z_2(K)$ is a multiple of the Euclidean ball. We define the isotropic constant of K by

$$L_K := \left(\frac{|Z_2(K)|}{|B_2^n|} \right)^{1/n}. \tag{2.10}$$

So, K is isotropic if and only if $Z_2(K) = L_K B_2^n$. Note that K is isotropic if and only if $L_K^n \mathbf{1}_{\frac{K}{L_K}}$ is isotropic. We define the isotropic constant of $\mu \in \mathcal{P}_{[n]}$ by $L_\mu := f(0)^{\frac{1}{n}}$. We refer to [45], [22] and [50] for additional information on isotropic convex bodies.

2.6 The bodies $K_p(\mu)$. A natural way to pass from log-concave measures to convex bodies was introduced by K. Ball in [3]. Here, we will give the definition in a somewhat more general setting: Let $\mu \in \mathcal{P}_{[n]}$ and assume that $0 \in \mathrm{supp}(\mu)$. For every $p > 0$ we define a set $K_p(\mu)$ as follows:

$$K_p(\mu) := \left\{ x \in \mathbb{R}^n : p \int_0^\infty f_\mu(rx) r^{p-1} dr \geqslant f_\mu(0) \right\}. \tag{2.11}$$

It is clear that $K_p(\mu)$ is a star shaped body with gauge function

$$\|x\|_{K_p(\mu)} := \left(\frac{p}{f_\mu(0)} \int_0^\infty f_\mu(rx) r^{p-1} dr \right)^{-1/p}. \tag{2.12}$$

2.7 The parameter $k_*(C)$. Let C be a symmetric convex body in $\mathbb{R}^n$. Define $k_*(C)$ as the largest positive integer $k \leqslant n$ for which

$$\mu_{n,k}\left(F \in G_{n,k} : \frac{1}{2} W(C)(B_2^n \cap F) \subseteq P_F(C) \subseteq 2W(C)(B_2^n \cap F) \right) \geqslant \frac{n}{n+k}.$$

Thus, $k_*(C)$ is the maximal dimension k such that a "random" k-dimensional projection of C is 4-Euclidean.

The parameter $k_*(C)$ is completely determined by the global parameters $W(C)$ and $R(C)$: There exist absolute constants $c_1, c_2 > 0$ such that

$$c_1 n \frac{W(C)^2}{R(C)^2} \leqslant k_*(C) \leqslant c_2 n \frac{W(C)^2}{R(C)^2} \tag{2.13}$$

for every symmetric convex body C in $\mathbb{R}^n$. The lower bound appears in Milman's proof of Dvoretzky's theorem (see [44]) and the upper bound was proved in [47].

We will need the following result (see [40], [35], [38] for a proof):

Proposition 2.1. *Let C be a symmetric convex body in $\mathbb{R}^n$ and $p > 0$. Then,*

(i) $W_p(C) \simeq W(C)$ *for all* $p \leqslant k_*(C)$.

(ii) $W_p(C) \simeq \sqrt{p/n}\, R(C)$ *for all* $k_*(C) \leqslant p \leqslant n$.

(iii) $W_p(C) \simeq R(C)$ *for all* $p \geqslant n$.

(iv) $W_{-cp_0}(C) \simeq W_{p_0}(C)$, *where* $p_0 = k_*(C)$ *and* $c > 0$ *is an absolute constant.*

2.8 The parameters $q_*(\mu), q_{-c}(\mu)$. Let $\mu \in \mathcal{P}_{[n]}$ and $p > -n$, $p \neq 0$. We define the quantities $I_p(\mu)$ by

$$I_p(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^p d\mu(x) \right)^{1/p}. \tag{2.14}$$

As before, if K is a compact set of volume 1, we write $I_p(K)$ instead of $I_p(\mathbf{1}_K)$.

The following result is proved in [50]:

Proposition 2.2. *Let μ be a* log*-concave centered measure on $\mathbb{R}^n$ and let $1 \leqslant p \leqslant n/2$. Then,*

$$I_{-p}(\mu) \simeq \sqrt{\frac{n}{p}} W_{-p}(Z_p(\mu)) \tag{2.15}$$

and

$$I_p(\mu) \simeq \sqrt{\frac{n}{p}} W_p(Z_p(\mu)). \tag{2.16}$$

It is known (see [50, Proposition 4.7]) that for any convex body K of volume 1 and for all $p > -n/2$, $p \neq 0$,

$$I_p(K) \geqslant I_p(\widetilde{B_2^n}) \simeq \sqrt{n}. \tag{2.17}$$

Then, Proposition 2.2 implies that

$$W(Z_p(K)) \geqslant W_{-p}(Z_p(K)) \geqslant c\sqrt{p}, \tag{2.18}$$

for $1 \leqslant p \leqslant n/2$, where $c > 0$ is an absolute constant.

Let μ be a centered log-concave measure in $\mathbb{R}^n$ and let $\delta \geqslant 1$. The parameters

$$q_{-c}(\mu, \delta) := \sup \left\{ 0 < r \leqslant \frac{n}{2} : I_{-r}(\mu) \geqslant \frac{1}{\delta} I_2(\mu) \right\} \tag{2.19}$$

and

$$q_*(\mu, \delta) := \sup \left\{ 0 < r \leqslant n : k_*(Z_r(\mu)) \geqslant \frac{r}{\delta} \right\} \tag{2.20}$$

played a crucial role in [49], [16]. In the sequel, we collect some basic facts for these two parameters.

Lemma 2.3. *Let C be a symmetric convex body in $\mathbb{R}^n$ and assume that $p, \delta \geqslant 1$ satisfy $W(C) \geqslant \frac{1}{\delta} W_p(C)$. Then, $k_*(C) \geqslant c\frac{p}{\delta^2}$.*

Proof. We may assume that $p \geqslant k_*(C)$ (otherwise, we have nothing to prove). Then,

$$\delta W(C) \geqslant W_p(C) \simeq \sqrt{p/n} R(C). \tag{2.21}$$

But this implies that

$$p \leqslant c\delta^2 n \left(\frac{W(C)}{R(C)} \right)^2. \tag{2.22}$$

This completes the proof. $\square$

We say that the centered convex body K of volume 1 has small diameter (with constant $\alpha \geqslant 1$) if

$$R(K) \leqslant \alpha\, I_2(K). \tag{2.23}$$

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. We define $V = K \cap 4I_2(K)B_2^n$ and $\bar{K} = \tilde{V}$. It is easy to check that $\bar{K}$ is a body of small diameter and $I_2(K) \simeq I_2(\bar{K})$. Moreover, we have the following.

Lemma 2.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant q \leqslant n/2$. Then,*

$$I_{-q}(K) \leqslant 2I_{-q}(\bar{K}). \tag{2.24}$$

Proof. Recall that $\bar{K} := |V|^{-\frac{1}{n}}V$, where $V := K \cap 4I_2(K)B_2^n$. Note that $|V| \geqslant \frac{15}{16}$. So, we have that

$$I_{-q}^{-q}(\bar{K}) = \int_{\bar{K}} \frac{1}{\|x\|_2^q}dx = \frac{|V|^{\frac{q}{n}}}{|V|}\int_V \frac{1}{\|x\|_2^q}dx \leqslant \frac{1}{|V|^{1-\frac{q}{n}}} I_{-q}^{-q}(K). \tag{2.25}$$

This implies that $I_{-q}(K) \leqslant 2I_{-q}(\bar{K})$. □

Proposition 2.5. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$ and let $\delta \geqslant 1$. Then,*

(i) $q_*(\bar{K}, c_1\delta^2) \geqslant q_{-c}(\bar{K}, \delta) \geqslant q_{-c}(K, c_2\delta)$.

(ii) $q_{-c}(K, c\delta) \geqslant \frac{q_*(K,\delta)}{c\delta}$.

(iii) *If* $p \geqslant q_*(K, \delta)$ *then* $I_2(K) \leqslant cR(Z_p(K)) \leqslant R(K)$.

(iv) *If* $p_2 \geqslant p_1 \geqslant q_*(\bar{K}, \delta)$ *then* $k_*(Z_{p_2}(\bar{K})) \geqslant ck_*(Z_{p_1}(\bar{K}))$.

(v) *If* $\delta_1, \delta_2 \geqslant 1$ *and* $q_*(\bar{K}, \delta_1\delta_2) \leqslant cn$, *then* $q_*(\bar{K}, \delta_1\delta_2) \geqslant \delta_2 q_*(\bar{K}, \delta_1)$.

(vi) *If K is isotropic, then* $q_*(K, \delta) \geqslant c\sqrt{n}$.

(vii) $q_{-c}(K, cL(K)) \geqslant n/2$ *and* $q_*(\bar{K}, cL(K)^2) \geqslant n$,

where $L(K) := \frac{I_2(K)}{\sqrt{n}}$.

Proof. We first note that since $\bar{K}$ has small diameter we have

$$W_p(Z_p(\bar{K})) \simeq \sqrt{\frac{p}{n}} I_p(\bar{K}) \simeq \sqrt{p}L(K) \tag{2.26}$$

for all $1 \leqslant p \leqslant n$.

(i) Let $q := q_{-c}(\bar{K}, \delta)$. Then,

$$\begin{aligned} W(Z_q(\bar{K})) &\geqslant W_{-q}(Z_q(\bar{K})) \simeq \sqrt{q/n}I_{-q}(\bar{K}) \\ &\geqslant \sqrt{q/n}\frac{\sqrt{n}L(\bar{K})}{\delta} \simeq \frac{\sqrt{q}L(\bar{K})}{\delta} \\ &\simeq \frac{W_q(Z_q(\bar{K}))}{\delta}. \end{aligned}$$

Then, Lemma 2.3 shows that $k_*(Z_q(\bar{K})) \geqslant c\frac{q}{\delta^2}$, and this implies that $q_*(\bar{K}, c_1\delta^2) \geqslant q = q_{-c}(\bar{K}, \delta)$.

Moreover, by Lemma 2.4 we have that $q_{-c}(\bar{K}, \delta) \geqslant q_{-c}(K, c_2\delta)$.

(ii) Let $r := q_*(K, \delta)$ (we may assume that $r \geqslant 2\delta$). Then, $k_*(Z_r(K)) \geqslant r/\delta$. So,

$$\begin{aligned} \sqrt{\frac{r}{\delta n}}\sqrt{n}L(K) &\leqslant \sqrt{\frac{r}{\delta n}} I_{\frac{r}{\delta}}(K) \simeq W_{\frac{r}{\delta}}(Z_{\frac{r}{\delta}}(K)) \\ &\leqslant W_{\frac{r}{\delta}}(Z_r(K)) \leqslant cW_{-c\frac{r}{\delta}}(Z_r(K)) \\ &\leqslant c\delta W_{-c\frac{r}{\delta}}(Z_{c\frac{r}{\delta}}(K)) \leqslant c\delta\sqrt{\frac{r}{\delta n}} I_{-c\frac{r}{\delta}}(K). \end{aligned}$$

This implies that $q_{-c}(K, c\delta) \geqslant \frac{q_*(K,\delta)}{c\delta}$ as claimed.

(iii) Since

$$\sqrt{\frac{p}{\delta n}}\sqrt{n}L(K) \leqslant \sqrt{\frac{p}{\delta n}} I_{\frac{p}{\delta}}(K) \simeq W_{\frac{p}{\delta}}(Z_{\frac{p}{\delta}}(K)) \simeq \sqrt{\frac{p}{\delta n}} R(Z_p(K)), \tag{2.27}$$

we get (iii).

(iv) From (iii) we see that, for $p \geqslant q_*(\bar{K}, \delta)$, we have $R(Z_p(\bar{K})) \simeq \sqrt{n}L(K)$. It follows that, for $p_2 \geqslant p_1 \geqslant q_*(\bar{K}, \delta)$,

$$k_*(Z_{p_2}(\bar{K})) \simeq \frac{W(Z_{p_2}(\bar{K}))^2}{L(K)^2} \geqslant c\frac{W(Z_{p_1}(\bar{K}))^2}{L(K)^2} \simeq k_*(Z_{p_1}(\bar{K})). \tag{2.28}$$

(v) Let $r_1 := q_*(\bar{K}, \delta_1)$. Then, $k_*(Z_{r_1}(\bar{K})) \geqslant \frac{r_1}{\delta_1}$. We may assume that, if $r_2 := q_*(\bar{K}, \delta_1\delta_2)$, then $k_*(Z_{r_2}(\bar{K})) = \frac{r_2}{\delta_1\delta_2}$. The result follows from (iv).

(vi) From (iii) we see that $\sqrt{n}L_K \leqslant cq_*(K, \delta)L_K$.

(vii) For $1 \leqslant p \leqslant \frac{n}{2}$ we have

$$I_{-p}(K) \geqslant I_{-p}(\widetilde{B_2^n}) \simeq \sqrt{n} \simeq \frac{I_2(K)}{L(K)}, \tag{2.29}$$

and hence, $q_{-c}(K, cL(K)) \geqslant n/2$. Moreover, using (iii), (2.13) and (2.18), we have that for $p \geqslant q_*(\bar{K}, \delta)$,

$$k_*(Z_p(\bar{K})) \geqslant \frac{W(Z_p(\bar{K}))^2}{L(K)^2} \geqslant \frac{cp}{L(K)^2}. \tag{2.30}$$

This completes the proof. □

Finally, we will need the main result of [16]:

Proposition 2.6. *There exists an absolute constant $c > 0$ such that for every $n \geqslant 1$ and $\delta \geqslant 1$,*

$$\sup_{K \text{ isotropic in } \mathbb{R}^n} L_K \leqslant \sup_{K \text{ isotropic in } \mathbb{R}^n} c\delta\sqrt{\frac{n}{q_{-c}(\bar{K}, \delta)}}\sqrt{\log c\frac{n}{q_{-c}(\bar{K}, \delta)}}. \tag{2.31}$$

3 Supergaussian directions

Let μ be a probability measure on $\mathbb{R}^n$, and let $\theta \in S^{n-1}$ and $1 \leqslant p \leqslant n$. We define the supergaussian constant $\overline{sg}_{\mu,p}(\theta)$ in the direction of θ at level p, as the least value of $r > 0$ for which

$$\mu\left(\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geqslant t m_\theta\}\right) \geqslant e^{-r^2 t^2} \tag{3.1}$$

for all $1 \leqslant t \leqslant \frac{\sqrt{p}}{r}$. We also write m_θ for the median of $|\langle \cdot, \theta\rangle|$ with respect to μ.

In the case $p = n$ we simply write $\overline{sg}_\mu(\theta)$; we call this value "supergaussian constant of θ". If C is a compact set of volume 1 we write $\overline{sg}_C(\theta)$ instead of $\overline{sg}_{\mathbf{1}_C}(\theta)$.

Let μ be a log-concave centered probability measure in $\mathbb{R}^n$ and let $1 \leqslant p \leqslant n$. We define a quantity $\bar{g}_{\mu,p}(\theta)$ as follows:

$$\bar{g}_{\mu,p}(\theta) := \inf\left\{\delta > 0 : h_{Z_q(\mu)}(\theta) \geqslant \frac{\sqrt{q}}{\delta} h_{Z_1(\mu)}(\theta) \text{ for all } 1 \leqslant q \leqslant p\right\}. \tag{3.2}$$

Again, we define $\bar{g}_\mu(\theta) = \bar{g}_{\mu,n}(\theta)$ and if K is a centered convex body of volume 1, we write $\bar{g}_K(\theta)$ instead of $\bar{g}_{\mathbf{1}_K}(\theta)$. We have the following:

Lemma 3.1. *Let μ be a* log-*concave centered probability measure on $\mathbb{R}^n$. For all $1 \leqslant p \leqslant n$,*

$$\overline{sg}_{\mu,p}(\theta) \simeq \bar{g}_{\mu,p}(\theta). \tag{3.3}$$

Proof. Let $1 \leqslant p \leqslant n$ and set $r_1 := \bar{g}_{\mu,p}(\theta)$ and $r_2 := \overline{sg}_{\mu,p}(\theta)$. For every $q \leqslant p$ we have

$$h_{Z_q(\mu)}(\theta) \geqslant \frac{\sqrt{q}}{r} h_{Z_1(K)}(\theta) \tag{3.4}$$

and

$$h_{Z_{2q}(\mu)}(\theta) \simeq h_{Z_q(\mu)}(\theta). \tag{3.5}$$

Using the Paley–Zygmund inequality we see that, for every $q \geqslant 1$,

$$\mu\left(\left\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geqslant \frac{1}{2} h_{Z_q(\mu)}(\theta)\right\}\right) \geqslant e^{-q}. \tag{3.6}$$

Under our assumptions, we have

$$\mu\left(\left\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geqslant \frac{\sqrt{q}}{2r_1} h_{Z_1(K)}(\theta)\right\}\right) \geqslant e^{-q} \tag{3.7}$$

for every $1 \leqslant q \leqslant p$.

We set $t := \frac{\sqrt{q}}{2r_1}$. Then, if $\frac{1}{\sqrt{2}r_1} \leqslant t \leqslant \frac{\sqrt{p}}{2r_1}$, we have that

$$\mu(\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geqslant t h_{Z_2(\mu)}(\theta)\}| \geqslant e^{-4r_1^2 t^2}. \tag{3.8}$$

This implies that $r_2 \leqslant cr_1$. On the other hand, for all $q \geqslant 2$,

$$\begin{aligned}
h^q_{Z_q(\mu)}(\theta) &= qh^q_{Z_2(\mu)}(\theta)\int_0^\infty s^{q-1}\mu(\{x\in\mathbb{R}^n : |\langle x,\theta\rangle| \geqslant sh_{Z_2(\mu)}(\theta)\})ds \\
&\geqslant qh^q_{Z_2(\mu)}(\theta)\int_0^{\frac{\sqrt{p}}{r_2}} s^{q-1}e^{-s^2r_2^2}ds \\
&\geqslant q\frac{h^q_{Z_2(\mu)}(\theta)}{r_2^q}\int_0^{\sqrt{p}} s^{q-1}e^{-s^2}ds \\
&= \left(\frac{h_{Z_2(\mu)}(\theta)}{r_2}\right)^q q\left(\int_0^\infty s^{q-1}e^{-s^2}ds - \int_{\sqrt{p}}^\infty s^{q-1}e^{-s^2}ds\right) \\
&\geqslant \left(c\frac{\sqrt{q}h_{Z_2(\mu)}(\theta)}{r_2}\right)^q,
\end{aligned}$$

taking into account the fact that, for every $\beta \geqslant \sqrt{\frac{q}{2}} \geqslant 1$,

$$2\int_\beta^\infty s^{q-1}e^{-s^2}ds \leqslant e^{-\beta^2+1}\beta^q. \tag{3.9}$$

Since $Z_2(\mu) \simeq Z_1(\mu)$, we conclude that $r_1 \leqslant cr_2$. □

Working with the quantities $\bar{g}_{\mu,p}(\theta)$ instead of $\overline{sg}_{\mu,p}(\theta)$ is much more convenient for us. Lemma 3.1 shows that, in the case of log-concave measures, these two quantities are equivalent up to an absolute constant.

We also define the following quantities:

$$\begin{aligned}
\mathbb{E}\bar{g}_{\mu,p} &:= \int_{S^{n-1}} \bar{g}_{\mu,p}(\theta)d\sigma(\theta) \\
a_{\mu,p} &:= \sup_{\theta\in S^{n-1}} \bar{g}_{\mu,p}(\theta) \\
\underline{g}_{\mu,p} &:= \inf_{\theta\in S^{n-1}} \bar{g}_{\mu,p}(\theta).
\end{aligned}$$

Note that the quantities $\underline{g}_{\mu,p}$ and $a_{\mu,p}$ are SL_n-invariant: for every $T \in SL_n$, we have

$$\underline{g}_{\mu\circ T,p} = \underline{g}_{\mu,p}\ ,\ a_{\mu\circ T,p} = a_{\mu,p}. \tag{3.10}$$

Indeed, one can easily check that if $T \in SL_n$ then $\bar{g}_{\mu\circ T}(\theta) = \bar{g}_\mu(\widetilde{T^*\theta})$, where $\widetilde{x} := \frac{x}{\|x\|_2}$.

Intuitively, the directions in which a convex body has "large subgaussian constant" are the ones in which K "resembles" a cone. In particular, it follows by a result of Berwald (see [5]) that for all symmetric convex bodies of volume 1 and for all $0 < p \leqslant q$, one has that

$$\frac{h_{Z_q(K)}(\theta)}{h_{Z_p(K)}(\theta)} \leqslant \frac{h_{Z_q(C_\theta)}(\theta)}{h_{Z_p(C_\theta)}(\theta)}, \tag{3.11}$$

where C_θ is a "double cone" in the direction of θ, i.e $C_\theta := \mathrm{conv}\{A, \theta, -\theta\}$ for some $A \subset \theta^\perp$, normalized to have volume 1.

On the other hand, the directions in which a convex body has "large super-gaussian constant" are the ones in which K "resembles" a "cylinder" Q_θ in the direction of θ, i.e. $Q_\theta := aA \times r\theta$ for some $A \subset \theta^\perp$, where $|A| = 1$ and $a, r > 0$ satisfy $ar = 1$. More precisely, we have the following:

Proposition 3.2. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ and let $\theta \in S^{n-1}$. Then,*

(i) For all $0 < p \leqslant q$,

$$\frac{h_{Z_q(K)}(\theta)}{h_{Z_p(K)}(\theta)} \geqslant \frac{h_{Z_q(Q_\theta)}(\theta)}{h_{Z_p(Q_\theta)}(\theta)} = \frac{(p+1)^{\frac{1}{p}}}{(q+1)^{\frac{1}{q}}}, \tag{3.12}$$

with equality if and only if $K = Q_\theta$.

(ii) For all $1 \leqslant p \leqslant n$,

$$\bar{g}_{K,p}(\theta) \leqslant \bar{g}_{Q_\theta,p}(\theta) \leqslant c\sqrt{p}, \tag{3.13}$$

where $c > 0$ is an absolute constant.

For the proof of Proposition 3.2 we will use the next lemma; a proof can be found in [12].

Lemma 3.3. *Let g be a non-negative, finite, not identically zero, integrable, strictly decreasing on its support, convex function on $[0, \infty)$. Then, the function*

$$q \mapsto \left(q(q+1) \int_0^\infty t^{q-1} \frac{g(t)}{g(0)} dt \right)^{1/q} \tag{3.14}$$

is strictly increasing in $(0, \infty)$, unless if $g(t) := a\mathbf{1}_{[0,\infty)}(1 - rt)$ for some $a, r > 0$.

Proof of Proposition 3.2. (i) Let $t \in [0, h_K(\theta)]$. We define a function g by

$$g(t) := g_K(t) := |\{x \in K : |\langle x, \theta \rangle| \geqslant t\}| = 2 \int_t^\infty f(s) ds, \tag{3.15}$$

where $f(s) := |\{x \in K : \langle x, \theta \rangle = s\}|$. By the Brunn–Minkowski inequality, f is log-concave and, since K is symmetric, f is decreasing and attains its maximum at 0. So, g is a strictly decreasing convex function on its support. Note that $g(0) = |K| = 1$. Moreover,

$$h^q_{Z_q(K)}(\theta) = q \int_0^\infty t^{q-1} g(t) dt. \tag{3.16}$$

Note that $g_{Q_\theta}(t) := a\mathbf{1}_{[0,\frac{1}{r}]}(1 - rt)$ and $h_{Z_q(Q_\theta)}(\theta) = (q+1)^{-\frac{1}{q}}$. So,

$$\frac{h_{Z_q(K)}(\theta)}{h_{Z_q(Q_\theta)}(\theta)} = \left(q(q+1) \int_0^\infty t^{q-1} g(t) dt \right)^{1/q} \tag{3.17}$$

and the result follows from Lemma 3.3. Assertion (ii) follows from the definition of $\bar{g}_{K,p}(\theta)$. $\square$

As an immediate consequence of Proposition 3.2 we get that, for all $\theta \in S^{n-1}$,

$$\bar{g}_K(\theta) \leqslant c\sqrt{n}. \tag{3.18}$$

Intuition suggests that it is not possible that all directions are "cylindrical". So, we expect that there exist directions with supergaussian constant much smaller than $\sqrt{n}$.

We conclude this section by pointing out the following immediate consequence of the definitions: For all $1 \leqslant q_1 \leqslant q_2 \leqslant n$ and $\theta \in S^{n-1}$,

$$\bar{g}_{\mu,q_1}(\theta) \leqslant \bar{g}_{\mu,q_2}(\theta) \leqslant \sqrt{\frac{q_2}{q_1}}\bar{g}_{\mu,q_1}(\theta). \tag{3.19}$$

4 Proof of Theorem 1.1

We will use the L_p versions of the Blaschke–Santaló inequality and of the Busemann–Petty inequality obtained by Lutwak–Zhang [41] and Lutwak, Yang and Zhang [42] respectively (see also [15] for a proof in the convex case).

Theorem 4.1. *Let K be a star-shaped, with respect to the origin, body of volume 1 in $\mathbb{R}^n$. Then,*

$$|Z_p^\circ(K)|^{1/n} \leqslant |Z_p^\circ(\widetilde{B_2^n})|^{1/n} \tag{4.1}$$

and

$$|Z_p(K)|^{1/n} \geqslant |Z_p(\widetilde{B_2^n})|^{1/n}, \tag{4.2}$$

with equality if and only if K is a centered ellipsoid of volume 1.

Using K. Ball's bodies $K_p(\mu)$ we can extend Theorem 4.1 to the case of probability measures $\mu \in \mathcal{P}_{[n]}$.

We will use the following lemma (see [45, p.76] for a proof).

Lemma 4.2. *Let f be a bounded measurable, non-negative function on $[0,\infty]$. Then, for every $0 < p \leqslant q$ we have*

$$\left(\frac{p}{\|f\|_\infty}\int_0^\infty t^{p-1}f(t)dt\right)^{1/p} \leqslant \left(\frac{q}{\|f\|_\infty}\int_0^\infty t^{q-1}f(t)dt\right)^{1/q}. \tag{4.3}$$

There is equality if and only if $f := \|f\|_\infty \mathbf{1}_{[0,a]}$ for some $a > 0$.

Proposition 4.3. *Let $\mu \in \mathcal{P}_{[n]}$ and let f_μ be its density. If $0 \in \mathrm{supp}(\mu)$ and $\|f_\mu\|_\infty \leqslant 1$, then*

$$|Z_p(\mu)| \geqslant |Z_p(\widetilde{B_2^n})| \quad \text{and} \quad |Z_p^\circ(\mu)| \leqslant |Z_p^\circ(\widetilde{B_2^n})|, \tag{4.4}$$

with equality if and only if $f_\mu := \mathbf{1}_{\mathcal{E}}$ for some centered ellipsoid $\mathcal{E}$ of volume 1.

Proof. Recall that for any $\mu \in \mathcal{P}_{[n]}$, the set $K_p(\mu)$ is star-shaped with respect to the origin and

$$\|x\|_{K_p(\mu)} := \left(\frac{p}{f_\mu(0)} \int_0^\infty f_\mu(rx) r^{p-1} dr \right)^{-1/p}. \tag{4.5}$$

Integration in polar coordinates gives

$$|K_n(\mu)| = \frac{1}{f_\mu(0)} \int_{\mathbb{R}^n} f_\mu(x) dx = \frac{1}{f_\mu(0)} \tag{4.6}$$

and

$$Z_p(\widetilde{K_{n+p}(\mu)}) |K_{n+p}(\mu)|^{\frac{1}{p}+\frac{1}{n}} f_\mu(0)^{1/p} = Z_p(\mu). \tag{4.7}$$

Lemma 4.2 implies that if $q \geqslant p > 1$, then

$$\|x\|_{K_q(\mu)} \leqslant \left(\frac{\|f_\mu\|_\infty}{f_\mu(0)} \right)^{\frac{1}{p}-\frac{1}{q}} \|x\|_{K_p(\mu)}. \tag{4.8}$$

We have equality only if $f_{\mu,x}(r) := f_\mu(rx)$ is constant for every $x \in S^{n-1}$, or equivalently, if $\frac{f_\mu}{\|f_\mu\|_\infty}$ is the indicator function of some star-shaped set in $\mathbb{R}^n$. It follows that

$$|K_{n+p}(\mu)| \geqslant \left(\frac{f_\mu(0)}{\|f_\mu\|_\infty} \right)^{n(\frac{1}{n}-\frac{1}{n+p})} |K_n(\mu)| = \left(\frac{f_\mu(0)}{\|f_\mu\|_\infty} \right)^{\frac{p}{n+p}} \frac{1}{f_\mu(0)}, \tag{4.9}$$

which implies

$$|K_{n+p}(\mu)|^{\frac{1}{p}+\frac{1}{n}} f_\mu^{\frac{1}{p}}(0) \geqslant \left(\frac{f_\mu(0)}{\|f_\mu\|_\infty} \right)^{\frac{1}{n}} f_\mu^{-\frac{n+p}{np}}(0) f_\mu^{\frac{1}{p}}(0) = \|f_\mu\|_\infty^{-\frac{1}{n}}. \tag{4.10}$$

This shows that

$$Z_p(\mu) \supseteq \|f_\mu\|_\infty^{-\frac{1}{n}} Z_p(\widetilde{K_{n+p}(\mu)}) \text{ and } Z_p^\circ(\mu) \subseteq \|f_\mu\|_\infty^{\frac{1}{n}} Z_p^\circ(\widetilde{K_{n+p}(\mu)}). \tag{4.11}$$

Taking volumes and using Theorem 4.1, we conclude that

$$|Z_p(\mu)| \geqslant \|f_\mu\|_\infty^{-1} |Z_p(\widetilde{B_2^n})| \ \text{ and } \ |Z_p^\circ(\mu)| \leqslant \|f_\mu\|_\infty |Z_p^\circ(\widetilde{B_2^n})|. \tag{4.12}$$

This proves the Proposition. $\square$

Using Theorem 4.1 we can show that if K is an isotropic convex body, then $\bar{g}_K(\theta) \leqslant CL_K$ for a random direction θ.

Proposition 4.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for every $t \geqslant 1$,*

$$\sigma\{\theta \in S^{n-1} : \bar{g}_K(\theta) \leqslant c_1 t L_K\} \geqslant 1 - \frac{\log_2 n}{t^n}, \tag{4.13}$$

where $C_1 > 0$ is an absolute constant.

Proof. Since K is convex, we have that

$$h_{Z_p(K)}(\theta) \simeq h_{Z_{2p}(K)}(\theta), \tag{4.14}$$

for every $p \geqslant 1$ and $\theta \in S^{n-1}$. So, we can write

$$\bar{g}_K(\theta) \simeq L_K \sup_{k \leqslant \log_2 n} \frac{2^{\frac{k}{2}}}{h_{Z_{2^k}(K)}(\theta)}, \tag{4.15}$$

where we have also used the fact that K is isotropic.

A direct computation shows that for $p \leqslant n$,

$$|Z_p^{\circ}(\widetilde{B_2^n})|^{1/n} \simeq \frac{\omega_n^{1/n}}{\sqrt{p}}. \tag{4.16}$$

Using polar coordinates, we get from Theorem 4.1 that

$$\omega_n \int_{S^{n-1}} \frac{d\sigma(\theta)}{h^n_{Z_p(K)}(\theta)} \leqslant \frac{\omega_n}{c^n(\sqrt{p})^n}. \tag{4.17}$$

By Markov's inequality we get that, for every $t \geqslant 1$,

$$\sigma\left\{\theta \in S^{n-1} : h_{Z_p(K)}(\theta) \leqslant \frac{C}{t}\sqrt{p}\right\} \leqslant \frac{1}{t^n}. \tag{4.18}$$

This implies that

$$\sigma\left\{\theta \in S^{n-1} : h_{Z_{2^k}(K)}(\theta) \geqslant \frac{C}{tL_K} 2^{\frac{k}{2}} L_K \ , k = 1, \dots, \log_2 n\right\} \geqslant 1 - \frac{\log_2 n}{t^n}, \tag{4.19}$$

and hence,

$$\sigma\{\theta \in S^{n-1} : \bar{g}_K(\theta) \leqslant C_1 t L_K\} \geqslant 1 - \frac{\log_2 n}{t^n}, \tag{4.20}$$

as claimed. $\square$

Proof of Theorem 1.1. Let K be an isotropic convex body in $\mathbb{R}^n$. From Proposition 3.2 we have that $\bar{g}_K(\theta) \leqslant c\sqrt{n}$. Let

$$A := \{\theta \in S^{n-1} : \underline{g}_K(\theta) \leqslant 2c_1 L_K\}. \tag{4.21}$$

From Proposition 4.4 we have that $\sigma(A) \geqslant 1 - \frac{\log_2 n}{2^n}$. Therefore,

$$\begin{aligned}
\mathbb{E}\bar{g}_K &= \int_{S^{n-1}} \bar{g}_K(\theta) d\sigma(\theta) \\
&= \int_A \bar{g}_K(\theta) d\sigma(\theta) + \int_{S^{n-1}\setminus A} \bar{g}_K(\theta) d\sigma(\theta) \\
&\leqslant 2c_1 L_K + c\sqrt{n}\sigma(S^{n-1} \setminus A) \\
&\leqslant 2c_1 L_K + \frac{\sqrt{n}\log_2 n}{2^n} \\
&\leqslant 3c_1 L_K,
\end{aligned}$$

since $L_K \geqslant c' > 0$ for all convex bodies (see [45]).

Moreover, if K is a centered convex body of volume 1 and K_{iso} is the isotropic image of K, by (3.10) we have that

$$\overline{g}_K = \overline{g}_{K_{iso}} = \inf_{\theta \in S^{n-1}} \overline{g}_{K_{iso}}(\theta) \leqslant \mathbb{E}\left(\overline{g}_{K_{iso}}\right) \leqslant 3c_1 L_K. \tag{4.22}$$

Now, Lemma 3.1 completes the proof. □

Note. (i) Using Proposition 4.3 instead of Theorem 4.1 one can prove the same result for any log-concave measure. Recall that $L_\mu = f(0)^{\frac{1}{n}}$. We omit the details.

(ii) The only place where convexity is needed in the proof of Proposition 4.4 is in assuming (4.14). So, the same result holds true for any $\mu \in \mathcal{P}_{[n]}$ which has some "regularity" as the one expressed in (4.14).

5 Proof of Theorem 1.2

As we have already mentioned in the Introduction, J. Bourgain ([11]) has proved that convex bodies for which every direction is subgaussian have bounded isotropic constant (by a function which depends only on the subgaussian constant). We will show that the same holds true in the "supergaussian" case; the proof is much easier.

Proposition 5.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and centre of mass at the origin. Let $a_K := \sup_{\theta \in S^{n-1}} \overline{g}_K(\theta)$. Then,*

$$L_K \leqslant c_1 a_K, \tag{5.1}$$

where $c_1 > 0$ is an universal constant.

Proof. Recall that a_K is SL_n-invariant. So, we can assume that K is isotropic. Then, for every $\theta \in S^{n-1}$,

$$h_K(\theta) \geqslant c_1 h_{Z_n(K)}(\theta) \geqslant c_2 \frac{\sqrt{n} L_K}{\overline{g}_K(\theta)} \geqslant c_2 \frac{\sqrt{n} L_K}{a_K}. \tag{5.2}$$

Since $|K| = 1$, there exists $\theta_0 \in S^{n-1}$ such that

$$h_K(\theta_0) \leqslant h_{\widetilde{B_2^n}}(\theta_0) \simeq \sqrt{n}. \tag{5.3}$$

It follows that $L_K \leqslant c_3 a_K$. □

Let K be an isotropic convex body in $\mathbb{R}^n$. We write $\bar{K}$ for the isotropic convex body of small diameter created by K, i.e. the isotropic image of $K \cap 4\sqrt{n} L_K B_2^n$. Note that $R(\bar{K}) \leqslant c\sqrt{n} L_{\bar{K}}$. Moreover, if $A \subset K$ with $|A| \geqslant \frac{1}{2}$, then for every $p \geqslant 1$ and $\theta \in S^{n-1}$,

$$h_{Z_p(K)}(\theta) \geqslant \left(\int_A |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}} = |A|^{\frac{1}{p} + \frac{1}{n}} h_{Z_p(\tilde{A})}(\theta) \geqslant c h_{Z_p(\tilde{A})}(\theta). \tag{5.4}$$

In particular, we have that $Z_p(\bar{K}) \subseteq cZ_p(K)$, where $c>0$ is a universal constant. Note that $L_K \simeq L_{\bar{K}}$. By the definition of $\bar{g}_{K,p}$, we have that for every $p \geqslant 1$,

$$\bar{g}_{K,p} \leqslant c\bar{g}_{\bar{K},p}. \tag{5.5}$$

Lemma 5.2. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $\bar{K}$ be defined as above. Then,*

(i) $q_*\left(\bar{K}, c_1\mathbb{E}^2(\bar{g}_{\bar{K}})\right) \geqslant n$,

(ii) For any $\delta \geqslant 2$, $\mathbb{E}(\bar{g}_{\bar{K},q_*(\bar{K},\delta)}) \leqslant c_2\sqrt{\delta}$,

where $c_1, c_2 > 0$ are absolute constants.

Proof. (i) Let $E := \mathbb{E}(\bar{g}_{\bar{K}})$. By definition we have that, for all $1 \leqslant p \leqslant n$,

$$h_{Z_p(\bar{K})}(\theta) \geqslant \frac{\sqrt{p}}{\bar{g}_{\bar{K}}(\theta)} L_{\bar{K}}. \tag{5.6}$$

Integrating over S^{n-1} we see that

$$W_{-1}^{-1}(Z_p(\bar{K})) = \int_{S^{n-1}} h^{-1}_{Z_p(\bar{K})}(\theta)d\sigma(\theta) \leqslant \frac{1}{\sqrt{p}L_{\bar{K}}} \int_{S^{n-1}} \bar{g}_{\bar{K}}(\theta)d\sigma(\theta) = \frac{E}{\sqrt{p}L_{\bar{K}}}. \tag{5.7}$$

By Hölder's inequality we get that, for all $1 \leqslant p \leqslant n$,

$$W(Z_p(\bar{K})) \geqslant W_{-1}(Z_p(\bar{K})) \geqslant \frac{\sqrt{p}L_{\bar{K}}}{E}. \tag{5.8}$$

Recall that

$$R(Z_p(\bar{K})) \leqslant R(\mathrm{conv}(\{\bar{K}, -\bar{K}\})) \leqslant c'\sqrt{n}L_{\bar{K}}. \tag{5.9}$$

So, by Milman's formula (2.13), for all $1 \leqslant p \leqslant n$,

$$k_*(Z_p(\bar{K})) \simeq n\left(\frac{W(Z_p(\bar{K}))}{R(Z_p(\bar{K}))}\right)^2 \geqslant c''\frac{p}{E^2}. \tag{5.10}$$

This proves assertion (i).

(ii) Let $q_1 := q_*(\bar{K},\delta)$ and $q_0 := q_*(\bar{K},2)$. Note that $\sqrt{n} \leqslant q_0 \leqslant q_1$. For all $1 \leqslant p \leqslant q_0$,

$$W(Z_p(\bar{K})) \simeq W_p(Z_p(\bar{K})) \simeq \sqrt{\frac{p}{n}}I_p(\bar{K}) \geqslant c\sqrt{p}L_{\bar{K}}, \tag{5.11}$$

where we have used Propositions 2.1 and 2.2.

Moreover,

$$R(Z_{q_0}(\bar{K})) \simeq \sqrt{\frac{n}{q_0}}W_{q_0}(Z_{q_0}(\bar{K})) \simeq I_{q_0}(\bar{K}) \geqslant c\sqrt{n}L_{\bar{K}}. \tag{5.12}$$

By definition we have that, for all $q_0 \leqslant p \leqslant q_1$, $k_*(Z_p(\bar{K})) \geqslant \frac{p}{\delta}$, or, using (2.13),

$$W(Z_p(\bar{K})) \geqslant c\sqrt{\frac{p}{n}}\frac{R(Z_p(\bar{K}))}{\sqrt{\delta}} \geqslant c\sqrt{\frac{p}{n}}\frac{R(Z_{q_0}(\bar{K}))}{\sqrt{\delta}} \geqslant c'\frac{\sqrt{p}L_{\bar{K}}}{\sqrt{\delta}}. \tag{5.13}$$

Thereore, for all $1 \leqslant p \leqslant q_1$,

$$W(Z_p(\bar{K})) \geqslant c\frac{\sqrt{p}L_{\bar{K}}}{\sqrt{\delta}}. \tag{5.14}$$

Moreover, recall that $k_*(Z_p(\bar{K})) \geqslant \sqrt{n}$ for all $1 \leqslant p \leqslant n$. A standard argument, based on the concentration of measure on S^{n-1}, shows that for $1 \leqslant p \leqslant q_1$,

$$\sigma\left(\{\theta \in S^{n-1} : h_{Z_p(\bar{K})}(\theta) \geqslant c'\frac{\sqrt{p}L_{\bar{K}}}{\sqrt{\delta}}\}\right) \geqslant 1 - e^{-c\sqrt{n}}. \tag{5.15}$$

Since $Z_p(\bar{K}) \simeq Z_{2p}(\bar{K})$, we get

$$\begin{aligned}\sigma\left(\left\{\theta \in S^{n-1} : h_{Z_p(\bar{K})}(\theta) \geqslant c'\frac{\sqrt{p}L_{\bar{K}}}{\sqrt{\delta}} \text{ for all } p \in [1, q_1]\right\}\right) &\geqslant 1 - \log_2(q_1)e^{-c\sqrt{n}} \\ &\geqslant 1 - e^{-c'\sqrt{n}}.\end{aligned}$$

In other words,

$$\sigma\left(\{\theta \in S^{n-1} : \bar{g}_{\bar{K},q_1}(\theta) \leqslant c''\sqrt{\delta}\}\right) \geqslant 1 - e^{-c'\sqrt{n}}. \tag{5.16}$$

Let

$$A := \{\theta \in S^{n-1} : \bar{g}_{\bar{K},q_1}(\theta) \leqslant c''\sqrt{\delta}\}. \tag{5.17}$$

Then,

$$\begin{aligned} E &= \int_{S^{n-1}} \bar{g}_{\bar{K},q_1}(\theta)d\sigma(\theta) \\ &= \int_A \bar{g}_{\bar{K},q_1}(\theta)d\sigma(\theta) + \int_{S^{n-1}\setminus A} \bar{g}_{\bar{K},q_1}(\theta)d\sigma(\theta) \\ &\leqslant c\sqrt{\delta} + \sigma(S^{n-1} \setminus A)\sqrt{n} \\ &\leqslant c'\sqrt{\delta}. \end{aligned}$$

This proves assertion (ii). $\square$

We can give now a second proof of Theorem 1.1.

Proposition 5.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

(i) $\mathbb{E}(\bar{g}_{K,\sqrt{n}}) \leqslant c_1$,

(ii) $\mathbb{E}(\bar{g}_K) \leqslant c_1\sqrt[4]{n}$,

(iii) $\mathbb{E}(\bar{g}_K) \leqslant c_2 L_K$,

where $c_1, c_2 > 0$ *are absolute constants.*

Proof. Let $\bar{K}$ be defined as above. Using (5.5) we have that

$$\mathbb{E}(\bar{g}_{K,p}) \leqslant c\mathbb{E}(\bar{g}_{\bar{K},p}) \tag{5.18}$$

for all $1 \leqslant p \leqslant n$. By Proposition 2.5 (vi) we have that $q_*(K, c) \geqslant \sqrt{n}$. Then, Lemma 5.2 shows that

$$\mathbb{E}(\bar{g}_{K,\sqrt{n}}) \leqslant c\mathbb{E}(\bar{g}_{\bar{K},\sqrt{n}}) \leqslant c_1. \tag{5.19}$$

This proves (i), while (ii) follows from (i) and (3.19). Finally, using Proposition 2.5 (i) and (vii), we have that $q_*(K, cL_K^2) \geqslant n$, and by Lemma 5.2,

$$\mathbb{E}(\bar{g}_K) \leqslant c\mathbb{E}(\bar{g}_{\bar{K}}) \leqslant c_2 L_K. \tag{5.20}$$

This completes (iii). $\square$

Proposition 5.3 (iii) provides an alternative proof of Theorem 1.1, although the reader may notice that the measure estimate for the directions in S^{n-1} with super-gaussian constant bounded by L_K is much better in Proposition 4.4. Moreover, the convexity plays more important role in the proof of the Proposition 5.3. The proof of Proposition 5.3 can be certainly carried out in the case of log-concave measures.

Proof of Theorem 1.2. By Proposition 2.5 (ii) we have that, for any $\delta \geqslant 1$, $q_{-c}(\bar{K}, \delta) \geqslant \frac{q_*(\bar{K}, c\delta)}{c\delta}$. Then, by Lemma 5.2,

$$q_{-c}\left(\bar{K}, c_1\mathbb{E}^2(\bar{g}_{\bar{K}})\right) \geqslant \frac{q_*\left(\bar{K}, c_1\mathbb{E}^2(\bar{g}_{\bar{K}})\right)}{c_1\mathbb{E}^2(\bar{g}_{\bar{K}})} \geqslant \frac{cn}{c_1\mathbb{E}^2(\bar{g}_{\bar{K}})}. \tag{5.21}$$

Now, we apply Proposition 2.6: we have

$$\begin{aligned}
\sup_{K \text{ isotropic}} L_K &\leqslant c \sup_{K \text{ isotropic}} \left(\mathbb{E}^2(\bar{g}_{\bar{K}}) \sqrt{\frac{n}{q_{-c}\left(\bar{K}, c_1\mathbb{E}^2(\bar{g}_{\bar{K}})\right)}} \sqrt{\log \frac{n}{q_{-c}\left(\bar{K}, c_1\mathbb{E}^2(\bar{g}_{\bar{K}})\right)}}\right) \\
&\leqslant c' \sup_{K \text{ isotropic}} \mathbb{E}^3(\bar{g}_{\bar{K}}) \sqrt{\log \mathbb{E}^2(\bar{g}_{\bar{K}})} \\
&\leqslant c' \sup_{K \text{ isotropic}} \mathbb{E}^3(\bar{g}_K) \sqrt{\log \mathbb{E}^2(\bar{g}_K)}.
\end{aligned}$$

This completes the proof. $\square$

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