

Inequalities for the quermassintegrals of sections of convex bodies

Dimitris-Marios Liakopoulos

Abstract

We provide general estimates which compare the quermassintegrals of a convex body K in \mathbb{R}^n with the averages of the corresponding quermassintegrals of the k -codimensional sections of K over $G_{n,n-k}$. An example is the inequality

$$\alpha_{n,k,j} \frac{W_j(K)}{|K|} \leq \int_{G_{n,n-k}} \frac{W_j(K \cap F)}{|K \cap F|} d\nu_{n,n-k}(F) \leq \beta_{n,k,j} \frac{W_j(K)}{|K|}$$

where the constants $\alpha_{n,k,j}$ and $\beta_{n,k,j}$ depend only on n, k and j , which holds true for any centrally symmetric convex body K in \mathbb{R}^n and any $0 \leq j \leq n - k - 1 \leq n - 1$. Using these estimates we obtain some positive results for suitable versions of the slicing problem for the quermassintegrals of a convex body.

1 Introduction

In this article we provide a number of general double-sided inequalities comparing the quermassintegrals of a convex body K in \mathbb{R}^n with the averages of the corresponding quermassintegrals of the k -codimensional sections of K over $G_{n,n-k}$. The starting point for studying this type of question is the slicing problem which asks if there exists an absolute constant $C_1 > 0$ such that for every $n \geq 2$ and every convex body K in \mathbb{R}^n with barycenter at the origin (we call these convex bodies centered) one has

$$|K|^{\frac{n-1}{n}} \leq C_1 \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|$$

where $|\cdot|$ denotes volume in the appropriate dimension. It is well-known that this problem is equivalent to the question if there exists an absolute constant $C_2 > 0$ such that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leq C_2$$

for all $n \geq 1$, where L_K is the isotropic constant of K (we refer the reader to [3] for background information on isotropic convex bodies). Bourgain proved in [2] that $L_n \leq c_1 \sqrt[4]{n} \log n$, and Klartag [14] improved this bound to $L_n \leq c_2 \sqrt[4]{n}$. After breakthrough work of Chen [6], Klartag and Lehec have recently established in [15] the polylogarithmic in the dimension bound $L_n \leq c_3 (\ln n)^4$. Even more recently, the method of [15] was slightly refined in [13] where it was shown that $L_n \leq c_4 (\ln n)^{2.2226}$. A closer examination of the equivalence of the two questions shows that

$$|K|^{\frac{n-1}{n}} \leq c_3 L_n \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|$$

for every centered convex body K in \mathbb{R}^n . The natural generalization of the problem, the so-called lower dimensional slicing problem, can be formulated as follows: Let $1 \leq k \leq n - 1$ and let $\alpha_{n,k}$ be the smallest positive constant $\alpha > 0$ such that

$$|K|^{\frac{n-k}{n}} \leq \alpha^k \max_{H \in G_{n,n-k}} |K \cap H|$$

for every centered convex body K in \mathbb{R}^n , where $G_{n,k}$ is the Grassmann manifold of all k -dimensional subspaces of \mathbb{R}^n . Then the question is if there exists an absolute constant $C_4 > 0$ such that $\alpha_{n,k} \leq C_3$ for all n and k .

The slicing problem for surface area, instead of volume, is the question if there exists a constant $\alpha_{n,k}$ such that

$$(1.1) \quad S(K) \leq \alpha_{n,k}^k |K|^{\frac{k}{n}} \max_{F \in G_{n,n-k}} S(K \cap F)$$

for every centrally symmetric convex body K in \mathbb{R}^n , where $S(A)$ denotes the surface area of a convex body A in the appropriate dimension. A negative answer to this question was given in [4]. In fact, this was also done for the natural generalization of the problem in which surface area is replaced by quermassintegrals of other orders. Recall that if K is a convex body in \mathbb{R}^n then the function $|K + \lambda B_2^n|$, where B_2^n is the Euclidean unit ball in \mathbb{R}^n , is a polynomial in $\lambda \in [0, \infty)$. We have

$$|K + \lambda B_2^n| = \sum_{j=0}^n \binom{n}{j} W_j(K) \lambda^j,$$

where $W_j(K)$ is the j -th quermassintegral of K . This is the classical Steiner formula (see Section 2 for background information on mixed volumes and quermassintegrals). The surface area $S(K)$ of K satisfies

$$S(K) = nW_1(K).$$

Given $n \geq 2$, $1 \leq k \leq n$ and $0 \leq j \leq n - k - 1$, one may ask if there exists a constant $\alpha_{n,k,j} > 0$ such that

$$(1.2) \quad W_j(K) \leq \alpha_{n,k,j}^k |K|^{\frac{k}{n}} \max_{F \in G_{n,n-k}} W_j(K \cap F)$$

for every centered (or centrally symmetric) convex body K in \mathbb{R}^n . It was proved in [4] that one cannot expect an upper bound of the form (1.2) even if sections are replaced by projections. More precisely, if $n \geq 2$, $1 \leq k \leq n$ and $0 \leq j \leq n - k - 1$ then for every $\alpha > 0$ there exists a convex body K in \mathbb{R}^n such that

$$(1.3) \quad W_j(K) > \alpha |K|^{\frac{k}{n}} \max_{F \in G_{n,n-k}} W_j(P_F(K)) \geq \alpha |K|^{\frac{k}{n}} \max_{F \in G_{n,n-k}} W_j(K \cap F),$$

where $P_F(K)$ denotes the orthogonal projection of K onto F .

In Section 3 we explore the possibility to obtain some positive results for variants of this question. Our first main result is the next theorem for the class of centered convex bodies.

Theorem 1.1. *Let K be a centered convex body in \mathbb{R}^n . For every $0 \leq j \leq n - k - 1 \leq n - 1$ we have that*

$$\alpha_{n,k,j} \left(\frac{n - k - j + 1}{n + 1} \right)^{n-k-j} W_j(K) \leq \int_{G_{n,n-k}} |P_{F^\perp}(K)| W_j(K \cap F) d\nu_{n,n-k}(F) \leq \alpha_{n,k,j} \binom{n-j}{k} W_j(K),$$

where $\alpha_{n,k,j} = \frac{\omega_{n-k} \omega_{n-j}}{\omega_{n-k-j} \omega_n}$.

Using the left hand side inequality of Theorem 1.1 one immediately gets

$$\alpha_{n,k,j} \left(\frac{n - k - j + 1}{n + 1} \right)^{n-k-j} W_j(K) \leq s_k(K) \cdot \max_{F \in G_{n,n-k}} W_j(K \cap F),$$

where

$$s_k(K) := \int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F).$$

By Kubota's formula (see (2.4) in Section 2) we have $s_k(K) = \frac{\omega_k}{\omega_n} W_{n-k}(K)$, and hence we obtain:

Corollary 1.2. *Let K be a centered convex body in \mathbb{R}^n . For every $0 \leq j \leq n - k - 1 \leq n - 1$ we have that*

$$(1.4) \quad \gamma_{n,k,j} \left(\frac{n-k-j+1}{n+1} \right)^{n-k-j} W_j(K) \leq W_{n-k}(K) \cdot \max_{F \in G_{n,n-k}} W_j(K \cap F),$$

where $\gamma_{n,k,j} = \frac{\omega_{n-k}\omega_{n-j}}{\omega_{n-k-j}\omega_k}$.

This estimate, which is in the spirit of results from [4], shows that we can have some version of (1.2) if we make the additional assumption that the natural parameter $W_{n-k}^{1/k}(K)/|K|^{1/n}$ of the body is well-bounded. A different variant of (1.4) is obtained in [4], which involves the parameter

$$t(K) = \left(\frac{|K|}{|r(K)B_2^n|} \right)^{\frac{1}{n}}$$

where $r(K)$ is the inradius of K , i.e. the largest value of $r > 0$ for which there exists $x_0 \in K$ such that $x_0 + rB_2^n \subseteq K$. It is proved in [4] that if K is a convex body in \mathbb{R}^n with $0 \in \text{int}(K)$ then, for all $1 \leq j \leq n - k \leq n - 1$ we have that

$$(1.5) \quad \frac{W_j(K)}{|K|} \leq t(K)^j \max_{F \in G_{n,n-k}} \frac{W_j(K \cap F)}{|K \cap F|}.$$

Here, we improve (1.5) in two ways. First, we compare $\frac{W_j(K)}{|K|}$ with the average of $\frac{W_j(K \cap F)}{|K \cap F|}$ over all $F \in G_{n,n-k}$. Second, we show that the two quantities are equivalent up to constants depending only on the dimensions n, k and j , with no dependence on any other parameter of the body.

Theorem 1.3. *Let K be a centrally symmetric convex body in \mathbb{R}^n . For any $0 \leq j \leq n - k - 1 \leq n - 1$ we have that*

$$\alpha_{n,k,j} \binom{n}{k}^{-1} \frac{W_j(K)}{|K|} \leq \int_{G_{n,n-k}} \frac{W_j(K \cap F)}{|K \cap F|} d\nu_{n,n-k}(F) \leq \alpha_{n,k,j} \binom{n-j}{k} \frac{W_j(K)}{|K|}.$$

In Section 4 we study a variant of the slicing problem. Our starting point is a surface area variant of the equivalence of the isomorphic Busemann–Petty problem with the slicing problem: Assuming that there is a constant γ_n such that if K and D are centrally symmetric convex bodies in \mathbb{R}^n that satisfy

$$S(K \cap \xi^\perp) \leq S(D \cap \xi^\perp)$$

for all $\xi \in S^{n-1}$, then $S(K) \leq \gamma_n S(D)$, one can see that there is some constant $c(n)$ such that

$$(1.6) \quad S(K) \leq c(n) S(K)^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(K \cap \xi^\perp)$$

for every convex body K in \mathbb{R}^n . It was proved in [4] that an inequality of this type holds true in general. If K is a convex body in \mathbb{R}^n then

$$S(K) \leq A_n S(K)^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(K \cap \xi^\perp)$$

where $A_n > 0$ is a constant depending only on n . Actually, the result is first proved for an arbitrary ellipsoid and then it is extended to any convex body, using John's theorem. Here, we give a direct proof of a more general result showing that an inequality analogous to (1.6) holds for any k and j , where k is the codimension of the subspaces and j is the order of the quermassintegral that we consider. Note that (1.6) corresponds to the case $k = j = 1$ of the next theorem.

Theorem 1.4. *Let K be a centrally symmetric convex body in \mathbb{R}^n . For every $0 \leq j \leq n - k - 1 \leq n - 1$ we have that*

$$W_j(K)^{n-k-j} \leq \alpha_{n,k,j} \max_{F \in G_{n,n-k}} W_j(K \cap F)^{n-j},$$

where $\alpha_{n,k,j} > 0$ is a constant depending only on n, k and j .

Theorem 1.4 is a consequence of another double-sided inequality. Using the generalized Blaschke-Petkantschin formula and integral-geometric results of Dann, Paouris and Pivovarov from [7] we show that if K is a centrally symmetric convex body in \mathbb{R}^n then for all $0 \leq j \leq n - k - 1 \leq n - 1$ we have that

$$(1.7) \quad c_{n,k,j}|K|^{n-k}W_{k+j}(K) \leq \int_K \cdots \int_K W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) dx_{n-k} \cdots dx_1 \leq \delta_{n,k,j}|K|^{n-k}W_{k+j}(K),$$

where $c_{n,k,j}$ and $\delta_{n,k,j}$ are constants depending only on n, k and j .

Using a variant of (1.7) we also obtain the next result.

Theorem 1.5. *Let K be a centrally symmetric convex body in \mathbb{R}^n . Then, for all $0 \leq j \leq n - 1$ and $N \geq n + 1$ we have that*

$$c_{n,N,j}|K|^N W_j(K) \leq \int_K \cdots \int_K W_j(\text{conv}\{x_1, \dots, x_N\}) dx_N \cdots dx_1$$

where $c_{n,N,j}$ is a constant depending only on n, N and j .

Theorem 1.5 is related to a result of Hartzoulaki and Paouris from [12] which asserts that, among all convex bodies K of volume 1 in \mathbb{R}^n , the expected value

$$\int_K \cdots \int_K W_j(\text{conv}\{x_1, \dots, x_N\}) dx_N \cdots dx_1$$

is minimized when K is the Euclidean ball D_n of volume 1. In other words, for every convex body K in \mathbb{R}^n we have

$$(1.8) \quad A_{n,N,j}|K|^{N+\frac{n-j}{n}} \leq \int_K \cdots \int_K W_j(\text{conv}\{x_1, \dots, x_N\}) dx_N \cdots dx_1$$

for some constant $A_{n,N,j} > 0$, with equality if and only if $K = D_n$ (see also [17] for the characterization of the cases of equality). This inequality generalizes to the setting of quermassintegrals a well-known result of Groemer [11] (see also [21] and [10]) which concerned the expected value of the volume of $\text{conv}\{x_1, \dots, x_N\}$. All these results are proved via Steiner symmetrization, and hence, the left hand-side of (1.8) involves only the volume of K and is not sensitive to the value of $W_j(K)$. Since

$$W_j(K) \geq |K|^{\frac{n-j}{n}}$$

by the Aleksandrov inequalities, Theorem 1.5 provides a stronger estimate if we ignore the values of the constants.

2 Mixed volumes and Quermassintegrals

We work in \mathbb{R}^n , which is equipped with the standard inner product $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} . The Grassmann manifold $G_{n,k}$ of all k -dimensional subspaces of \mathbb{R}^n is equipped with the Haar probability measure $\nu_{n,k}$. For every $1 \leq k \leq n - 1$ and $F \in G_{n,k}$ we write P_F for the orthogonal projection from \mathbb{R}^n onto F .

A convex body in \mathbb{R}^n is a compact convex subset K of \mathbb{R}^n with non-empty interior. We say that K is centrally symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its barycenter $\frac{1}{|K|} \int_K x dx$ is at the origin. The support function of a convex body K is defined by $h_K(y) = \max\{\langle x, y \rangle : x \in K\}$, and the mean width of K is

$$w(K) = \int_{S^{n-1}} h_K(\xi) d\sigma(\xi).$$

The circumradius of K is the quantity $R(K) = \max\{\|x\|_2 : x \in K\}$ i.e. the smallest $R > 0$ for which $K \subseteq RB_2^n$. We write $r(K)$ for the inradius of K , the largest $r > 0$ for which there exists $x_0 \in K$ such that $x_0 + rB_2^n \subseteq K$. If $0 \in \text{int}(K)$ then we define the polar body K° of K by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.$$

The cone probability measure μ_K on the boundary $\partial(K)$ of a convex body K with $0 \in \text{int}(K)$ is defined by

$$\mu_K(B) = \frac{|\{rx : x \in B, 0 \leq r \leq 1\}|}{|K|}$$

for all Borel subsets B of $\partial(K)$. We shall use the identity

$$(2.1) \quad \int_{\mathbb{R}^n} f(x) dx = n |K| \int_0^\infty r^{n-1} \int_{\partial(K)} f(rx) d\mu_K(x) dr$$

which holds for every integrable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (see [16, Proposition 1]).

Minkowski's theorem, and definition of the mixed volumes, states that if K_1, \dots, K_m are non-empty, compact convex subsets of \mathbb{R}^n , then the volume of $\lambda_1 K_1 + \dots + \lambda_m K_m$ is a homogeneous polynomial of degree n in $\lambda_i > 0$. One can write

$$|\lambda_1 K_1 + \dots + \lambda_m K_m| = \sum_{1 \leq i_1, \dots, i_n \leq m} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \dots \lambda_{i_n},$$

where the coefficients $V(K_{i_1}, \dots, K_{i_n})$ are invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \dots, K_{i_n})$ is the mixed volume of K_{i_1}, \dots, K_{i_n} . In particular, if K and D are two convex bodies in \mathbb{R}^n then the function $|K + \lambda D|$ is a polynomial in $\lambda \in [0, \infty)$:

$$|K + \lambda D| = \sum_{j=0}^n \binom{n}{j} V_{n-j}(K, D) \lambda^j,$$

where $V_{n-j}(K, D) = V((K, n-j), (D, j))$ is the j -th mixed volume of K and D (we use the notation (D, j) for D, \dots, D j -times). If $D = B_2^n$ then we set $W_j(K) := V_{n-j}(K, B_2^n) = V((K, n-j), (B_2^n, j))$; this is the j -th quermassintegral of K . The intrinsic volumes $V_j(K)$ of K are defined for $0 \leq j \leq n$ by

$$(2.2) \quad V_j(K) = \frac{\binom{n}{j}}{\omega_{n-j}} W_{n-j}(K).$$

Thus, Steiner's formula can be written in the following two equivalent ways:

$$|K + \lambda B_2^n| = \sum_{j=0}^n \binom{n}{j} W_j(K) \lambda^j = \sum_{j=0}^n \omega_{n-j} V_j(K) \lambda^{n-j}.$$

The mixed volume $V_{n-1}(K, D)$ can be expressed as

$$(2.3) \quad V_{n-1}(K, D) = \frac{1}{n} \int_{S^{n-1}} h_D(\theta) d\sigma_K(\theta),$$

where σ_K is the surface area measure of K ; this is the Borel measure on S^{n-1} defined by

$$\sigma_K(A) = m(\{x \in \text{bd}(K) : \text{the outer normal to } K \text{ at } x \text{ belongs to } A\}),$$

where m is the Hausdorff measure on $\text{bd}(K)$. In particular, the surface area $S(K) := \sigma_K(S^{n-1})$ of K satisfies

$$S(K) = nW_1(K).$$

Kubota's integral formula expresses the quermassintegral $W_j(K)$ as an average of the volumes of $(n-j)$ -dimensional projections of K :

$$(2.4) \quad W_j(K) = \frac{\omega_n}{\omega_{n-j}} \int_{G_{n,n-j}} |P_F(K)| d\nu_{n,n-j}(F).$$

Applying this formula for $j = n-1$ we see that

$$W_{n-1}(K) = \omega_n w(K).$$

Aleksandrov's inequalities (see [5] and [19]) imply that if we set

$$(2.5) \quad Q_j(K) = \left(\frac{W_{n-j}(K)}{\omega_n} \right)^{\frac{1}{j}} = \left(\frac{1}{\omega_j} \int_{G_{n,j}} |P_F(K)| d\nu_{n,j}(F) \right)^{\frac{1}{j}},$$

then $j \mapsto Q_j(K)$ is decreasing. In particular, for every $1 \leq j \leq n-1$ we have

$$(2.6) \quad \text{vrad}(K) := \left(\frac{|K|}{\omega_n} \right)^{\frac{1}{n}} \leq \left(\frac{1}{\omega_j} \int_{G_{n,j}} |P_F(K)| d\nu_{n,j}(F) \right)^{\frac{1}{j}} \leq w(K).$$

Another consequence of Aleksandrov's inequalities is the following Brunn-Minkowski inequality for the quermassintegrals: one has

$$(2.7) \quad W_j(K+D)^{\frac{1}{n-j}} \geq W_j(K)^{\frac{1}{n-j}} + W_j(D)^{\frac{1}{n-j}}$$

for all $j = 0, \dots, n-1$ and any pair of convex bodies K and D in \mathbb{R}^n .

We refer to the books [9] and [19] for basic facts from the Brunn-Minkowski theory and to the books [1] and [3] for basic facts from asymptotic convex geometry.

3 Proof of Theorem 1.1 and Theorem 1.3

In this section we prove Theorem 1.1 and Theorem 1.3. Consider the set $A_{n,k}$ of k -dimensional affine subspaces of \mathbb{R}^n equipped with the Haar probability measure $\mu_{n,k}$. Our starting point is Crofton's formula (see [20, Theorem 5.1.1]).

Theorem 3.1. *Let K be a convex body in \mathbb{R}^n . For any $1 \leq j \leq k \leq n-1$ we have that*

$$\int_{A_{n,k}} V_j(K \cap F) d\mu_{n,k}(F) = \frac{\binom{k}{j} \omega_k \omega_{n-k+j}}{\binom{n}{k-j} \omega_j \omega_n} V_{n-k+j}(K).$$

Using (2.2) for K and $K \cap F$, $F \in G_{n,k}$, we rewrite the assertion of Theorem 3.1 as follows:

$$\int_{A_{n,k}} W_{k-j}(K \cap F) d\mu_{n,k}(F) = \frac{\omega_k \omega_{n-k+j}}{\omega_j \omega_n} W_{k-j}(K).$$

Equivalently, for all $0 \leq j \leq n-k-1 \leq n-1$,

$$\int_{A_{n,n-k}} W_j(K \cap F) d\mu_{n,n-k}(F) = \frac{\omega_{n-k} \omega_{n-j}}{\omega_{n-k-j} \omega_n} W_j(K).$$

Note that (see e.g. [20, Section 5.1])

$$(3.1) \quad \begin{aligned} \int_{A_{n,n-k}} W_j(K \cap F) d\mu_{n,n-k}(F) &= \int_{G_{n,n-k}} \int_{F^\perp} W_j(K \cap (x+F)) dx d\nu_{n,n-k}(F) \\ &= \int_{G_{n,k}} \int_F W_j(K \cap (x+F^\perp)) dx d\nu_{n,k}(F). \end{aligned}$$

We shall use the next set of inequalities of Rogers-Shephard type.

Lemma 3.2. *Let K be a centered convex body in \mathbb{R}^n . For every $F \in G_{n,k}$ and $0 \leq j \leq n - k - 1 \leq n - 1$ we have that*

$$\begin{aligned} \binom{n-j}{k}^{-1} |P_F(K)| W_j(K \cap F^\perp) &\leq \int_F W_j(K \cap (x + F^\perp)) dx \\ &\leq \left(\frac{n+1}{n-k-j+1} \right)^{n-k-j} |P_F(K)| W_j(K \cap F^\perp). \end{aligned}$$

Proof. We follow the proof of the Rogers-Shephard inequality (see [18]). For the left hand-side inequality, let $x \in P_F(K)$ and $t = \|x\|_{P_F(K)} \in [0, 1]$, where $\|\cdot\|_{P_F(K)}$ is the Minkowski functional of $P_F(K)$. Then, there exists $z \in F^\perp$ such that $x + tz \in tK$, therefore

$$K \cap (x + F^\perp) \supseteq (1-t)(K \cap F^\perp) + x + tz.$$

Then,

$$\begin{aligned} W_j(K \cap (x + F^\perp)) &= V(K \cap (x + F^\perp); n-k-j, B_2^{n-k}; j) \geq V((1-t)(K \cap F^\perp); n-k-j, B_2^{n-k}; j) \\ &= (1-t)^{n-k-j} V(K \cap F^\perp; n-k-j, B_2^{n-k}; j) = (1-t)^{n-k-j} W_j(K \cap F^\perp). \end{aligned}$$

It follows that

$$W_j(K \cap (x + F^\perp)) \geq (1 - \|x\|_{P_F(K)})^{n-k-j} W_j(K \cap F^\perp).$$

Integrating with respect to $x \in F$ we obtain

$$\begin{aligned} \int_F W_j(K \cap (x + F^\perp)) dx &= \int_{P_F(K)} W_j(K \cap (x + F^\perp)) dx \\ &\geq W_j(K \cap F^\perp) \int_F (1 - \|x\|_{P_F(K)})^{n-k-j} \mathbb{1}_{P_F(K)}(x) dx, \end{aligned}$$

because the support of $x \mapsto W_j(K \cap (x + F^\perp))$ is the projection $P_F(K)$ of K onto F . Now, changing to polar coordinates with respect to the cone measure $\mu_{\partial P_F(K)}$ of $\partial P_F(K)$ and using (2.1), we write

$$\begin{aligned} \int_F (1 - \|x\|_{P_F(K)})^{n-k-j} \mathbb{1}_{P_F(K)}(x) dx &= k |P_F(K)| \int_{\partial P_F(K)} \int_0^1 t^{k-1} (1 - \|t\theta\|_{P_F(K)})^{n-k-1} dt d\mu_{\partial P_F(K)}(\theta) \\ &= k |P_F(K)| \int_0^1 t^{k-1} (1-t)^{n-k-j} dt = \binom{n-j}{k}^{-1} |P_F(K)|. \end{aligned}$$

This proves the left hand-side inequality of the lemma. For the right hand-side inequality we write

$$\int_F W_j(K \cap (x + F^\perp)) dx = \int_{P_F(K)} W_j(K \cap (x + F^\perp)) dx \leq |P_F(K)| \max_{x \in F} W_j(K \cap (x + F^\perp)).$$

Finally we use an inequality of Stephen and Yaskin (see [23]) for the quermassintegrals of centered convex bodies, namely,

$$\max_{x \in F} W_j(K \cap (x + F^\perp)) \leq \left(\frac{n+1}{n-k-j+1} \right)^{n-k-j} W_j(K \cap F^\perp),$$

which extends Fradelizi's inequality

$$(3.2) \quad \max_{x \in F} |K \cap (x + F^\perp)| \leq \left(\frac{n+1}{n-k+1} \right)^{n-k} |K \cap F^\perp|$$

from [8] that had settled the volume case. □

From Lemma 3.2 we can easily deduce Theorem 1.1.

Proof of Theorem 1.1. Let K be a centered convex body in \mathbb{R}^n . For every $0 \leq j \leq n - k - 1 \leq n - 1$ We integrate the inequalities of Lemma 3.2 with respect to $\nu_{n,k}$ and recalling (3.1) we obtain

$$(3.3) \quad \alpha_{n,k,j} \left(\frac{n-k-j+1}{n+1} \right)^{n-k-j} W_j(K) \leq \int_{G_{n,k}} |P_F(K)| W_j(K \cap F^\perp) d\nu_{n,k}(F) \leq \alpha_{n,k,j} \binom{n-j}{k} W_j(K),$$

where $\alpha_{n,k,j} = \frac{\omega_{n-k}\omega_{n-j}}{\omega_{n-k-j}\omega_n}$. Since

$$\int_{G_{n,k}} |P_F(K)| W_j(K \cap F^\perp) d\nu_{n,k}(F) = \int_{G_{n,n-k}} |P_{F^\perp}(K)| W_j(K \cap F) d\nu_{n,n-k}(F),$$

this is equivalent to the assertion of the theorem. \square

We pass now to the proof of Theorem 1.3. First we prove a version of the result for centered convex bodies.

Theorem 3.3. *Let K be a centered convex body in \mathbb{R}^n . For any $0 \leq j \leq n - k - 1 \leq n - 1$ we have*

$$\beta_{n,k,j} \frac{W_j(K)}{|K|} \leq \int_{G_{n,k}} \frac{W_j(K \cap F^\perp)}{|K \cap F^\perp|} d\nu_{n,k}(F) \leq \gamma_{n,k,j} \frac{W_j(K)}{|K|},$$

where $\beta_{n,k,j} = \alpha_{n,k,j} \left(\frac{n-k-j+1}{n+1} \right)^{n-k-j} \binom{n}{k}^{-1}$ and $\gamma_{n,k,j} = \alpha_{n,k,j} \binom{n-j}{k} \left(\frac{n+1}{n-k+1} \right)^{n-k}$.

Proof. From the left hand-side inequality of Theorem 1.1 and the Rogers-Shephard type inequality

$$(3.4) \quad |P_F(K)| |K \cap F^\perp| \leq \binom{n}{k} |K|$$

of Spingarn (see [22]) we obtain the left hand-side inequality of the theorem. Next, using Fubini's theorem and (3.2) we see that

$$(3.5) \quad |K| = \int_{P_F(K)} |K \cap (x + F^\perp)| dx \leq |P_F(K)| \max_{x \in P_F(K)} |K \cap (x + F^\perp)| \\ \leq \left(\frac{n+1}{n-k+1} \right)^{n-k} |P_F(K)| |K \cap F^\perp|,$$

and combining this estimate with the right hand-side inequality of Theorem 1.1 we obtain the right hand side inequality of the theorem. \square

For the centrally symmetric case we may use some additional observations.

Proof of Theorem 1.3. Assuming that K is centrally symmetric, we have

$$W_j(K \cap (x + F^\perp)) = W_j(K \cap (-x + F^\perp))$$

for every $x \in P_F(K)$. We also have $\frac{1}{2}[K \cap (x + F^\perp)] + \frac{1}{2}[K \cap (-x + F^\perp)] \subseteq K \cap F^\perp$, and hence, by the Brunn-Minkowski inequality (2.7) we get

$$W_j(K \cap (x + F^\perp)) \leq W_j(K \cap F^\perp)$$

for every $x \in P_F(K)$. It follows that

$$\int_F W_j(K \cap (x + F^\perp)) dx \leq |P_F(K)| W_j(K \cap F^\perp).$$

Integrating over all $F \in G_{n,k}$ and using (3.1) we get

$$\alpha_{n,k,j} W_j(K) \leq \int_{G_{n,k}} |P_F(K)| W_j(K \cap F^\perp) d\nu_{n,k}(F).$$

Applying (3.4) we see that

$$\alpha_{n,k,j} \binom{n}{k}^{-1} \frac{W_j(K)}{|K|} \leq \int_{G_{n,k}} \frac{W_j(K \cap F^\perp)}{|K \cap F^\perp|} d\nu_{n,k}(F).$$

Then, using the right hand side inequality of Theorem 1.1 and the fact that

$$|K| \leq |P_F(K)| \max_{x \in P_F(K)} |K \cap (x + F^\perp)| = |P_F(K)| |K \cap F^\perp|$$

holds for centrally symmetric convex bodies, we conclude that

$$\int_{G_{n,k}} \frac{W_j(K \cap F^\perp)}{|K \cap F^\perp|} d\nu_{n,k}(F) \leq \alpha_{n,k,j} \binom{n-j}{k} \frac{W_j(K)}{|K|}.$$

This shows that for any $0 \leq j \leq n-k-1 \leq n-1$ we have

$$\alpha_{n,k,j} \binom{n}{k}^{-1} \frac{W_j(K)}{|K|} \leq \int_{G_{n,k}} \frac{W_j(K \cap F^\perp)}{|K \cap F^\perp|} d\nu_{n,k}(F) \leq \alpha_{n,k,j} \binom{n-j}{k} \frac{W_j(K)}{|K|},$$

which is equivalent to the assertion of the theorem. \square

4 Proof of Theorem 1.4

In this section we prove Theorem 1.4. We start by introducing a number of tools that will be needed. Let f_1, \dots, f_q be non-negative, bounded, integrable functions on \mathbb{R}^d such that $\|f_j\|_1 > 0$ for every $j = 1, \dots, q$. Given a compact convex set $C \subset \mathbb{R}^q$ and $p \neq 0$, we define

$$\mathcal{F}_{C,p}(f_1, \dots, f_q) = \left(\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} |[x_1, \dots, x_q]C|^p \prod_{j=1}^q \frac{f_j(x_j)}{\|f_j\|_1} dx_1 \cdots dx_q \right)^{1/p},$$

where

$$[x_1, \dots, x_q]C = \left\{ \sum_{j=1}^q c_j x_j : c = (c_j) \in C \right\}$$

and $|[x_1, \dots, x_q]C|$ is the volume of this set. The next theorem is due to Dann, Paouris and Pivovarov (see [7, Theorem 4.4]).

Theorem 4.1. *Let q and d be positive integers. Let f be a non-negative, bounded integrable function on \mathbb{R}^d with $\|f\|_1 > 0$. Let $C \subset \mathbb{R}^q$ be a compact convex set and $p \geq 1$. We set $m = \min(q, d, \dim(C))$. Then,*

$$\mathcal{F}_{C,p}(f; q) \geq \left(\frac{\|f\|_1}{\omega_d \|f\|_\infty} \right)^{\frac{m}{d}} \mathcal{F}_{C,p}(\mathbb{1}_{B_2^q}; q),$$

where $\mathcal{F}_{C,p}(f; q) = \mathcal{F}_{C,p}(f, \dots, f)$ (q times).

We also introduce some notation. Given a compact convex set $L \subset \mathbb{R}^m \subset \mathbb{R}^n$, we denote by $W_j^{(n)}(L)$ the j -th quermassintegral of L in dimension n , which is defined by

$$W_j^{(n)}(L) = V(L; n-j, B_2^n; j).$$

In the case where the j -th quermassintegral of L is taken in dimension m our notation will be the usual one. Namely,

$$W_j(L) = W_j^{(m)}(L) = V(L; m - j, B_2^m; j).$$

The next lemma gives the relation between $W_j^{(n)}(L)$ and $W_j^{(m)}(L)$. It is probably well-known but we include a proof for completeness.

Lemma 4.2. *Let K be a compact convex set in \mathbb{R}^n and $F \in G_{n, n-k}$. Then,*

$$W_{k+j}^{(n)}(K \cap F) = \frac{\omega_{k+j} \binom{n-k}{j}}{\omega_j \binom{n}{k+j}} W_j^{(n-k)}(K \cap F)$$

for all $j = 1, \dots, n - k$.

Proof. We can give a proof by induction on k . Note that if we consider a compact convex set $T \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ then

$$\begin{aligned} \sum_{j=0}^n \binom{n}{j} W_j^{(n)}(T) \lambda^j &= |T + \lambda B_2^n| = \int_{-\lambda}^{\lambda} |(T + \lambda B_2^n) \cap (\mathbb{R}^{n-1} + te_n)|_{n-1} dt \\ &= \int_{-\lambda}^{\lambda} |T + (\lambda^2 - t^2)^{1/2} B_2^{n-1}|_{n-1} dt \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} W_j^{(n-1)}(T) \int_{-\lambda}^{\lambda} (\lambda^2 - t^2)^{j/2} dt \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} W_j^{(n-1)}(T) \lambda^{j+1} \frac{\omega_{j+1}}{\omega_j}. \end{aligned}$$

Comparing the coefficients of the two polynomials we see that

$$W_{j+1}^{(n)}(T) = \frac{\omega_{j+1}}{\omega_j} \frac{\binom{n-1}{j}}{\binom{n}{j+1}} W_j^{(n-1)}(T)$$

for all $j = 1, \dots, n - 1$. Now, induction shows that if $T \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ then

$$W_{k+j}^{(n)}(T) = \frac{\omega_{k+j} \binom{n-k}{j}}{\omega_j \binom{n}{k+j}} W_j^{(n-k)}(T)$$

for all $j = 1, \dots, n - k$. The lemma follows if we identify $F \in G_{n, n-k}$ with \mathbb{R}^{n-k} and apply the above formula to $T = K \cap F$. \square

With these tools we are able to prove the next theorem.

Theorem 4.3. *Let K be a centrally symmetric convex body in \mathbb{R}^n . For all $0 \leq j \leq n - k - 1 \leq n - 1$ we have that*

$$c_{n, k, j} |K|^{n-k} W_{k+j}(K) \leq \int_K \cdots \int_K W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) dx_{n-k} \cdots dx_1 \leq \delta_{n, k, j} |K|^{n-k} W_{k+j}(K),$$

where $c_{n, k, j} = \frac{\omega_j}{\omega_{k+j} \omega_{n-k-j}} \frac{1}{\binom{n-k}{j}} \mathcal{F}_{C, 1}(\mathbb{1}_{B_2^{n-k-j}}; n - k)$ with $C = \text{conv}\{0, e_1, \dots, e_{n-k}\}$ and

$$\delta_{n, k, j} = \frac{\omega_j}{\omega_{k+j}} \frac{\binom{n}{k+j}}{\binom{n-k}{j}}.$$

Proof. First we prove the left hand side inequality. Let $F \in G_{n,n-k-j}$ and define $f : F \rightarrow \mathbb{R}^+$ by

$$f(x) = \pi_F(\mathbb{1}_K)(x) = |K \cap (x + F^\perp)|.$$

Then $\|f\|_1 = \int_F |f(x)| dx = \int_F |K \cap (x + F^\perp)| dx = |K|$ and $\|f\|_\infty = |K \cap F^\perp|$. We set $d = n - k - j$, $q = n - k$, $p = 1$, $C = \text{conv}\{0, e_1, \dots, e_{n-k}\}$ and $m = \min\{n - k, n - k - j, n - k\} = n - k - j$. From Theorem 4.1 we have that

$$\begin{aligned} & \frac{1}{|K|^{n-k}} \int_K \cdots \int_K |P_F(\text{conv}\{0, x_1, \dots, x_{n-k}\})| dx_{n-k} \cdots dx_1 \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\text{conv}\{0, P_F(x_1), \dots, P_F(x_{n-k})\}| \prod_{j=1}^{n-k} \frac{\mathbb{1}_K(x_j)}{|K|} dx_{n-k} \cdots dx_1 \\ &= \int_F \cdots \int_F |\text{conv}\{0, x_1, \dots, x_{n-k}\}| \prod_{j=1}^{n-k} \frac{\pi_F(\mathbb{1}_K)(x_j)}{|K|} dx_{n-k} \cdots dx_1 \\ &= \int_F \cdots \int_F |\text{conv}\{0, x_1, \dots, x_{n-k}\}| \prod_{j=1}^{n-k} \frac{f(x_j)}{\|f\|_1} dx_{n-k} \cdots dx_1 \\ &= \mathcal{F}_{C,1}(f; n - k) \geq \left(\frac{\|f\|_1}{\omega_{n-k-j} \|f\|_\infty} \right)^{\frac{n-k-j}{n-k-j}} \mathcal{F}_{C,1}(\mathbb{1}_{B_2^{n-k-j}}; n - k) \\ &= \frac{F_{C,1}(\mathbb{1}_{B_2^{n-k-j}}; n - k)}{\omega_{n-k-j}} \frac{|K|}{|K \cap F^\perp|} \\ &\geq \frac{F_{C,1}(\mathbb{1}_{B_2^{n-k-j}}; n - k)}{\omega_{n-k-j} \binom{n}{n-k-j}} |P_F(K)|, \end{aligned}$$

where the last inequality comes from the Rogers-Shephard inequality $|K| \leq |P_F(K)| |K \cap F^\perp|$. Integrating both sides of this inequality over $G_{n,n-k-j}$ we get

$$\frac{F_{C,1}(\mathbb{1}_{B_2^{n-k-j}}; n - k)}{\omega_{n-k-j}} |K|^{n-k} W_{k+j}(K) \leq \int_K \cdots \int_K W_{k+j}^{(n)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) dx_{n-k} \cdots dx_1.$$

Using the formula of Lemma 4.2 we obtain the result.

For the right hand side inequality we observe that $P_F(\text{conv}\{0, x_1, \dots, x_{n-k}\}) \subseteq P_F(K)$ for every $x_1, \dots, x_{n-k} \in K$, therefore we may write

$$\begin{aligned} & \int_K \cdots \int_K |P_F(\text{conv}\{0, x_1, \dots, x_{n-k}\})| dx_{n-k} \cdots dx_1 \leq \int_K \cdots \int_K |P_F(K)| dx_{n-k} \cdots dx_1 \\ &= |K|^{n-k} |P_F(K)|. \end{aligned}$$

Integrating both sides of this inequality over $G_{n,n-k-j}$ as before, we get

$$\int_K \cdots \int_K W_{k+j}^{(n)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) dx_{n-k} \cdots dx_1 \leq |K|^{n-k} W_{k+j}(K),$$

and using the formula of Lemma 4.2 we obtain the result. \square

We shall combine Theorem 4.3 with the generalized Blaschke-Petkantschin formula (see [20, Theorem 7.2]).

Lemma 4.4. Let $1 \leq s \leq n-1$. There exists a constant $p(n, s) > 0$ such that, for any non negative bounded Borel measurable function $g : (\mathbb{R}^n)^s \rightarrow \mathbb{R}$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} g(x_1, \dots, x_s) dx_s \cdots dx_1 \\ &= p(n, s) \int_{G_{n,s}} \int_F \cdots \int_F g(x_1, \dots, x_s) |\text{conv}\{0, x_1, \dots, x_s\}|^{n-s} dx_s \cdots dx_1 d\nu_{n,s}(F). \end{aligned}$$

Theorem 4.5. Let K be a centrally symmetric convex body in \mathbb{R}^n . For every $0 \leq j \leq n-k-1 \leq n-1$ we have that

$$W_{k+j}(K) \leq c_{n,k,j}^{-1} \max_{F \in G_{n,n-k}} W_j(K \cap F)$$

where $c_{n,k,j}$ is the constant of Theorem 4.3.

Proof. From the left hand side inequality of Theorem 4.3 and the Blaschke-Petkantschin formula with $s = n-k$ we get

$$\begin{aligned} c_{n,k,j} |K|^{n-k} W_{k+j}(K) &\leq \int_K \cdots \int_K W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) dx_{n-k} \cdots dx_1 \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) \prod_{i=1}^{n-k} \mathbb{1}_K(x_i) dx_{n-k} \cdots dx_1 \\ &= p(n, n-k) \int_{G_{n,n-k}} \int_F \cdots \int_F W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) |\text{conv}\{0, x_1, \dots, x_{n-k}\}|^k \prod_{i=1}^{n-k} \mathbb{1}_K(x_i) \\ &\quad dx_{n-k} \cdots dx_1 d\nu_{n,n-k}(F) \\ &= p(n, n-k) \int_{G_{n,n-k}} \int_{K \cap F} \cdots \int_{K \cap F} W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) |\text{conv}\{0, x_1, \dots, x_{n-k}\}|^k \\ &\quad dx_{n-k} \cdots dx_1 d\nu_{n,n-k}(F) \\ &\leq p(n, n-k) \int_{G_{n,n-k}} W_j(K \cap F) \int_{K \cap F} \cdots \int_{K \cap F} |\text{conv}\{0, x_1, \dots, x_{n-k}\}|^k dx_{n-k} \cdots dx_1 d\nu_{n,n-k}(F) \\ &\leq \max_{F \in G_{n,n-k}} W_j(K \cap F) \cdot p(n, n-k) \int_{G_{n,n-k}} \int_{K \cap F} \cdots \int_{K \cap F} |\text{conv}\{0, x_1, \dots, x_{n-k}\}|^k \\ &\quad dx_{n-k} \cdots dx_1 d\nu_{n,n-k}(F) \\ &= |K|^{n-k} \max_{F \in G_{n,n-k}} W_j(K \cap F), \end{aligned}$$

which gives the claim of the theorem. \square

Now, Theorem 1.4 follows from Theorem 4.5 and Aleksandrov's inequalities.

Proof of Theorem 1.4. Let K be a centrally symmetric convex body in \mathbb{R}^n and $0 \leq j \leq n-k-1 \leq n-1$. By Aleksandrov's inequalities we have

$$(4.1) \quad \omega_n^k W_j(K)^{n-k-j} \leq W_{k+j}(K)^{n-j}.$$

Combining this fact with the inequality of Theorem 4.5 we get

$$W_j(K)^{n-k-j} \leq (\omega_n^k c_{n,k,j}^{n-j})^{-1} \max_{F \in G_{n,n-k}} W_j(K \cap F)^{n-j}$$

and the proof is complete. \square

Assuming that K is centered, we may repeat the proof of Theorem 4.3, the only difference being that for the function $f(x) = \pi_F(\mathbb{1}_K)(x) = |K \cap (x + F^\perp)|$ we have

$$\|f\|_\infty \leq \left(\frac{n+1}{k+j+1}\right)^{k+j} |K \cap F^\perp|$$

by Fradelizi's inequality (3.5). Then, we have the following analogues of Theorem 4.5 and Theorem 1.4.

Corollary 4.6. *Let K be a centered convex body in \mathbb{R}^n . Then, for all $0 \leq j \leq n-k-1 \leq n-1$ we have that*

$$c'_{n,k,j} |K|^{n-k} W_{k+j}(K) \leq \int_K \cdots \int_K W_j^{(n-k)}(\text{conv}\{0, x_1, \dots, x_{n-k}\}) dx_{n-k} \cdots dx_1 \leq \delta_{n,k,j} |K|^{n-k} W_{k+j}(K),$$

where $c'_{n,k,j} = c_{n,k,j} \left(\frac{n+1}{k+j+1}\right)^{-(k+j)}$ and $\delta_{n,k,j}$ is the constant in Theorem 4.3.

Corollary 4.7. *Let K be a centered convex body in \mathbb{R}^n . Then, for all $0 \leq j \leq n-k-1 \leq n-1$ we have that*

$$W_{k+j}(K) \leq (c'_{n,k,j})^{-1} \max_{E \in G_{n,n-k}} W_j(K \cap E)$$

and

$$W_j(K)^{n-k-j} \leq (\omega_n^k c'_{n,k,j}{}^{n-j})^{-1} \max_{E \in G_{n,n-k}} W_j(K \cap E)^{n-j}$$

where $c'_{n,k,j}$ is the constant in Corollary 4.6.

Following the proof of Theorem 4.3 we can also obtain the following.

Theorem 4.8. *Let K be a centrally symmetric convex body in \mathbb{R}^n . Then, for all $0 \leq j \leq n-1$ and $N \geq n+1$ we have that*

$$(4.2) \quad c_{n,N,j} |K|^N W_j(K) \leq \int_K \cdots \int_K W_j(\text{conv}\{x_1, \dots, x_N\}) dx_N \cdots dx_1$$

where $c_{n,N,j} = \frac{1}{\omega_{n-j} \binom{n}{j}} \mathcal{F}_{C,1}(\mathbb{1}_{B_2^{n-j}}; N)$ with $C = \text{conv}\{e_1, \dots, e_N\}$. If K is not assumed centrally symmetric then we obtain a similar inequality with constants $c'_{n,N,j} = c_{n,N,j} \left(\frac{n+1}{j+1}\right)^{-j}$.

Remark 4.9. Hartzoulaki and Paouris proved in [12] the inequality

$$(4.3) \quad A_{n,N,j,p} |K|^{N+\frac{(n-j)p}{n}} \leq \int_K \cdots \int_K W_j(\text{conv}\{x_1, \dots, x_N\})^p dx_1 \cdots dx_N$$

for every convex body K in \mathbb{R}^n and any $N \geq n+1$ and $p > 0$, where

$$A_{n,N,j,p} = \int_{D_n} \cdots \int_{D_n} W_j(\text{conv}\{x_1, \dots, x_N\})^p dx_1 \cdots dx_N$$

and D_n is the Euclidean ball of volume 1 in \mathbb{R}^n . From the Aleksandrov-Fenchel inequality

$$\omega_n^{\frac{j}{n}} |K|^{\frac{n-j}{n}} \leq W_j(K)$$

and (4.2) we see that in the case $p = 1$, apart from the values of the constants $c_{n,N,j}$ and $c'_{n,N,j}$, Theorem 4.8 provides a strengthened version of (4.3) as explained in the introduction.

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DIMITRIS-MARIOS LIAKOPOULOS: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis 157-84, Athens, Greece.

E-mail: dimliakol@gmail.com