On the mean value of the area of a random polygon in a plane convex body

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Abstract

Let K be a convex body in $\mathbb{R}^2$ with area $A(K) = 1$. For every $n \geq 3$ we consider the expected value $m(K,n)$ of the area of the convex hull of n points chosen uniformly from K:

$$m(K,n) = \int_{y_1 \in K} \cdots \int_{y_n \in K} A(\text{co}\{y_1, \dots, y_n\})\, dy_n \dots dy_1.$$

We prove that for every $n \geq 3$, $m(K,n)$ is maximized (over all bodies of area 1) if and only if K is a triangle.

1 Introduction

Let K be a convex body in Euclidean space $\mathbb{R}^d, d \geq 2$, with volume $V(K) = 1$, and $n \geq d+1$ be a natural number. We select n independent random points $y_1, y_2, \dots, y_n$ from K (we assume they all have the uniform distribution in K). Their convex hull $\text{co}\{y_1, y_2, \dots, y_n\}$ is a random polytope in K with at most n vertices. Consider the expected value of the volume of this polytope

$$m(K,n) = \int_{y_1 \in K} \cdots \int_{y_n \in K} V(\text{co}\{y_1, \dots, y_n\}) dy_n \dots dy_1. \tag{1}$$

It is easy to see that if $U : \mathbb{R}^d \to \mathbb{R}^d$ is a volume preserving affine transformation, then for every convex body K with $V(K) = 1$, $m(K,n) = m(U(K), n)$.

It is also well known (see John [7]), that there exists a constant $C(d)$, depending only on the dimension d of the space, such that if K is a convex body in $\mathbb{R}^d$ with $V(K) = 1$, then there is a volume preserving affine transformation U with $U(K) \subseteq B(o, C(d))$, the ball with center at the origin o and radius $C(d)$.

From the compactness of the space of compact convex subsets of $B(o, C(d))$ with the Hausdorff metric and the fact that the functional $m : K \to m(K,n)$ is continuous in this metric (see Groemer [3]), it follows that there exist K_1, K_2 with

$$V(K_1) = V(K_2) = 1 \quad \text{and} \quad m(K_1, n) \leq m(K, n) \leq m(K_2, n),$$

for every convex body K in $\mathbb{R}^d$ with $V(K) = 1$.

The problem is to find those K which minimize or maximize this mean value $m(K, n)$, if $d \geq 2, n \geq d + 1$ are given.

Blaschke [1, 2] has proved that if $d = 2, n = 3$,

$$\frac{35}{48\pi^2} \leq m(K, n) \leq \frac{1}{12},$$

and we have equality on the left hand side only when K is an ellipse, while on the right hand side we have equality only when K is a triangle.

Groemer [3, 4] solved the problem of minimizing $m(K, n)$ by showing that: "if $d \geq 2, n \geq d+1$, then $m(K, n)$ attains its minimum value when, and only when, K is an ellipsoid".

In the opposite direction, Dalla and Larman [5] showed that for $d = 2$, and for every $n \geq 3$, $m(K, n) \leq m(T, n)$ for every plane convex body with area $A(K) = 1$, where T is a triangle with $A(T) = 1$. They also showed that the inequality is strict if K is a polygon with more than three vertices.

We shall complete this last result, by proving in a different way that the inequality is strict whenever K is a plane convex body which is not a triangle. That is, we prove the following.

Theorem. *Let K be a plane convex body with area $A(K) = 1$. Then, if T is a triangle with $A(T) = 1$, and $n \geq 3$,*

$$m(K, n) < m(T, n),$$

unless K too is a triangle, in which case equality clearly holds.

Let us say a few words about the proof. If K is any plane convex body and G is any line in the plane, we write $L = P_G(K)$ for the orthogonal projection of K onto G. We may assumee that G is the x-axis of the plane and, taking a line G' parallel to G if needed, that K is contained in the positive halfplane. So,

$$K = \{y = (x, t) : a \leq x \leq b,\ f(x) \leq t \leq g(x)\}, \tag{2}$$

where f is convex, g is concave, and $0 \leq f \leq g$ on $L = [a, b]$.

Consider the transformation $S_G : K \to K_G$, where

$$K_G = \{y = (x, t) : a \leq x \leq b,\ 0 \leq t \leq g(x) - f(x)\}.$$

It is clear that K_G is a plane convex body and easy to see that $A(K_G) = A(K)$ (S_G is known as the Schüttelung operation).

In Section 2 we prove that the mean value $m(K, n)$ increases under the transformation S_G. More precisely we have

Proposition 1. *For every line G in the plane, and every plane convex body K, if $n \geq 3$ then*

$$m(K,n) \leq m(K_G,n). \tag{3}$$

In Section 3 we answer the question of strict inequality in (3). The key step is the following.

Proposition 2. *If K is not a triangle, then there exists a line G in the plane such that for every $n \geq 3$*

$$m(K,n) < m(K_G,n). \tag{4}$$

Proposition 2 and our remarks on the existence of a "maximizing" K imply our Theorem.

2 The mean value $m(K,n)$ increases under the transformation S_G

In what follows, we assume that K is in the form (2). If $x \in [a,b]$ we denote by H_x the line which is perpendicular to G and passes through x. Then, (1) becomes

$$m(K,n) = \int_{y_1=(x_1,t_1)\in K} \cdots \int_{y_n=(x_n,t_n)\in K} A(\mathrm{co}\{(x_1,t_1),\ldots,(x_n,t_n)\})dy_n \ldots dy_1$$

$$= \int_a^b \cdots \int_a^b \left[\int_{t_1\in H_{x_1}\cap K} \cdots \int_{t_n\in H_{x_n}\cap K} A(\mathrm{co}\{(x_i,t_i), i \leq n\})dt_n \ldots dt_1 \right] dx_n \ldots dx_1.$$

If $x_1 < x_2 < \ldots < x_n$, we define

$$M(x_1,\ldots,x_n) = \int_{t_1\in H_{x_1}\cap K} \cdots \int_{t_n\in H_{x_n}\cap K} A(\mathrm{co}\{(x_1,t_1),\ldots,(x_n,t_n)\})dt_n \ldots dt_1.$$

Since the set of $\{(x_1,t_1),\ldots,(x_n,t_n)\}$ for which $x_i = x_j$ for some $i \neq j$ is of measure zero in K^n, in order to prove Proposition 1 it suffices to prove that

$$M(x_1,\ldots,x_n) \leq M_G(x_1,\ldots,x_n), \tag{5}$$

where,

$$M_G(x_1,\ldots,x_n) = \int_{t_1\in H_{x_1}\cap K_G} \cdots \int_{t_n\in H_{x_n}\cap K_G} A(\mathrm{co}\{(x_1,t_1),\ldots,(x_n,t_n)\})dt_n \ldots dt_1.$$

Let $l_i = p_i' = (g(x_i) - f(x_i))/2$ (half of the length of $H_{x_i} \cap K$ or $H_{x_i} \cap K_G$) and $p_i = (g(x_i) + f(x_i))/2, i = 1, 2, \ldots, n$. Then,

$$M(x_1, \ldots, x_n) = \int_{|t_1 - p_1| \leq l_1} \cdots \int_{|t_n - p_n| \leq l_n} A(\mathrm{co}\{(x_1, t_1), \ldots, (x_n, t_n)\}) dt_n \ldots dt_1$$

$$= \int_{|z_1| \leq l_1} \cdots \int_{|z_n| \leq l_n} A(\mathrm{co}\{(x_1, p_1 + z_1), \ldots, (x_n, p_n + z_n)\}) dz_n \ldots dz_1$$

$$= \int_{|z_1| \leq l_1} \cdots \int_{|z_n| \leq l_n} A(\mathrm{co}\{(x_1, p_1 - z_1), \ldots, (x_n, p_n - z_n)\}) dz_n \ldots dz_1$$

$$= \frac{1}{2} \int_{|z_1| \leq l_1} \cdots \int_{|z_n| \leq l_n} [A(\mathrm{co}\{(x_i, p_i + z_i)\}) + A(\mathrm{co}\{(x_i, p_i - z_i)\})] dz_n \ldots dz_1.$$

In exactly the same way, we get

$$M_G(x_1, \ldots, x_n) = \frac{1}{2} \int_{|z_1| \leq l_1} \cdots \int_{|z_n| \leq l_n} [A(\mathrm{co}\{(x_i, p_i' + z_i)\}) + A(\mathrm{co}\{(x_i, p_i' - z_i)\})].$$

So, (5) will be true if for every $z_1, \ldots, z_n$ with $|z_i| \leq l_i$, the following inequality holds:

$$A(\mathrm{co}\{(x_i, p_i + z_i)\}) + A(\mathrm{co}\{(x_i, p_i - z_i)\}) \leq A(\mathrm{co}\{(x_i, p_i' + z_i)\}) + A(\mathrm{co}\{(x_i, p_i' - z_i)\}). \tag{6}$$

After these preliminary remarks, we pass to the

Proof of Proposition 1. Let $|z_i| \leq l_i, i = 1, \ldots, n$. Then, we can find $\lambda_i \in [0, 1], i = 1, \ldots, n$ and $\mu_i = 1 - \lambda_i$ such that

$$p_i + z_i = \lambda_i f(x_i) + \mu_i g(x_i).$$

It is easy to see that

$$p_i - z_i = \mu_i f(x_i) + \lambda_i g(x_i),$$
$$p_i' + z_i = \mu_i g(x_i) - \mu_i f(x_i),$$
$$p_i' - z_i = \lambda_i g(x_i) - \lambda_i f(x_i).$$

For the proof of Proposition 1 it suffices to show that

$$A(R) + A(Q) \leq A(R') + A(Q'), \tag{7}$$

where

$$R = \mathrm{co}\{(x_i, \lambda_i f(x_i) + \mu_i g(x_i))\},$$
$$Q = \mathrm{co}\{(x_i, \mu_i f(x_i) + \lambda_i g(x_i))\},$$
$$R' = \mathrm{co}\{(x_i, \mu_i g(x_i) - \mu_i f(x_i))\},$$

$$Q' = \mathrm{co}\{(x_i, \lambda_i g(x_i) - \lambda_i f(x_i))\}.$$

In general, if $X = \mathrm{co}\{(x_i, t_i), i = 1, \dots, n\}$ we shall say that:

(i) (x_i, t_i) is an upper vertex of X, if

$$j < i < k \quad \Rightarrow \quad t_i > \frac{x_k - x_i}{x_k - x_j} t_j + \frac{x_i - x_j}{x_k - x_j} t_k;$$

(ii) (x_i, t_i) is a lower vertex of X, if

$$j < i < k \quad \Rightarrow \quad t_i < \frac{x_k - x_i}{x_k - x_j} t_j + \frac{x_i - x_j}{x_k - x_j} t_k.$$

With this definition, (x_1, t_1) and (x_n, t_n) are both upper and lower vertices of X. If $I \subseteq \{1, 2, \dots, n\}$ is of the form $I = \{i_0 = 1 < i_1 < \dots < i_{k-1} < i_k = n\}$, we define

$$E_X(I) = \frac{1}{2} \sum_{s=1}^{k} (x_{i_s} - x_{i_{s-1}})(t_{i_{s-1}} + t_{i_s}),$$

the area between the x-axis and the broken line with vertices $(x_i, t_i), i \in I$. In this notation, if I is the index set of the upper vertices of X and J is the index set of the lower vertices of X, we note that

$$A(X) = E_X(I) - E_X(J).$$

Finally, for any function $h : [a, b] \to \mathbb{R}$ and $I = \{i_0 = 1 < i_1 < \dots < i_{k-1} < i_k = n\}$, we write

$$E_h(I) = \frac{1}{2} \sum_{s=1}^{k} (x_{i_s} - x_{i_{s-1}})(h(x_{i_{s-1}}) + h(x_{i_s})).$$

Lemma 1. *Let I, J and K, L be the index sets of the upper and lower vertices of R and Q respectively, and I', J' and K', L' be the index sets of the upper and lower vertices of R' and Q' respectively. Then,*

(α) $I \cup L \subseteq I'$, $K \cup J \subseteq K'$,

(β) $I \cap L \supseteq L'$, $K \cap J \supseteq J'$.

Proof: If $i \in I$ and $\rho < i < \sigma,\ \rho, \sigma \in \{1, 2, \dots, n\}$, we have

$$\lambda_i f(x_i) + \mu_i g(x_i) > \frac{x_\sigma - x_i}{x_\sigma - x_\rho}(\lambda_\rho f(x_\rho) + \mu_\rho g(x_\rho)) + \frac{x_i - x_\rho}{x_\sigma - x_\rho}(\lambda_\sigma f(x_\sigma) + \mu_\sigma g(x_\sigma))$$

and

$$-f(x_i) \geq \frac{x_\sigma - x_i}{x_\sigma - x_\rho}(-f(x_\rho)) + \frac{x_i - x_\rho}{x_\sigma - x_\rho}(-f(x_\sigma)).$$

So,

$$\mu_i g(x_i) - \mu_i f(x_i) > \frac{x_\sigma - x_i}{x_\sigma - x_\rho}(\mu_\rho g(x_\rho) - \mu_\rho f(x_\rho)) + \frac{x_i - x_\rho}{x_\sigma - x_\rho}(\mu_\sigma g(x_\sigma) - \mu_\sigma f(x_\sigma)).$$

Thus $i \in I'$ and hence $I \subseteq I'$. It is equally easy to see that

$$I \subseteq I', \quad K \subseteq K', \quad L \supseteq L', \quad J \supseteq J'. \tag{8}$$

Next, we define the sets

$$R'' = \text{co}\{(x_i, \lambda_i f(x_i) + \mu_i g(x_i) - g(x_i))\} = \text{co}\{(x_i, \lambda_i f(x_i) - \lambda_i g(x_i))\} = -Q'$$

and

$$Q'' = \text{co}\{(x_i, \mu_i f(x_i) + \lambda_i g(x_i) - g(x_i))\} = \text{co}\{(x_i, \mu_i f(x_i) - \mu_i g(x_i))\} = -R'.$$

If I'', J'' and K'', L'' are the index sets of the upper and lower vertices of R'' and Q'' respectively, it is clear that

$$I'' = L', \quad J'' = K', \quad K'' = J', \quad L'' = I'. \tag{9}$$

But, just as in the proof of (8), one can see that

$$J \subseteq J'', \;\; L \subseteq L'', \;\; K \supseteq K'', \;\; I \supseteq I''. \tag{10}$$

For example, if $j \in J$ and $\rho < j < \sigma, \; \rho, \sigma \in \{1, 2, \ldots, n\}$ we have

$$\lambda_j f(x_j) + \mu_j g(x_j) < \frac{x_\sigma - x_j}{x_\sigma - x_\rho}(\lambda_\rho f(x_\rho) + \mu_\rho g(x_\rho)) + \frac{x_j - x_\rho}{x_\sigma - x_\rho}(\lambda_\sigma f(x_\sigma) + \mu_\sigma g(x_\sigma))$$

and

$$-g(x_j) \leq \frac{x_\sigma - x_j}{x_\sigma - x_\rho}(-g(x_\rho)) + \frac{x_j - x_\rho}{x_\sigma - x_\rho}(-g(x_\sigma)).$$

So,

$$\lambda_j f(x_j) - \lambda_j g(x_j) < \frac{x_\sigma - x_j}{x_\sigma - x_\rho}(\lambda_\rho f(x_\rho) - \lambda_\rho g(x_\rho)) + \frac{x_j - x_\rho}{x_\sigma - x_\rho}(\lambda_\sigma f(x_\sigma) - \lambda_\sigma g(x_\sigma)).$$

That is, $j \in J''$ and $J \subseteq J''$. Inclusions (8), (9) and (10) imply our Lemma 1. □

We continue with the proof of (7), namely

$$A(R) + A(Q) \leq A(R') + A(Q'),$$

or, equivalently,

$$E_R(I) - E_R(J) + E_Q(K) - E_Q(L) \leq E_{R'}(I') - E_{R'}(J') + E_{Q'}(K') - E_{Q'}(L').$$

It suffices to show that

$$E_R(I) - E_Q(L) \le E_{R'}(I') - E_{Q'}(L'), \tag{11}$$

$$E_Q(K) - E_R(J) \le E_{Q'}(K') - E_{R'}(J'), \tag{12}$$

and this is accomplished in the following

Lemma 2. *If I, J, K, L and I', J', K', L' are as in Lemma 1, then inequalities* (11) *and* (12) *hold.*

Proof: Both inequalities are proved in the same way, so we restrict ourselves to the proof of

$$E_R(I) + E_{Q'}(L') \le E_{R'}(I') + E_Q(L).$$

Since $I \cup L$ is a subset of I' and the points $(x_{i'}, \mu_{i'} g(x_{i'}) - \mu_{i'} f(x_{i'})), i' \in I'$ are in a concave position (they are the upper vertices of R'), we have

$$E_{R'}(I') \ge E_{R'}(I \cup L).$$

So it is enough to prove

$$E_R(I) + E_{Q'}(L') \le E_{R'}(I \cup L) + E_Q(L). \tag{13}$$

The four regions in (13) are bounded by the segment $[x_1, x_n]$ on the x-axis, the lines $x = x_1$, $x = x_n$, and the four broken lines $c_i : [x_1, x_n] \to \mathbb{R}, i = 1, 2, 3, 4$, with

c_1 having vertices at the points $(x_i, \lambda_i f(x_i) + \mu_i g(x_i)), i \in I$,
c_2 having vertices at the points $(x_i, \lambda_i g(x_i) - \lambda_i f(x_i)), i \in L'$,
c_3 having vertices at the points $(x_i, \mu_i g(x_i) - \mu_i f(x_i)), i \in I \cup L$,
c_4 having vertices at the points $(x_i, \lambda_i g(x_i) + \mu_i f(x_i)), i \in L$.

Also, from Lemma 1, $L' \subseteq I \cap L$ $(\subseteq I \cup L)$. If k_s, k_{s+1} are consecutive indices from $I \cup L$, all four c_i are linear on $[x_{k_s}, x_{k_{s+1}}]$. It follows that (13) will be true if for every $k \in I \cup L$,

$$c_1(x_k) + c_2(x_k) \le c_3(x_k) + c_4(x_k). \tag{14}$$

Let $L' = \{\rho_0 = 1 < \rho_1 < \ldots < \rho_\nu = n\}$. If $\rho_s < \rho_{s+1}$ are two consecutive indices from L', we shall verify (14) for every $k \in I \cup L$ with $\rho_s \le k \le \rho_{s+1}$.

We distinguish four cases:

(α) $k = \rho_s$ or $k = \rho_{s+1}$. Then $k \in I, k \in L', k \in I \cup L$ and $k \in L$; so,

$$\begin{aligned} c_1(x_k) + c_2(x_k) &= \lambda_k f(x_k) + \mu_k g(x_k) + \lambda_k g(x_k) - \lambda_k f(x_k) = g(x_k) \\ &= \mu_k g(x_k) - \mu_k f(x_k) + \lambda_k g(x_k) + \mu_k f(x_k) \\ &= c_3(x_k) + c_4(x_k). \end{aligned}$$

(β) $k \in I \cap L \backslash L'$. Then $k \in I, k \in L, k \in I \cup L$; so,

$$c_2(x_k) = \frac{x_{\rho_{s+1}} - x_k}{x_{\rho_{s+1}} - x_{\rho_s}}(\lambda_{\rho_s} g(x_{\rho_s}) - \lambda_{\rho_s} f(x_{\rho_s}))$$

$$+\frac{x_k - x_{\rho_s}}{x_{\rho_{s+1}} - x_{\rho_s}}(\lambda_{\rho_{s+1}} g(x_{\rho_{s+1}}) - \lambda_{\rho_{s+1}} f(x_{\rho_{s+1}}))$$

and (14) becomes

$$\lambda_k f(x_k) + \mu_k g(x_k) + c_2(x_k) \le \mu_k g(x_k) - \mu_k f(x_k) + \lambda_k g(x_k) + \mu_k f(x_k)$$

or, equivalently,

$$c_2(x_k) \le \lambda_k g(x_k) - \lambda_k f(x_k).$$

The last inequality holds because the points $(x_{\rho_s}, \lambda_{\rho_s} g(x_{\rho_s}) - \lambda_{\rho_s} f(x_{\rho_s}))$ and $(x_{\rho_{s+1}}, \lambda_{\rho_{s+1}}$
$g(x_{\rho_{s+1}}) - \lambda_{\rho_{s+1}} f(x_{\rho_{s+1}}))$ are consecutive lower vertices of Q', $\rho_s < k < \rho_{s+1}$, and the point $(x_k, \lambda_k g(x_k) - \lambda_k f(x_k))$ lies in Q' but it is not a lower vertex of it.

(γ) $k \in I \backslash L$. Then $k \in I, k \in I \cup L$. Let $l_\tau < l_{\tau+1}$ be two consecutive indices from L such that $\rho_s \le l_\tau < k < l_{\tau+1} \le \rho_{s+1}$. If $A = x_{\rho_{s+1}} - x_{\rho_s}$, $B = x_{l_\tau} - x_{\rho_s}$, $\Gamma = x_k - x_{l_\tau}$, $\Delta = x_{l_{\tau+1}} - x_k$, then

$$c_2(x_k) = \frac{A - B - \Gamma}{A}(\lambda_{\rho_s} g(x_{\rho_s}) - \lambda_{\rho_s} f(x_{\rho_s})) + \frac{B + \Gamma}{A}(\lambda_{\rho_{s+1}} g(x_{\rho_{s+1}}) - \lambda_{\rho_{s+1}} f(x_{\rho_{s+1}})),$$

and

$$c_4(x_k) = \frac{\Delta}{\Gamma + \Delta}(\lambda_{l_\tau} g(x_{l_\tau}) + \mu_{l_\tau} f(x_{l_\tau})) + \frac{\Gamma}{\Gamma + \Delta}(\lambda_{l_{\tau+1}} g(x_{l_{\tau+1}}) + \mu_{l_{\tau+1}} f(x_{l_{\tau+1}})),$$

and (14) becomes

$$\lambda_k f(x_k) + \mu_k g(x_k) + c_2(x_k) \le \mu_k g(x_k) - \mu_k f(x_k) + c_4(x_k),$$

i.e,

$$c_2(x_k) \le c_4(x_k) - f(x_k). \tag{15}$$

Since f is a convex function,

$$f(x_k) \le \frac{\Delta}{\Gamma + \Delta} f(x_{l_\tau}) + \frac{\Gamma}{\Gamma + \Delta} f(x_{l_{\tau+1}})$$

so, in order to verify (15), we only need to check that

$$c_2(x_k) \le \frac{\Delta}{\Gamma + \Delta}(\lambda_{l_\tau} g(x_{l_\tau}) - \lambda_{l_\tau} f(x_{l_\tau})) + \frac{\Gamma}{\Gamma + \Delta}(\lambda_{l_{\tau+1}} g(x_{l_{\tau+1}}) - \lambda_{l_{\tau+1}} f(x_{l_{\tau+1}})). \tag{16}$$

But as in case (β),

$$\lambda_{l_\tau} g(x_{l_\tau}) - \lambda_{l_\tau} f(x_{l_\tau}) \geq \frac{A-B}{A}(\lambda_{\rho_s} g(x_{\rho_s}) - \lambda_{\rho_s} f(x_{\rho_s})) + \frac{B}{A}(\lambda_{\rho_{s+1}} g(x_{\rho_{s+1}}) - \lambda_{\rho_{s+1}} f(x_{\rho_{s+1}})), \tag{17}$$

and

$$\lambda_{l_{\tau+1}} g(x_{l_{\tau+1}}) - \lambda_{l_{\tau+1}} f(x_{l_{\tau+1}}) \geq \frac{A-B-\Gamma-\Delta}{A}(\lambda_{\rho_s} g(x_{\rho_s}) - \lambda_{\rho_s} f(x_{\rho_s})) + \frac{B+\Gamma+\Delta}{A}(\lambda_{\rho_{s+1}} g(x_{\rho_{s+1}}) - \lambda_{\rho_{s+1}} f(x_{\rho_{s+1}})), \tag{18}$$

with equality in (17), (18) if $l_\tau = \rho_s$ or $l_{\tau+1} = \rho_{s+1}$ respectively and strict inequality otherwise. From (17), (18) we conclude that (16) holds.

(δ) $k \in L \backslash I$. Then $k \in L,\ k \in I \cup L$. We find $i_\tau < i_{\tau+1}$ two consecutive indices from I so that $\rho_s \leq i_\tau < k < i_{\tau+1} \leq \rho_{s+1}$. Then we compute $c_1(x_k),\ c_2(x_k)$ and proceed as in case (γ). □

By Lemma 2, inequality (7) is true. So, we have proved Proposition 1.

3 If K is not a triangle, there is a line G such that $m(K,n)$ strictly increases under the transformation S_G

Let K, K_G be as in section 1. We define

$$R_0 = Q_0 = \mathrm{co}\{(x_i, p_i)\} = \mathrm{co}\{(x_i, \frac{f(x_i)+g(x_i)}{2})\},$$

with I_0 the index set of its upper vertices and J_0 the index set of its lower vertices, and

$$R_0' = Q_0' = \mathrm{co}\{(x_i, p_i')\} = \mathrm{co}\{(x_i, \frac{g(x_i)-f(x_i)}{2})\},$$

with I_0' the index set of its upper vertices and J_0' the index set of its lower vertices.

In the next lemma we find necessary conditions for $A(R_0) = A(R_0')$ to be true.

Lemma 3. *If $A(R_0) = A(R_0')$, then we have (α) and (β) below.*

(α) *The following conditions are all satisfied.*

(i) $(x_j, g(x_j)), j \in J_0$ *are collinear.*

(ii) $(x_i, f(x_i)), i \in I_0$ *are collinear.*

(iii) *If* $j \in J_0 \backslash I_0$ *and* $i_k < j < i_{k+1}$ *where* i_k, i_{k+1} *are two consecutive indices from* I_0*, then* $(x_{i_k}, g(x_{i_k}))$*,* $(x_j, g(x_j))$ *and* $(x_{i_{k+1}}, g(x_{i_{k+1}}))$ *are collinear.*

(iv) *If* $i \in I_0 \backslash J_0$ *and* $j_k < i < j_{k+1}$ *where* j_k, j_{k+1} *are two consecutive indices from* J_0*, then* $(x_{j_k}, f(x_{j_k}))$*,* $(x_i, f(x_i))$ *and* $(x_{j_{k+1}}, f(x_{j_{k+1}}))$ *are collinear.*

(β) $I'_0 = I_0 \cup J_0$.

Proof: (α) From Lemma 1, $I_0 \cup J_0 \subseteq I'_0$ (set $\lambda_i = \mu_i = 1/2$, $i = 1, \ldots, n$). We also have $J'_0 = \{1, n\}$ because the points (x_i, p'_i), $i = 1, \ldots, n$ of R'_0 are in a concave position. Since $A(R_0) = A(R'_0)$, we must have

$$E_{R_0}(I_0) - E_{R_0}(J_0) = E_{R'_0}(I'_0) - E_{R'_0}(J'_0)$$

or, equivalently,

$$E_f(I_0) + E_g(I_0) - E_f(J_0) - E_g(J_0) = E_g(I'_0) - E_f(I'_0) + E_f(J'_0) - E_g(J'_0),$$

i.e,

$$[E_g(I'_0) - E_g(I_0)] + [E_g(J_0) - E_g(J'_0)] + [E_f(J'_0) - E_f(I_0)] \\ + [E_f(J_0) - E_f(I'_0)] = 0.$$

Since f is convex, g is concave and $I'_0 \supseteq I_0 \cup J_0$, $I_0 \cap J_0 \supseteq J'_0$, the four summands in the above equality are non-negative. It follows that

(i)$'$ $E_g(J_0) = E_g(J'_0)$,

(ii)$'$ $E_f(J'_0) = E_f(I_0)$,

(iii)$'$ $E_g(I'_0) = E_g(I_0)$,

(iv)$'$ $E_f(J_0) = E_f(I'_0)$.

From condition (i)$'$, since $J'_0 = \{1, n\}$ and g is concave, $(x_j, g(x_j)), j \in J_0$ are collinear. From condition (ii)$'$, since $J'_0 = \{1, n\}$ and f is convex, $(x_i, f(x_i)), i \in I_0$ are collinear. Condition (iii)$'$, since $I'_0 \supseteq I_0$ and g is concave, implies that for any $j \in J_0 \backslash I_0$ the points $(x_{i_k}, g(x_{i_k})), (x_j, g(x_j))$ and $(x_{i_{k+1}}, g(x_{i_{k+1}}))$, where i_k, i_{k+1} consecutive indices from I_0 with $i_k < j < i_{k+1}$, are collinear.

Condition (iv)$'$, since $I'_0 \supseteq J_0$ and f is convex, implies that for any $i \in I_0 \backslash J_0$ the points $(x_{j_k}, f(x_{j_k})), (x_i, f(x_i))$ and $(x_{j_{k+1}}, f(x_{j_{k+1}}))$ where j_k, j_{k+1} are consecutive indices from J_0 with $j_k < i < j_{k+1}$, are collinear.

(β) From Lemma 1, $I'_0 \supseteq I_0 \cup J_0$. If $I_0 \cup J_0 = W$, and $W \neq I'_0$, then

$$A(R''_0) < A(R'_0),$$

where $R''_0 = \mathrm{co}\{(x_i, p'_i), i \in W\}$. Also, from inequality (7) (taking W instead of $\{1, \ldots, n\}$),

$$A(R_0) \leq A(R''_0).$$

This contradicts our hypothesis $A(R_0) = A(R'_0)$, and proves

$$I'_0 = I_0 \cup J_0. \quad \square$$

Proof of Proposition 2. Let K be a plane convex body with more than three extreme points and let A, B, C and D be four of them. Then A, B, C and D form a convex quadrilateral $ABCD$ in the plane. We choose G to be the perpendicular to the diagonal AC of $ABCD$.

We may assume that G is the x-axis and

$$K = \{y = (x,t) : a \leq x \leq b,\ f(x) \leq t \leq g(x)\},$$

where:

(i) $P_G(K) = [a,b]$;

(ii) $0 \leq f \leq g$, f is convex, g is concave on $[a,b]$;

(iii) $P_G(A) = P_G(C) = x$ and $a < x < b$; and

(iv) $A = (x, g(x)),\ C = (x, f(x))$.

Now, let $S_G : K \to K_G$, where

$$K_G = \{y = (x,t) : a \leq x \leq b,\ 0 \leq t \leq g(x) - f(x)\},$$

and choose $a \leq x_1^* < x_2^* = x < x_3^* < \ldots < x_n^* \leq b$ (we can do this because $a < x < b$).

Set $p_i^* = (f(x_i^*) + g(x_i^*))/2$ and $p_i^{*'} = (g(x_i^*) - f(x_i^*))/2$. We shall prove that

$$A(R_0^*) = A(\mathrm{co}\{(x_i^*, p_i^*)\}) < A(\mathrm{co}\{(x_i^*, p_i^{*'})\}) = A(R_0^{*'}). \tag{19}$$

Suppose that equality holds. From Lemma 3(β), we must have $I_0' = I_0 \cup J_0$. It is easy to see that $2 \in I_0'$. Let k be the next index from I_0', that is, 1,2 and k are the three first indices of I_0'. Since $1 \in I_0 \cap J_0$, we have to examine four cases:

(i) $\{1,2,k\} \subseteq J_0$. Then, by (i) of Lemma 3(α), the three points $(x_1^*, g(x_1^*))$, $(x, g(x)) = A$ and $(x_k^*, g(x_k^*))$ are collinear, which is false because A is an extreme point of K.

(ii) $\{1,2,k\} \subseteq I_0$. Then, by (ii) of Lemma 3(α), the three points $(x_1^*, f(x_1^*))$, $(x, f(x)) = C$ and $(x_k^*, f(x_k^*))$ are collinear, which is false because C is an extreme point of K.

(iii) $\{1,2\} \subseteq J_0$ and $\{1,k\} \subseteq I_0$. Then, by (iii) of Lemma 3(α), the three points $(x_1^*, g(x_1^*))$, $(x, g(x)) = A$ and $(x_k^*, g(x_k^*))$ are collinear, which is false because A is an extreme point of K.

(iv) $\{1,2\} \subseteq I_0$ and $\{1,k\} \subseteq J_0$. Then, by (iv) of Lemma 3(α), the three points $(x_1^*, f(x_1^*))$, $(x, f(x)) = C$ and $(x_k^*, f(x_k^*))$ are collinear, which is false because C is an extreme point of K.

So, (19) is true. But the integrands defining $M(x_1^*, \ldots, x_n^*)$ are continuous functions of $z_1, \ldots, z_n$ and satisfy for every $|z_i| \leq l_i$ the inequality (see (6) above)

$$A(\mathrm{co}\{(x_i^*, p_i^* + z_i)\}) + A(\mathrm{co}\{(x_i^*, p_i^* - z_i)\})$$
$$\leq A(\mathrm{co}\{(x_i^*, p_i^{*'} + z_i)\}) + A(\mathrm{co}\{(x_i^*, p_i^{*'} - z_i)\}).$$

We proved that this inequality is strict for $z_1 = \ldots = z_n = 0$, i.e,

$$A(\mathrm{co}\{(x_i^*, p_i^*)\}) < A(\mathrm{co}\{(x_i^*, p_i^{*'})\}),$$

and this implies that

$$M(x_1^*, \ldots, x_n^*) < M_G(x_1^*, \ldots, x_n^*).$$

But, for every $a \leq x_1 < \ldots < x_n \leq b$ we have

$$M(x_1, \ldots, x_n) \leq M_G(x_1, \ldots, x_n).$$

Since $M(x_1, \ldots, x_n)$ and $M_G(x_1, \ldots, x_n)$ are continuous functions of $x_1, \ldots, x_n$, (20) implies that

$$m(K, n) = \int_a^b \ldots \int_a^b M(x_1, \ldots, x_n)\, dx_n \ldots dx_1$$

$$< \int_a^b \ldots \int_a^b M_G(x_1, \ldots, x_n)\, dx_n \ldots dx_1 = m(K_G, n),$$

and the proof of Proposition 2 is complete. □

4 Remarks

(i) The case $n = 3$ is much simpler. If we define

$$\mathbf{e} = (1, 1, 1), \quad \mathbf{x} = (x_1, x_2, x_3), \quad \mathbf{z} = (z_1, z_2, z_3),$$

$$\mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{p}' = (p_1', p_2', p_3'),$$

and $D(z_1, z_2, z_3) = D(\mathbf{z}) = \frac{1}{2}\det(\mathbf{e}, \mathbf{x}, \mathbf{z})$, then

$$M(x_1, x_2, x_3) = \frac{1}{2} \int_{|z_1| \leq l_1} \ldots \int_{|z_3| \leq l_3} [|D(\mathbf{z} + \mathbf{p})| + |D(\mathbf{z} - \mathbf{p})|] dz_3 \ldots dz_1$$

$$= \frac{1}{2} \int_{|z_1| \leq l_1} \ldots \int_{|z_3| \leq l_3} [|D(\mathbf{z}) + D(\mathbf{p})| + |D(\mathbf{z}) - D(\mathbf{p})|] dz_3 \ldots dz_1$$

$$= \int_{|z_1| \leq l_1} \ldots \int_{|z_3| \leq l_3} \max[|D(\mathbf{z})|, |D(\mathbf{p})|] dz_3 \ldots dz_1$$

and

$$M_G(x_1, x_2, x_3) = \int_{|z_1|\le l_1} \dots \int_{|z_3|\le l_3} \max[|D(\mathbf{z})|, |D(\mathbf{p}')|] dz_3 \dots dz_1.$$

So, inequality (5) becomes

$$|D(\mathbf{p})| \le |D(\mathbf{p}')|.$$

We continue as in the proof of Proposition 2.

(ii) The crucial property of the triangle, related to our method of proof, seems to be the following: "if T is a triangle and G is any line in the plane, write T in the form (2):

$$T = \{y = (x, t) : a \le x \le b,\ f(x) \le t \le g(x)\}.$$

Then, either f or g must be linear on $P_G(T) = [a, b]$".

(iii) Buchta [8] has obtained the exact value of $m(T, n)$:

$$m(T, n) = 1 - \frac{2}{n+1} \sum_{k=1}^{n} \frac{1}{k}.$$

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