Distances between classical positions of centrally symmetric convex bodies

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Abstract

We study some classical positions (minimal surface area position, minimal mean width position, John's position, Löwner's position and the isotropic position) of a centrally symmetric convex body K in $\mathbb{R}^n$. Using their isotropic characterizations, we provide upper bounds for the "trace distance" of any two of them. Most of these bounds are of the order of $\sqrt{n}$.

1 Introduction

Let $\mathcal{SK}_{[n]}$ denote the class of all centrally symmetric convex bodies of volume 1 in $\mathbb{R}^n$ (in the sequel we call them just *symmetric* for simplicity). If $K \in \mathcal{SK}_{[n]}$ then the family of *positions* of K is the set $\{T(K) : T \in SL(n)\}$. The aim of this note is to discuss and to compare some of the classical positions of a body $K \in \mathcal{SK}_{[n]}$ which are extensively used in the study of finite dimensional normed spaces. A common feature of all these positions is that they appear as solutions of extremal problems of the following type: we are given a functional f on convex bodies and we ask for the maximum or minimum of the map $T \mapsto f(T(K))$ over all $T \in SL(n)$. The positions which are listed below appear as solutions of such problems:

(i) The *isotropic position* $K_{(i)}$ of K minimizes the functional

$$T \mapsto I_2(T(K)) = \left(\int_{T(K)} \|x\|_2^2 dx \right)^{1/2}.$$

(ii) The *minimal surface area position* $K_{(s)}$ of K minimizes the surface area functional $T \mapsto \partial(T(K))$.

(iii) The *minimal mean width position* $K_{(w)}$ of K minimizes the mean width functional $T \mapsto w(T(K))$.

(iv) *John's position* $K_{(j)}$ of K maximizes the inradius functional $T \mapsto r(T(K))$.

(v) *Löwner's position* $K_{(\ell)}$ of K minimizes the circumradius functional $T \mapsto R(T(K))$.

For relevant definitions of these functionals see Section 2.

A second important common feature of all these positions is that they admit an *isotropic characterization* (we provide background information in Section 2). Moreover, they are essentially uniquely determined; if $K_{(x)}$ is in one of these classical positions then $K'_{(x)}$ is in the same position if and only if there exists $U \in O(n)$ such that $K'_{(x)} = U(K_{(x)})$. In this note, positions which have these properties will be called *isotropic.*

Our aim is to study some natural notions of distance between positions of a body $K \in \mathcal{SK}_{[n]}$ and to provide upper bounds for them, for all possible pairs of the classical positions which were introduced above. We are mainly interested in the *trace distance* which we now define.

Definition 1.1. Let $K \in \mathcal{SK}_{[n]}$ and let $K_{(x)}$, $K_{(y)}$ be two isotropic positions of K. There exists $T \in SL(n)$ such that $T(K_{(x)}) = K_{(y)}$. Then, we define the *trace distance* of $K_{(x)}$ and $K_{(y)}$ by

$$d_{\rm tr}(K_{(x)}, K_{(y)}) := \frac{{\rm tr}(\sqrt{T^*T})}{n}. \tag{1.1}$$

Since all isotropic positions are uniquely determined up to orthogonal transformations, for any $U, V \in O(n)$ we have that UTV maps the position of $K_{(x)}$ to the position of $K_{(y)}$. So, we may assume that T is symmetric and positive definite, and hence $\sqrt{T^*T} = T$. Actually, we may assume that T is diagonal with positive entries $\lambda_1, \cdots, \lambda_n$. We can also arrange the entries λ_i in increasing order. This diagonal matrix is uniquely determined by the pair of positions we study. Finally, we define the symmetric distance

$$D_{\rm tr}(K_{(x)}, K_{(y)}) := \max\{d_{\rm tr}(K_{(x)}, K_{(y)}), d_{\rm tr}(K_{(y)}, K_{(x)})\}. \tag{1.2}$$

In Section 3 we collect general arguments that lead to upper bounds for the distance $d_{\rm tr}(K_{(x)}, K_{(y)})$. They all exploit the isotropic characterization of the classical positions. In order to give a flavor of the results, we list some of them in the next theorem.

Theorem 1.2. *Let $K \in \mathcal{SK}_{[n]}$. Then, we have*

$$\begin{aligned}
d_{\rm tr}(K_{(i)}, K_{(x)}) &\leqslant \frac{c_1 I_2(K_{(x)})}{I_2(K_{(i)})} &&\text{and}\quad d_{\rm tr}(K_{(x)}, K_{(i)}) \leqslant \frac{c_2\sqrt{n}}{r(K_{(x)})}\\
d_{\rm tr}(K_{(x)}, K_{(s)}) &\leqslant \frac{\partial(K_{(x)})}{\partial(K_{(s)})} &&\text{and}\quad d_{\rm tr}(K_{(s)}, K_{(x)}) \leqslant \frac{c_3 \partial(K_{(s)}) I_2(K_{(x)})}{n}\\
d_{\rm tr}(K_{(w)}, K_{(x)}) &\leqslant \frac{w(K_{(x)})}{w(K_{(w)})} &&\text{and}\quad d_{\rm tr}(K_{(x)}, K_{(w)}) \leqslant \frac{c_4 w(K_{(w)})}{r(K_{(x)})}\\
d_{\rm tr}(K_{(x)}, K_{(j)}) &\leqslant \frac{r(K_{(j)})}{r(K_{(x)})} &&\text{and}\quad d_{\rm tr}(K_{(j)}, K_{(x)}) \leqslant \frac{R(K_{(x)})}{r(K_{(j)})}\\
d_{\rm tr}(K_{(\ell)}, K_{(x)}) &\leqslant \frac{R(K_{(x)})}{R(K_{(\ell)})} &&\text{and}\quad d_{\rm tr}(K_{(x)}, K_{(\ell)}) \leqslant \frac{R(K_{(\ell)})}{r(K_{(x)})},
\end{aligned}$$

where $c_i > 0$ are absolute constants.

In Section 2 we recall some simple estimates on the basic geometric parameters of bodies in classical position. Combined with the estimates of Theorem 1.2, they lead to upper bounds for the trace distance. We provide short proofs in Section 4; the results are summarized in the next table.

Theorem 1.3. *For every $K \in \mathcal{SK}_{[n]}$ we have the following upper bounds for the distance $d_{\rm tr}(K_{(x)}, K_{(y)})$:*

	$K_{(i)}$	$K_{(s)}$	$K_{(w)}$	$K_{(j)}$	$K_{(\ell)}$
$K_{(i)}$	1	$\frac{\sqrt{n}}{L_K}$	$\frac{\sqrt{n}\log n}{L_K}$	$\frac{\sqrt{n}}{L_K}$	$\frac{\sqrt{n}}{L_K}$
$K_{(s)}$	$\sqrt{n}L_K$	1	$\frac{\sqrt{n}\log n}{r_s(n)}$	$\frac{\sqrt{n}}{r_s(n)}$	$\frac{n}{r_s(n)}$
$K_{(w)}$	$\sqrt[4]{n}L_K$	$\sqrt{n}$	1	$\sqrt{n}$	$\sqrt{n}$
$K_{(j)}$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{n}\log n$	1	$\sqrt{n}$
$K_{(\ell)}$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{n}\log n$	$\sqrt{n}$	1

where $r_s(n) = \min\{r(K_{(s)}) : K \in \mathcal{SK}_{[n]}\}$ and L_K is the isotropic constant of K.

It is often useful to be able to combine two different positions of a body K or to compare their behaviour with respect to basic functionals. For example, it is natural to ask for the exact dependence on n of the quantity

$$I_2^{(x)}(n) = \max\left\{I_2(K_{(x)}) : K \in \mathcal{SK}_{[n]}\right\}, \tag{1.3}$$

where $x \in \{j, \ell, w, s\}$. In other words, to see how the other four positions behave with respect to the "I_2-functional". We discuss this question in Section 5.

Theorem 1.4. *Let $K \in \mathcal{SK}_{[n]}$. Then, we have the estimates*

(i) $\frac{c_1 n}{\sqrt{\log n}} \leqslant I_2^{(j)}(n) \leqslant c_2 n$,

(ii) $I_2^{(s)}(n) \geqslant \frac{c_3 n}{\sqrt{\log n}}$,

(iii) $c_4 n^{1-\epsilon} \leqslant I_2^{(\ell)}(n) \leqslant c_5 n$,

where the left hand-side inequality in (iii) *is satisfied for all small enough $\epsilon > 0$ provided that $n \geqslant n_0(\epsilon)$, and $c_1, \dots, c_5 > 0$ denote absolute constants.*

The same question arises for the other functionals that we study in this note: surface area, mean width, inradius and circumradius. In Section 5 we briefly discuss them as well, and we provide examples on the sharpness of the estimates in Theorem 1.3. We conclude with some natural questions which arise from this work; the answer to them would clarify the picture completely.

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2 Classical positions of convex bodies

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, D_n for the Euclidean ball of volume 1 and S^{n-1} for the unit sphere. We denote the unit ball of ℓ_p^n by B_p^n, $1\leqslant p\leqslant\infty$. In particular, we also write Q_n for the cube $B_\infty^n=[-1,1]^n$ and $C_n=\left[-\frac{1}{2},\frac{1}{2}\right]^n$ for the cube of volume 1. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto an $(n-1)$-dimensional subspace F. We also define $B_F:=B_2^n\cap F$ and $S_F:=S^{n-1}\cap F$. Finally, we write $\overline{A}$ for the homothetic image of volume 1 of a symmetric convex body $A\subseteq\mathbb{R}^n$, i.e. $\overline{A}:=\frac{A}{|A|^{1/n}}$.

The letters c,c',c_1,c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a\simeq b$, we mean that there exist absolute constants $c_1,c_2>0$ such that $c_1a\leqslant b\leqslant c_2a$. Also, if $K,L\subseteq\mathbb{R}^n$ we will write $K\simeq L$ if there exist absolute constants $c_1,c_2>0$ such that $c_1K\subseteq L\subseteq c_2K$.

We refer to the books [4] and [23] for basic facts from the Brunn-Minkowski theory and to the books [17], [20] and [24] for basic facts from the local theory of normed spaces. We also refer to [16] and [5] for more information on isotropic convex bodies.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x\in K$ implies that $-x\in K$. In this note, we fix an orthonormal basis $\{e_1,\ldots,e_n\}$ of $\mathbb{R}^n$ and say that a convex body $K\in\mathcal{SK}_{[n]}$ is unconditional if $T(K)=K$ for every diagonal matrix $T=\mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_n)$ with $\varepsilon_i=\pm1$.

The support function of a convex body K is defined by $h_K(y)=\max\{\langle x,y\rangle : x\in K\}$, and the mean width of K is

$$w(K)=\int_{S^{n-1}}h_K(u)\,\sigma(du). \tag{2.1}$$

Note that $h_K(Ty)=h_{T^*(K)}(y)$ for all $y\in\mathbb{R}^n$. The circumradius of K is the quantity $R(K)=\max\{\|x\|_2 : x\in K\}$ i.e. the smallest $R>0$ for which $K\subseteq RB_2^n$. We write $r(K)$ for the inradius of K (the largest $r>0$ for which $rB_2^n\subseteq K$) and we define the polar body K° of K by $K^\circ:=\{y\in\mathbb{R}^n : \langle x,y\rangle\leqslant 1 \text{ for all } x\in K\}$.

Every symmetric convex body C in $\mathbb{R}^n$ is the unit ball $C=\{x\in\mathbb{R}^n : \|x\|_C\leqslant 1\}$ for a norm $\|\cdot\|_C$. We will use the fact that $h_C(x)=\|x\|_{C^\circ}$ for all $x\in\mathbb{R}^n$ and $\rho_C(x)\|x\|_C=1$ for all $x\neq 0$, where ρ_C is the radial function of C, defined by $\rho_C(x)=\max\{t>0 : tx\in C\}$ for all $x\neq 0$.

A Borel measure μ on S^{n-1} is called isotropic if

$$n\int_{S^{n-1}}\langle u,\theta\rangle^2 d\mu(u)=\mu(S^{n-1}) \tag{2.2}$$

for every $\theta\in S^{n-1}$. It is easily checked that μ is isotropic if and only if, for every

$i, j = 1, \ldots, n$,

$$n \int_{S^{n-1}} u_i u_j d\mu(u) = \mu(S^{n-1})\delta_{i,j}. \tag{2.3}$$

This is in turn equivalent to the fact that, for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_{S^{n-1}} \langle u, T(u) \rangle d\mu(u) = \frac{\mathrm{tr}(T)}{n} \mu(S^{n-1}). \tag{2.4}$$

Next, we introduce the classical positions that we are going to discuss; we set the notation and provide some background information.

2.1. *Minimal surface area position.* The surface measure σ_K of a convex body $K \in \mathcal{SK}_{[n]}$ is the Borel measure σ_K on S^{n-1} defined by $\sigma_K(A) = \nu(\{x \in \mathrm{bd}(K) : u_K(x) \in A\})$, where $u_K(x)$ is the outer unit normal vector to K at x, and ν is the $(n-1)$-dimensional Lebesgue measure on $\mathrm{bd}(K)$. The surface area of K is equal to $\partial(K) = \sigma_K(S^{n-1})$. We say that K has *minimal surface area* if $\partial(K) \leqslant \partial(T(K))$ for every $T \in SL(n)$. Petty ([18], see also [9]) proved that K has minimal surface area if and only if σ_K is isotropic. Equivalently, if

$$\partial(K) = n \int_{S^{n-1}} \langle u, \theta \rangle^2 \sigma_K(du) \tag{2.5}$$

for every $\theta \in S^{n-1}$. We write $K_{(s)}$ for the position of K which minimizes surface area. Note that there is no ambiguity; up to orthogonal transformations, we have a unique such body of volume 1 in the linear class of K. By the isoperimetric inequality

$$\partial(K) \geqslant n\omega_n^{1/n} |K|^{\frac{n-1}{n}} \tag{2.6}$$

(see e.g. [23] or [4]) for every $K \in \mathcal{SK}_{[n]}$ we have $\partial(K) \geqslant \partial(D_n) \geqslant c_1\sqrt{n}$ where $c_1 > 0$ is an absolute constant. K. Ball proved in [1] that $\partial(K_{(s)}) \leqslant \partial(C_n) = 2n$; this is the reverse isoperimetric inequality for $\mathcal{SK}_{[n]}$.

2.2. *Minimal mean width position.* We say that $K \in \mathcal{SK}_{[n]}$ is in *minimal mean width position* if $w(K) \leqslant w(T(K))$ for every $T \in SL(n)$. It was proved in [6] that K has minimal mean width if and only if

$$w(K) = n \int_{S^{n-1}} \langle u, \theta \rangle^2 h_K(u) \sigma(du) \tag{2.7}$$

for every $\theta \in S^{n-1}$. Equivalently, if and only if the measure $d\nu_K = h_K d\sigma$ is isotropic. We write $K_{(w)}$ for the position of K which minimizes mean width. Again, there is no ambiguity; we have uniqueness of $K_{(w)}$ up to orthogonal transformations. Urysohn's inequality $w(C) \geqslant (|C|/|B_2^n|)^{1/n}$ which holds true for every convex body C in $\mathbb{R}^n$ (see e.g. [20] for a proof) implies that $w(K_{(w)}) \geqslant w(D_n) \geqslant c_2\sqrt{n}$, where $c_2 > 0$ is an absolute constant. It is also known that every $K \in \mathcal{SK}_{[n]}$ has a position $\tilde{K}$ with $w(\tilde{K}) \leqslant c_3\sqrt{n}\log n$, where $c_3 > 0$ is an absolute constant (see §5.3 for a more detailed discussion, and [24], [20] and [7] for references). Therefore, $w(K_{(w)}) \leqslant w(\tilde{K}) \leqslant c_3\sqrt{n}\log n$.

2.3. *Isotropic position.* We say that $K \in \mathcal{SK}_{[n]}$ is in *isotropic position* if $I_2(K) \leqslant I_2(T(K))$ for every $T \in SL(n)$. This is equivalent to the existence of a constant $L_K > 0$ such that

$$\int_K \langle x,\theta\rangle^2 dx = L_K^2 \tag{2.8}$$

for every $\theta \in S^{n-1}$. We will make use of the close connection of the "moments of inertia" of a body $K \in \mathcal{SK}_{[n]}$ with the areas of its corresponding central hyperplane sections: for every $\theta \in S^{n-1}$ one has

$$\int_K \langle x,\theta\rangle^2 dx \simeq \int_K |\langle x,\theta\rangle|\, dx \simeq \frac{1}{|K\cap\theta^{\perp}|} \tag{2.9}$$

(see [16] or [5] for a proof of both assertions). We write $K_{(i)}$ for the isotropic position of K; again, $K_{(i)}$ is uniquely determined up to orthogonal transformations. It is easily checked that $L_K \geqslant L_{D_n} \geqslant c_4 > 0$ for every $K \in \mathcal{SK}_{[n]}$, where $c_4 > 0$ is an absolute constant. On the other hand, the well-known slicing problem asks if there exists an absolute constant $C > 0$ such that $L_K \leqslant C$ for every $K \in \mathcal{SK}_{[n]}$. Bourgain proved in [2] the general upper bound $L_K \leqslant c\sqrt[4]{n}\log n$, and the best known general estimate is currently $L_K \leqslant c\sqrt[4]{n}$; see [12] (and [13]).

2.4. *John's position.* In this article we say that $K \in \mathcal{SK}_{[n]}$ is in John's position if $r(K) \geqslant r(T(K))$ for every $T \in SL(n)$. This is the case when the ellipsoid of maximal volume inscribed in K is a multiple rB_2^n of the Euclidean unit ball B_2^n. We write $K_{(j)}$ for John's position of K; one can check that $K_{(j)}$ is uniquely determined up to orthogonal transformations. John's theorem [10] states that $K = K_{(j)}$ is in John's position if and only if $B_2^n \subseteq r^{-1}K_{(j)}$ and there exist $u_1,\dots,u_m \in \mathrm{bd}(r^{-1}K_{(j)}) \cap S^{n-1}$ and positive real numbers $c_1,\dots,c_m$ such that the identity operator can be decomposed in the form

$$I = \sum_{j=1}^m c_j u_j \otimes u_j, \tag{2.10}$$

where $(u_j \otimes u_j)(y) = \langle u_j, y\rangle u_j$. From this representation of the identity we get

$$\sum_{j=1}^m c_j \langle u_j,\theta\rangle^2 = 1 \tag{2.11}$$

for all $\theta \in S^{n-1}$. Therefore, if we consider the measure μ on S^{n-1} which is supported by $\{u_1,\dots,u_m\}$ and gives mass c_j to $\{u_j\}$, $j = 1,\dots,m$, then μ is isotropic.

2.5. *Löwner's position.* In analogy to 2.4 we say that a convex body $K \in \mathcal{SK}_{[n]}$ is in Löwner's position if $R(K) \leqslant R(T(K))$ for every $T \in SL(n)$. One can check that this holds true if and only if the ellipsoid of minimal volume containing K is a multiple RB_2^n of the Euclidean unit ball B_2^n. We write $K_{(\ell)}$ for Löwner's position of K; again, $K_{(\ell)}$ is uniquely determined up to orthogonal transformations. By John's theorem, $K = K_{(\ell)}$ is in Löwner's position if and only if $K_{(\ell)} \subseteq RB_2^n$ and there exist $u_1, \dots, u_m \in \mathrm{bd}(R^{-1}K_{(\ell)}) \cap S^{n-1}$ and positive real numbers $c_1, \dots, c_m$ such that the measure μ on S^{n-1} which is supported by $\{u_1, \dots, u_m\}$ and gives mass c_j to $\{u_j\}$, $j = 1, \dots, m$, is isotropic.

We close this Section with a Lemma which provides simple, but useful, bounds for the inradius and the circumradius of $K_{(i)}$, $K_{(j)}$ and $K_{(\ell)}$.

Lemma 2.1. *Let $K \in \mathcal{SK}_{[n]}$. Then, we have:*

(i) $R(K_{(i)}) \leqslant c_1 n L_K$ *and* $r(K_{(i)}) \geqslant L_K$.

(ii) $c_2\sqrt{n} \leqslant R(K_{(j)}) \leqslant c_3 n$, $c_4 \leqslant r(K_{(j)}) \leqslant c_5\sqrt{n}$ *and* $R(K_{(j)}) \leqslant \sqrt{n}\, r(K_{(j)})$.

(iii) $c_6\sqrt{n} \leqslant R(K_{(\ell)}) \leqslant c_7 n$, $c_8 \leqslant r(K_{(\ell)}) \leqslant c_9\sqrt{n}$ *and* $R(K_{(\ell)}) \leqslant \sqrt{n}\, r(K_{(\ell)})$.

Proof. (i) It is proved in [11] that every isotropic convex body $K_{(i)}$ in $\mathbb{R}^n$ is contained in the ball $(n+1)L_K B_2^n$. On the other hand, for every $u \in S^{n-1}$ we have

$$h_{K_{(i)}}(u) = \|\langle \cdot, u\rangle\|_{L^\infty(K)} \geqslant \|\langle \cdot, u\rangle\|_{L^2(K)} = L_K. \tag{2.12}$$

This shows that $r(K_{(i)}) \geqslant L_K$. Both bounds are sharp up to an absolute constant; this can be checked from the examples of $\overline{B}_1^n$ (the multiple of B_1^n of volume 1) and C_n respectively.

(ii) The fact that $R(K_{(j)}) \leqslant \sqrt{n}\, r(K_{(j)})$ is a consequence of John's theorem. Since $K_{(j)} \subseteq R(K_{(j)})B_2^n$ and $|K_{(j)}| = 1$ while $|B_2^n|^{1/n} \simeq 1/\sqrt{n}$, comparison of volumes shows that $R(K_{(j)}) \geqslant c_2\sqrt{n}$, and hence $r(K_{(j)}) \geqslant c_4 = c_2$. Similarly, from the fact that $r(K_{(j)})B_2^n \subseteq K_{(j)}$ we see that $r(K_{(j)}) \leqslant c_5\sqrt{n}$, and hence $R(K_{(j)}) \leqslant c_3 n$, where $c_3 = c_5$. All the bounds are sharp up to an absolute constant; this can be checked from the examples of $\overline{B}_1^n$ and C_n respectively.

(iii) We argue as in (ii). $\square$

Remark. It is easily checked that $R(K) \leqslant c_1\sqrt{n}\, w(K)$ for every symmetric convex body in $\mathbb{R}^n$. The concluding remarks of §2.2 show that

$$R(K_{(w)}) \leqslant c_1\sqrt{n}\, w(K_{(w)}) \leqslant c_2 n \log n. \tag{2.13}$$

We will discuss bounds for $r(K_{(w)})$, $r(K_{(s)})$ and $R(K_{(s)})$ in Section 5.

3 Bounds for the trace

In this Section we provide some general arguments that lead to upper bounds for the trace distance $d_{\mathrm{tr}}(K_{(x)}, K)$ or $d_{\mathrm{tr}}(K, K_{(x)})$, where $K_{(x)}$ is one of the five classical positions of K. Each one of the arguments is based on the isotropic condition which is satisfied by $K_{(x)}$.

3.1 On $d_{\rm tr}(K_{(i)}, K)$ and $d_{\rm tr}(K, K_{(i)})$

For any $C \in \mathcal{SK}_{[n]}$ we set

$$J(C) := \int_C \|x\|_1 dx. \tag{3.1}$$

Proposition 3.1. *Let $K_{(i)}$ be an isotropic convex body in $\mathbb{R}^n$. Let $K = T(K_{(i)})$ for some diagonal positive definite operator $T = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ in $SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{c_1 J(K)}{J(K_{(i)})}, \tag{3.2}$$

where $c_1 > 0$ is an absolute constant.

Proof. We will use the fact (see [16] or [5]) that $\int_{K_{(i)}} |\langle x, u\rangle|\, dx \simeq L_K$ for all $u \in S^{n-1}$. For every $j = 1, \dots, n$ we have

$$\lambda_j L_K \simeq \lambda_j \int_{K_{(i)}} |\langle x, e_j\rangle|\, dx = \int_K |\langle x, e_j\rangle|\, dx. \tag{3.3}$$

It follows that

$$\frac{\mathrm{tr}(T)}{n} \simeq \frac{1}{nL_K} \int_K \|x\|_1 dx. \tag{3.4}$$

Since $nL_K \simeq J(K_{(i)})$, this proves the proposition. $\square$

Remark. Note that $J(K) \leqslant \sqrt{n} I_2(K)$ and $J(K_{(i)}) \simeq \sqrt{n} I_2(K_{(i)}) \simeq nL_K$. Therefore, we also have

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{c_2 I_2(K)}{I_2(K_{(i)})}. \tag{3.5}$$

Proposition 3.2. *Let $K_{(i)}$ be an isotropic convex body. Assume that $K_{(i)} = T(K)$ for some symmetric and positive definite $T \in SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{c_3 \sqrt{n}}{r(K)}, \tag{3.6}$$

where $c_3 > 0$ is an absolute constant.

Proof. Since $K_{(i)}$ is isotropic, from the obvious analogue of (2.4) we have

$$[\mathrm{tr}(T)] L_K^2 = \int_{K_{(i)}} \langle x, Tx\rangle\, dx. \tag{3.7}$$

Using the fact that $\langle x, y\rangle \leqslant \|x\|_C h_C(y)$ for every $x, y \in \mathbb{R}^n$ and any $C \in \mathcal{SK}_{[n]}$ and taking into account the upper bound $O(nL_K)$ for $R(K_{(i)})$ from Lemma 2.1 (i), we write

$$\langle x, Tx\rangle \leqslant \|x\|_K h_K(Tx) = \frac{h_{K_{(i)}}(x)\|x\|_2}{r(K)} \leqslant \frac{R(K_{(i)})\|x\|_2}{r(K)} \leqslant \frac{c_4 nL_K \|x\|_2}{r(K)}. \tag{3.8}$$

Then, (3.7) gives

$$[\mathrm{tr}(T)]L_K^2 \leqslant \frac{c_4 n L_K}{r(K)} \int_{K_{(i)}} \|x\|_2 dx \leqslant \frac{c_4 n L_K \cdot \sqrt{n} L_K}{r(K)}, \tag{3.9}$$

and the result follows. □

3.2 On $d_{\mathrm{tr}}(K, K_{(s)})$ and $d_{\mathrm{tr}}(K_{(s)}, K)$

Proposition 3.3. *Let $K_{(s)}$ be a convex body which has minimal surface area. Assume that $K_{(s)} = T(K)$ for some symmetric and positive definite $T \in SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{\partial(K)}{\partial(K_{(s)})}. \tag{3.10}$$

Proof. Since the measure $\sigma_{K_{(s)}}$ is isotropic, we have that

$$\partial(K_{(s)}) \frac{\mathrm{tr}(T)}{n} = \int_{S^{n-1}} \langle u, Tu \rangle \, d\sigma_{K_{(s)}}(u) \leqslant \int_{S^{n-1}} \|Tu\|_2 d\sigma_{K_{(s)}}(u). \tag{3.11}$$

Since T is symmetric, we have $\|Tu\|_2 = h_{T(B_2^n)}(u)$. Using the integral representation

$$V(K, \ldots, K, C) = \frac{1}{n} \int_{S^{n-1}} h_C(u) d\sigma_K(u) \tag{3.12}$$

of the mixed volume $V(K, \ldots, K, C)$ for every pair of convex bodies K and C, and the fact that, for every affine transformation A of $\mathbb{R}^n$ and any n-tuple $K_1, \ldots, K_n$ of convex bodies we have

$$V(A(K_1), \ldots, A(K_n)) = |\det A| V(K_1, \ldots, K_n) \tag{3.13}$$

(see [23, Chapter 5] for both assertions) we get

$$\begin{aligned} \partial(K_{(s)}) \frac{\mathrm{tr}(T)}{n} &\leqslant \int_{S^{n-1}} h_{T(B_2^n)}(u) d\sigma_{K_{(s)}}(u) = nV(K_{(s)}, \ldots, K_{(s)}, T(B_2^n)) \\ &= n |\det T| V(K, \ldots, K, B_2^n) = \partial(K). \end{aligned} \tag{3.14}$$

This proves the proposition. □

Proposition 3.4. *Let $K_{(s)}$ be a convex body which has minimal surface area. Assume that $K = T(K_{(s)})$ for some diagonal operator $T = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ in $SL(n)$ with $\lambda_i > 0$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{c_1 \partial(K_{(s)})}{\sqrt{n}} \frac{J(K)}{n}, \tag{3.15}$$

where $c_1 > 0$ is an absolute constant.

Proof. We will use the fact (see [9]) that

$$\frac{\partial(K_{(s)})}{2n} \leqslant |P_{u^\perp}(K_{(s)})| \leqslant \frac{\partial(K_{(s)})}{2\sqrt{n}} \tag{3.16}$$

for every $u \in S^{n-1}$. To see this recall that, from Cauchy's formula (see [23]), the area of the $(n-1)$-dimensional projection $P_{u^\perp}(K_{(s)})$ of $K_{(s)}$ can be written in the form

$$|P_{u^\perp}(K_{(s)})| = \frac{1}{2}\int_{S^{n-1}} |\langle \theta, u\rangle| d\sigma_{K_{(s)}}(\theta), \tag{3.17}$$

and hence, the Cauchy-Schwarz inequality shows that

$$\begin{aligned} |P_{u^\perp}(K_{(s)})| &= \frac{1}{2}\int_{S^{n-1}} |\langle \theta, u\rangle|\, d\sigma_{K_{(s)}}(\theta) \\ &\leqslant \frac{1}{2}\left(\int_{S^{n-1}} |\langle \theta, u\rangle|^2 d\sigma_{K_{(s)}}(\theta)\right)^{1/2} \sqrt{\partial(K_{(s)})} \\ &= \frac{\partial(K_{(s)})}{2\sqrt{n}}, \end{aligned} \tag{3.18}$$

while the inequality $|\langle \theta, u\rangle| \geqslant \langle \theta, u\rangle^2$ (for $\theta, u \in S^{n-1}$) implies that

$$|P_{u^\perp}(K_{(s)})| \geqslant \frac{1}{2}\int_{S^{n-1}} \langle \theta, u\rangle^2 d\sigma_{K_{(s)}}(\theta) = \frac{\partial(K_{(s)})}{n}. \tag{3.19}$$

By a change of variables we check that, for every $i = 1, \ldots, n$,

$$\lambda_i \int_{K_{(s)}} |\langle x, e_i\rangle|\, dx = \int_K |\langle x, e_i\rangle|\, dx. \tag{3.20}$$

From (2.9) we have

$$\int_{K_{(s)}} |\langle x, \theta\rangle\, dx \simeq \frac{1}{|K_{(s)} \cap \theta^\perp|} \tag{3.21}$$

for every $\theta \in S^{n-1}$, so we finally get

$$\left(\int_{K_{(s)}} |\langle x, e_i\rangle\, dx\right)^{-1} \simeq |K_{(s)} \cap e_i^\perp| \leqslant |P_{e_i^\perp}(K_{(s)})| \leqslant \frac{\partial(K_{(s)})}{2\sqrt{n}}, \tag{3.22}$$

using (3.16) as well. Inserting this inequality into (3.20) and adding over all $i = 1, \ldots, n$, we get

$$\mathrm{tr}(T) \leqslant \frac{\partial(K_{(s)})}{2\sqrt{n}} \sum_{i=1}^{n} \int_K |\langle x, e_i\rangle|\, dx, \tag{3.23}$$

and the result follows. $\square$

3.3 On $d_{\mathrm{tr}}(K_{(w)}, K)$ and $d_{\mathrm{tr}}(K, K_{(w)})$

Proposition 3.5. *Let $K_{(w)}$ be a convex body which has minimal mean width. Assume that $K = T(K_{(w)})$ for some symmetric and positive definite $T \in SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{w(K)}{w(K_{(w)})}. \tag{3.24}$$

Proof. We may assume that K is smooth enough. Since $h_{K_{(w)}} d\sigma$ is isotropic, from [6, Theorem 3.1] we have that

$$\begin{aligned} w(K_{(w)})\frac{\mathrm{tr}(T)}{n} &= \int_{S^{n-1}} \langle \nabla h_{K_{(w)}}(u), Tu\rangle \, d\sigma(u) \\ &\leqslant \int_{S^{n-1}} h_{K_{(w)}}(Tu) d\sigma(u), \end{aligned} \tag{3.25}$$

using the fact that, in general, $\nabla h_C(u) \in C$ (in fact, it is the unique point on the boundary of C at which u is the outer normal to C). Since $h_{K_{(w)}}(Tu) = h_K(u)$, this gives

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{1}{w(K_{(w)})} \int_{S^{n-1}} h_K(u) d\sigma(u), \tag{3.26}$$

and (3.24) follows. $\square$

Proposition 3.6. *Let $K_{(w)}$ be a convex body which has minimal mean width. Assume that $K_{(w)} = T(K)$ for some symmetric and positive definite $T \in SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{c_1 w(K_{(w)})}{r(K)}, \tag{3.27}$$

where $c_1 > 0$ is an absolute constant.

Proof. Since $h_{K_{(w)}}(u) d\sigma(u)$ is isotropic, we have

$$w(K_{(w)})\frac{\mathrm{tr}(T)}{n} = \int_{S^{n-1}} \langle u, Tu\rangle h_{K_{(w)}}(u) \, \sigma(du). \tag{3.28}$$

We use the fact that

$$\langle u, Tu\rangle \leqslant h_{K_{(w)}}(u) \|Tu\|_{K_{(w)}}. \tag{3.29}$$

We have $K \supseteq r(K) B_2^n$, and hence

$$\|Tu\|_{K_{(w)}} = \|u\|_{T^{-1}(K_{(w)})} = \|u\|_K \leqslant \frac{1}{r(K)}. \tag{3.30}$$

Combining the above, and using the fact that $\|h_{K_{(w)}}\|^2_{L^2(S^{n-1})} \leqslant c_2\|h_{K_{(w)}}\|^2_{L^1(S^{n-1})}$ for some absolute constant $c_2>0$ (recall that $h_{K_{(w)}}$ is a norm and that the $L^1(S^{n-1})$ and $L^2(S^{n-1})$ of any norm on $\mathbb{R}^n$ are equivalent up to an absolute constant; see [17]), we get

$$\tag{3.31} \begin{aligned} w(K_{(w)})\frac{\mathrm{tr}(T)}{n} &\leqslant \frac{1}{r(K)}\int_{S^{n-1}} h^2_{K_{(w)}}(u)d\sigma(u) \\ &\leqslant \frac{c_2}{r(K)}\left(\int_{S^{n-1}} h_{K_{(w)}}(u)d\sigma(u)\right)^2 \\ &= \frac{c_2 w^2(K_{(w)})}{r(K)}, \end{aligned}$$

which gives (3.27). $\square$

3.4 On $d_{\mathrm{tr}}(K_{(j)},K)$ and $d_{\mathrm{tr}}(K,K_{(j)})$

Proposition 3.7. *Let $K_{(j)}$ be a convex body which is in John's position. Assume that $K_{(j)}=T(K)$ for some symmetric and positive definite $T\in SL(n)$. Then,*

$$\tag{3.32} \frac{\mathrm{tr}(T)}{n} \leqslant \frac{r(K_{(j)})}{r(K)}.$$

Proof. From John's theorem we know that there exist $c_i>0$ and $u_i\in S^{n-1}$ such that $\|r(K_{(j)})u_i\|_{K_{(j)}}=1$, $h_{K_{(j)}}(u_i)=r(K_{(j)})$ and $I=\sum_{i=1}^m c_iu_i\otimes u_i$. From the representation of the identity it follows that

$$\tag{3.33} \mathrm{tr}(T)=\sum_{i=1}^m c_i\langle u_i,Tu_i\rangle.$$

For every $1\leqslant i\leqslant m$ we write

$$\tag{3.34} \langle u_i,Tu_i\rangle \leqslant \|u_i\|_K h_K(Tu_i) \leqslant \frac{h_{K_{(j)}}(u_i)}{r(K)} = \frac{r(K_{(j)})}{r(K)}.$$

Finally, we use the fact that $\sum_{i=1}^m c_i=n$; this follows from (3.33) if we choose $T=I$. $\square$

Proposition 3.8. *Let $K_{(j)}$ be a convex body which is in John's position. Assume that $K=T(K_{(j)})$ for some symmetric and positive definite $T\in SL(n)$. Then,*

$$\tag{3.35} \frac{\mathrm{tr}(T)}{n} \leqslant \frac{R(K)}{r(K_{(j)})} \leqslant \frac{R(K)}{r(K)}.$$

Proof. As before, there exist $c_i>0$ and $u_i\in S^{n-1}$ such that $\|r(K_{(j)})u_i\|_{K_{(j)}}=1$ and $I=\sum_{i=1}^m c_iu_i\otimes u_i$. We have

$$\tag{3.36} \mathrm{tr}(T)=\sum_{i=1}^m c_i\langle u_i,Tu_i\rangle$$

and, for every $1 \leqslant i \leqslant m$, we write

$$\langle u_i, Tu_i\rangle \leqslant \|u_i\|_{K_{(j)}} h_{K_{(j)}}(Tu_i) = \frac{h_K(u_i)}{r(K_{(j)})} \leqslant \frac{R(K)}{r(K_{(j)})}. \tag{3.37}$$

Now, we use the fact that John's position maximizes the inradius, and hence $r(K_{(j)}) \geqslant r(K)$. This means that

$$\langle u_i, Tu_i\rangle \leqslant \frac{R(K)}{r(K)}, \tag{3.38}$$

and the result follows from $\sum_{i=1}^m c_i = n$. □

3.5 On $d_{\mathrm{tr}}(K_{(\ell)}, K)$ and $d_{\mathrm{tr}}(K, K_{(\ell)})$

Proposition 3.9. *Let $K_{(\ell)}$ be a convex body which is in Löwner's position. Assume that $K = T(K_{(\ell)})$ for some symmetric $T \in SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{R(K)}{R(K_{(\ell)})}. \tag{3.39}$$

Proof. We use the fact that C is in John's position if and only if $\overline{C^\circ}$ is in Löwner's position. Taking polars and using the affine invariance of the volume product $|C| \cdot |C^\circ|$ we have $\overline{K^\circ}_{(j)} = T(\overline{K^\circ})$. Then, applying Proposition 3.7 for $\overline{K^\circ}$, we see that

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{r(K^\circ_{(j)})}{r(K^\circ)} = \frac{R(K)}{R(K_\ell)}. \tag{3.40}$$

□

Proposition 3.10. *Let $K_{(\ell)}$ be a convex body which is in Löwner's position. Assume that $K_{(\ell)} = T(K)$ for some symmetric and positive definite $T \in SL(n)$. Then,*

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{R(K_{(\ell)})}{r(K)} \leqslant \frac{R(K)}{r(K)}. \tag{3.41}$$

Proof. Taking polars, we have $\overline{K^\circ} = T(\overline{K^\circ}_{(j)})$. Then, applying Proposition 3.8 for $\overline{K^\circ}$, we see that

$$\frac{\mathrm{tr}(T)}{n} \leqslant \frac{R(K^\circ)}{r(K^\circ_{(j)})} = \frac{R(K_{(\ell)})}{r(K)}. \tag{3.42}$$

Since Löwner's position minimizes the circumradius, the proof is complete. □

4 Upper bounds for $d_{\rm tr}(K_{(x)}, K_{(y)})$

Using the results of the previous Section, one can give the following upper bounds for $d_{\rm tr}(K_{(x)}, K_{(y)})$, where $x, y \in \{i, j, \ell, s, w\}$:

• $\mathbf{K_{(i)}}$ **and** $\mathbf{K_{(s)}}$: We have $d_{\rm tr}(K_{(i)}, K_{(s)}) \leqslant \partial(K_{(i)})/\partial(K_{(s)})$ from Proposition 3.3. We can give an upper bound for $\partial(K_{(i)})$, using the monotonicity of mixed volumes: for every convex body C in $\mathbb{R}^n$ with $0 \in \mathrm{int}(C)$, we have

$$\partial(C) = nV(C, \ldots, C, B_2^n) \leqslant nV\left(C, \ldots, C, \tfrac{1}{r(C)}C\right) = \frac{n|C|}{r(C)}. \tag{4.1}$$

Since $K_{(i)}$ is isotropic, from Lemma 2.1 (i) we have $r(K_{(i)}) \geqslant L_K$, and hence $\partial(K_{(i)}) \leqslant n/L_K$. Combining this inequality with the lower bound $\partial(K_{(s)}) \geqslant c_1\sqrt{n}$, we get

$$d_{\rm tr}(K_{(i)}, K_{(s)}) \leqslant \frac{\partial(K_{(i)})}{\partial(K_{(s)})} \leqslant \frac{n}{L_K \partial(K_{(s)})} \leqslant \frac{c_2\sqrt{n}}{L_K}. \tag{4.2}$$

On the other hand, from Proposition 3.4 we have

$$d_{\rm tr}(K_{(s)}, K_{(i)}) \leqslant \frac{\partial(K_{(s)})}{\sqrt{n}} \frac{J(K_{(i)})}{n} \leqslant \frac{L_K \partial(K_{(s)})}{\sqrt{n}} \leqslant 2L_K\sqrt{n}, \tag{4.3}$$

because

$$J(K_{(i)}) = \int_{K_{(i)}} \|x\|_1 dx = \sum_{j=1}^n \int_{K_{(i)}} |\langle x, e_j\rangle| \, dx \leqslant nL_K \tag{4.4}$$

and $\partial(K_{(s)}) \leqslant 2n$ by Ball's reverse isoperimetric inequality (see the concluding remark in §2.1).

• $\mathbf{K_{(i)}}$ **and** $\mathbf{K_{(w)}}$: From Proposition 3.6 and the fact that $r(K_{(i)}) \geqslant L_K$, we get

$$d_{\rm tr}(K_{(i)}, K_{(w)}) \leqslant \frac{c_1 w(K_{(w)})}{r(K_{(i)})} \leqslant \frac{c_2\sqrt{n}\log n}{L_K}. \tag{4.5}$$

On the other hand, from Proposition 3.5 we have

$$d_{\rm tr}(K_{(w)}, K_{(i)}) \leqslant \frac{w(K_{(i)})}{w(K_{(w)})} \leqslant c_3 \sqrt[4]{n} L_K, \tag{4.6}$$

if we employ the known upper bound $w(K) \leqslant c_4 n^{3/4} L_K$ for the mean width of an isotropic convex body in $\mathbb{R}^n$ (see [8] and the references therein) and the fact that $w(K_{(w)}) \geqslant c_5\sqrt{n}$ by Urysohn's inequality.

• $\mathbf{K_{(i)}}$ **and** $\mathbf{K_{(j)}}$: From Proposition 3.1 and the fact that

$$J(K_{(j)}) \leqslant \sqrt{n} I_2(K_{(j)}) \leqslant \sqrt{n} R(K_{(j)}) \leqslant c_1 n\sqrt{n}, \tag{4.7}$$

we get

$$d_{\mathrm{tr}}(K_{(i)},K_{(j)})\leqslant\frac{c_2J(K_{(j)})}{nL_K}\leqslant\frac{c_3\sqrt{n}}{L_K}. \tag{4.8}$$

On the other hand, from Proposition 3.2 we have

$$d_{\mathrm{tr}}(K_{(j)},K_{(i)})\leqslant\frac{c_4\sqrt{n}}{r(K_{(j)}))}\leqslant c_5\sqrt{n}, \tag{4.9}$$

because $r(K_{(j)})\geqslant c_6$ from Lemma 2.1 (ii).

• $\mathbf{K_{(i)}}$ **and** $\mathbf{K}_{(\ell)}$: From Proposition 3.1 and the fact that

$$J(K_{(\ell)})\leqslant\sqrt{n}I_2(K_{(\ell)})\leqslant\sqrt{n}R(K_{(\ell)})\leqslant c_1n\sqrt{n}, \tag{4.10}$$

we get

$$d_{\mathrm{tr}}(K_{(i)},K_{(\ell)})\leqslant\frac{c_2J(K_{(\ell)})}{nL_K}\leqslant\frac{c_3\sqrt{n}}{L_K}. \tag{4.11}$$

On the other hand, from Proposition 3.2 we have

$$d_{\mathrm{tr}}(K_{(\ell)},K_{(i)})\leqslant\frac{c_4\sqrt{n}}{r(K_{(\ell)}))}\leqslant c_5\sqrt{n}, \tag{4.12}$$

because $r(K_{(\ell)})\geqslant c_6$ from Lemma 2.1 (iii).

• $\mathbf{K_{(w)}}$ **and** $\mathbf{K_{(j)}}$: From Proposition 3.6 and the fact that $r(K_{(j)})\geqslant c_1$, we get

$$d_{\mathrm{tr}}(K_{(j)},K_{(w)})\leqslant\frac{c_2w(K_{(w)})}{r(K_{(j)})}\leqslant c_3\sqrt{n}\log n. \tag{4.13}$$

On the other hand, from Proposition 3.5 we have

$$d_{\mathrm{tr}}(K_{(w)},K_{(\ell)})\leqslant\frac{w(K_{(j)})}{w(K_{(w)})}\leqslant\frac{R(K_{(j)})}{w(K_{(w)})}\leqslant c_4\sqrt{n}, \tag{4.14}$$

if we use the fact that $R(K_{(j)})\leqslant c_5n$ and $w(K_{(w)})\geqslant c_6\sqrt{n}$.

• $\mathbf{K_{(w)}}$ **and** $\mathbf{K}_{(\ell)}$: From Proposition 3.6 and the fact that $r(K_{(\ell)})\geqslant c_1$, we get

$$d_{\mathrm{tr}}(K_{(\ell)},K_{(w)})\leqslant\frac{c_2w(K_{(w)})}{r(K_{(\ell)})}\leqslant c_3\sqrt{n}\log n. \tag{4.15}$$

On the other hand, from Proposition 3.5 we have

$$d_{\mathrm{tr}}(K_{(w)},K_{(\ell)})\leqslant\frac{w(K_{(\ell)})}{w(K_{(w)})}\leqslant\frac{R(K_{(\ell)})}{w(K_{(w)})}\leqslant c_4\sqrt{n}, \tag{4.16}$$

if we use the fact that $R(K_{(\ell)})\leqslant c_5n$ and $w(K_{(w)})\geqslant c_6\sqrt{n}$.

• $\mathbf{K_{(j)}}$ **and** $\mathbf{K}_{(\ell)}$: From Proposition 3.8 and the fact that $R(K_{(\ell)}) \leqslant R(K_{(j)})$, we get

$$d_{\rm tr}(K_{(j)}, K_{(\ell)}) \leqslant \frac{R(K_{(\ell)})}{r(K_{(j)})} \leqslant \frac{R(K_{(j)})}{r(K_{(j)})} \leqslant \sqrt{n} \tag{4.17}$$

using John's theorem in the end. On the other hand, from Proposition 3.9 we have

$$d_{\rm tr}(K_{(\ell)}, K_{(j)}) \leqslant \frac{R(K_{(j)})}{R(K_{(\ell)})} \leqslant c_1\sqrt{n}, \tag{4.18}$$

because $R(K_{(j)}) \leqslant c_2 n$ and $R(K_{(\ell)}) \geqslant c_3\sqrt{n}$ from Lemma 2.1.

• $\mathbf{K_{(s)}}$ **and** $\mathbf{K_{(w)}}$: It is proved in [15, Theorem 7.1] that $w(K_{(s)}) \leqslant c_1 n$. Since $w(K_{(w)}) \geqslant c_2\sqrt{n}$, from Proposition 3.5 we get

$$d_{\rm tr}(K_{(w)}, K_{(s)}) \leqslant \frac{w(K_{(s)})}{w(K_{(w)})} \leqslant c_3\sqrt{n}. \tag{4.19}$$

From Proposition 3.4 we get the bound $d_{\rm tr}(K_{(s)}, K_{(w)}) \leqslant c_4 I_2(K_{(w)})$ and from Proposition 3.6 we see that $d_{\rm tr}(K_{(s)}, K_{(w)}) \leqslant \frac{c_5\sqrt{n}\log n}{r(K_{(s)})}$. However, we do not have an upper bound for $I_2(K_{(w)})$ which is close to $\sqrt{n}$, and we do not have a lower bound for $r(K_{(s)})$ which is of the order of 1.

• $\mathbf{K_{(s)}}$ **and** $\mathbf{K_{(j)}}$: We use the fact that

$$\partial(K_{(j)}) \leqslant \frac{n}{r(K_{(j)})} \leqslant c_1 n, \tag{4.20}$$

because $r(K_{(j)}) \geqslant c_2$. Then, from Proposition 3.3 we get

$$d_{\rm tr}(K_{(j)}, K_{(s)}) \leqslant \frac{\partial(K_{(j)})}{\partial(K_{(s)})} \leqslant c_3\sqrt{n}. \tag{4.21}$$

From Proposition 3.7 we see that $d_{\rm tr}(K_{(s)}, K_{(j)}) \leqslant \frac{r(K_{(j)})}{r(K_{(s)})} \leqslant \frac{c_4\sqrt{n}}{r(K_{(s)})}$. However, we do not have a lower bound for $r(K_{(s)})$ which is of the order of 1.

• $\mathbf{K_{(s)}}$ **and** $\mathbf{K}_{(\ell)}$: As in the previous case, we use the fact that

$$\partial(K_{(\ell)}) \leqslant \frac{n}{r(K_{(\ell)})} \leqslant c_1 n, \tag{4.22}$$

because $r(K_{(\ell)}) \geqslant c_2$. Then, from Proposition 3.3 we get

$$d_{\rm tr}(K_{(\ell)}, K_{(s)}) \leqslant \frac{\partial(K_{(\ell)})}{\partial(K_{(s)})} \leqslant c_3\sqrt{n}. \tag{4.23}$$

From Proposition 3.10 we see that $d_{\rm tr}(K_{(s)}, K_{(\ell)}) \leqslant \frac{R(K_{(\ell)})}{r(K_{(s)})} \leqslant \frac{R(K_{(s)})}{r(K_{(s)})}$. However, it is not clear if one can have an upper bound which is close to $\sqrt{n}$ for this last quantity.

5 Examples and Questions

Most of the upper bounds that we have obtained for $d_{\rm tr}(K_{(x)}, K_{(y)})$ are in terms of the quantities $f(K_{(x)})$, where f is one of the functionals I_2, ∂, w, R or r, and $K_{(x)}$ is one of the classical positions of K. One is naturally led in two types of questions: the first one is to check if these upper bounds are sharp, while to second one is to estimate

$$f^{(x)}(n) = \max\{f(K_{(x)}) : K \in \mathcal{SK}_{[n]}\} \tag{5.1}$$

for each f and all these positions (note that the maximum should be replaced by a minimum in the case of the inradius functional r). We are mostly interested in the isotropic position; for this reason, below we give a sample of examples regarding $I_2^{(x)}(n)$ and $D_{\rm tr}(K_{(x)}, K_{(i)})$. Most of them are products of standard unconditional convex bodies (as in [22] and [15]) We conclude this paper with a discussion of some related open questions.

5.1 Bounds for $\boldsymbol{I_2(K_{(x)})}$

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$. Then,

$$I_2(K) \geqslant I_2(K_{(i)}) = \sqrt{n} L_K \tag{5.2}$$

with equality if and only if $K = K_{(i)}$ is in the isotropic position. In this Subsection we give bounds for the quantity $I_2^{(x)}(n) := \max\{I_2(K_{(x)}) : K \in \mathcal{SK}_{[n]}\}$, where $K_{(x)}$ denotes one of the other classical positions $K_{(s)}, K_{(w)}, K_{(j)}$ or $K_{(\ell)}$. We start with the minimal surface area position.

Proposition 5.1. *There exists an unconditional convex body $K_{(s)}$ of volume 1 in $\mathbb{R}^n$ which has minimal surface area and satisfies*

$$I_2(K_{(s)}) \geqslant \frac{cn}{\sqrt{\log n}}, \tag{5.3}$$

where $c > 0$ is an absolute constant.

Proof. Let $k, m \in \mathbb{N}$ with $k + m = n$ and $a, b > 0$ with $a^k b^m = 1$, and define $K_{(s)} := a\overline{B}_1^k \times bC_m$. It is proved in [15] that $K_{(s)}$ has minimal surface area if

$$a = \left(\partial_{\overline{B}_1^k}/(2k)\right)^{\frac{m}{k+m}} \text{ and } b = \left(2k/\partial_{\overline{B}_1^k}\right)^{\frac{k}{k+m}}. \tag{5.4}$$

We choose $m \simeq \frac{k}{\log k}$. Note that $k \leqslant n \leqslant 2k$. Then, since $\partial_{\overline{B}_1^k} \simeq \sqrt{k}$, we get that

$$a \simeq 1,\ b \simeq \sqrt{k} \simeq \sqrt{n}. \tag{5.5}$$

Using the fact that $a^k b^m = 1$, we compute

$$
\begin{aligned}
I_2^2(K_{(s)}) &= |a\overline{B}_1^k| \int_{bC_m} \|x\|_2^2 dx + |bC_m| \int_{a\overline{B}_1^k} \|y\|_2^2 dy \\
&= a^k b^{m+2} I_2^2(C_m) + b^m a^{k+2} I_2^2(\overline{B}_1^k) \\
&\simeq b^2 m + a^2 k \simeq km \simeq \frac{n^2}{\log n}.
\end{aligned} \tag{5.6}
$$

This proves the Proposition. □

Question 5.2. *To give a sharp upper bound for $I_2^{(s)}(n)$. Is it true that $I_2(K_{(s)}) \leqslant Cn$ for every $K \in \mathcal{SK}_{[n]}$?*

We know that if $K_{(j)}$ is in John's position then $R(K_{(j)}) \leqslant Cn$, and hence $I_2(K_{(j)}) \leqslant Cn$ for every $K \in \mathcal{SK}_{[n]}$. Our next result shows that $I_2^{(j)}(n) \geqslant \frac{cn}{\sqrt{\log n}}$.

Proposition 5.3. *There exists an unconditional convex body $K_{(j)}$ of volume 1 in $\mathbb{R}^n$ which is in John's position and satisfies*

$$
I_2(K_{(j)}) \geqslant \frac{cn}{\sqrt{\log n}}, \tag{5.7}
$$

where $c > 0$ is an absolute constant.

Proof. We use the following fact: if V is a symmetric convex body in $\mathbb{R}^m$ which is in John's position and if $Q_k = [-1,1]^k$, then $V \times Q_k$ is also in John's position in $\mathbb{R}^{m+k}$. To see this, we use induction on k. It is enough to show that $V_1 := V \times [-1,1]$ is in John's position. To this end, first note that $B_2^{m+1} \subseteq B_2^m \times [-1,1] \subseteq V_1$. Moreover, for every $x = (y,t) \in \mathbb{R}^{m+1}$ we have that

$$
x = y + te_{m+1} = \sum_{j=1}^m c_j \langle x, u_j\rangle u_j + \langle x, e_{m+1}\rangle e_{m+1}, \tag{5.8}
$$

using the decomposition of identity (2.10) for V. Since e_{m+1} is also a contact point for V_1, the proof is complete by John's theorem.

Using the previous claim we see that if $m + k = n$ then $B_2^m \times Q_k$ is in John's position in $\mathbb{R}^n$. We consider the body $K_{(j)} = \overline{B_2^m \times Q_k} = a^{-1}(B_2^m \times Q_k)$, where

$$
a = |B_2^m \times Q_k|^{1/n} = |B_2^m|^{1/n} |Q_k|^{1/n} \simeq m^{-\frac{m}{2n}}. \tag{5.9}
$$

Then, we get:

$$
\begin{aligned}
I_2^2(K_{(j)}) &= |a^{-1}B_2^m| \int_{a^{-1}Q_k} \|x\|_2^2\, dx + |a^{-1}Q_k| \int_{a^{-1}B_2^m} \|x\|_2^2\, dx \\
&= \left(\frac{|Q_k|^{1/k}}{a}\right)^2 \int_{C_k} \|y\|_2^2\, dy + \left(\frac{|B_2^m|^{1/m}}{a}\right)^2 \int_{\overline{B_2^m}} \|y\|_2^2\, dy \\
&\simeq |Q_k|^{2/k} a^{-2} k + |B_2^m|^{2/m} a^{-2} m \\
&\simeq a^{-2} k \simeq m^{\frac{m}{n}} k.
\end{aligned} \tag{5.10}
$$

Choosing $k \simeq \frac{n}{\log n}$ we obtain $I_2(K_{(j)}) \simeq \frac{n}{\sqrt{\log n}}$. $\square$

Similarly, we know that if $K_{(\ell)}$ is in Löwner's position then $R(K_{(\ell)}) \leqslant Cn$, and hence $I_2(K_{(\ell)}) \leqslant Cn$. The next example shows that $I_2^{(\ell)}(n)$ is approximately equal to n.

Proposition 5.4. *For every $\epsilon > 0$ there exists $n_0(\epsilon) \in \mathbb{N}$ with the following property: For any $n \geqslant n_0(\epsilon)$ there exists an unconditional convex body $K_{(\ell)}$ of volume 1 in $\mathbb{R}^n$ which is in Löwner's position and satisfies the following:*

$$I_2(K_{(\ell)}) \geqslant cn^{1-\epsilon}, \tag{5.11}$$

where $c > 0$ is an absolute constant.

Proof. Let $k+m=n$. There exist $\delta, b > 0$ such that $\delta^m b^k = 1$ and $K = \delta B_2^m \times bB_1^k$ is in Löwner's position. Since $R(B_2^m \times B_1^k) = \sqrt{2}$, we have $\delta^2 + b^2 = R^2(K) \leqslant 2$. This shows that $\max\{\delta, b\} \leqslant \sqrt{2}$ and, taking into account the condition $\delta^m b^k = 1$, we also have $\delta^{\frac{2m}{k}} \geqslant 1/2$ and $b^{\frac{2k}{m}} \geqslant 1/2$.

We consider the body $K_{(\ell)} = \overline{\delta B_2^m \times bB_1^k} = a^{-1}(\delta B_2^m \times bB_1^k)$, where

$$a = |\delta B_2^m \times bB_1^k|^{1/n} \simeq \delta^{\frac{m}{n}} b^{\frac{k}{n}} |B_2^m|^{1/n} |B_1^k|^{1/n} \simeq \delta^{\frac{m}{n}} b^{\frac{k}{n}} m^{-\frac{m}{2n}} k^{-\frac{k}{n}}. \tag{5.12}$$

Then, we get:

$$\begin{aligned} I_2^2(K_{(\ell)}) &= |a^{-1}\delta B_2^m| \int_{a^{-1}bB_1^k} \|x\|_2^2\, dx + |a^{-1}bB_1^k| \int_{a^{-1}\delta B_2^m} \|x\|_2^2\, dx \\ &= \left(\frac{b|B_1^k|^{1/k}}{a}\right)^2 \int_{\overline{B_1^k}} \|y\|_2^2\, dy + \left(\frac{\delta|B_2^m|^{1/m}}{a}\right)^2 \int_{\overline{B_2^m}} \|y\|_2^2\, dy \\ &\simeq |B_1^k|^{2/k} b^2 a^{-2} k + |B_2^m|^{2/m} \delta^2 a^{-2} m \\ &\simeq \frac{b^2a^{-2}}{k} + \delta^2 a^{-2} \geqslant \delta^2 a^{-2} \\ &\simeq \delta^{2\left(1-\frac{m}{n}\right)} b^{-\frac{2k}{n}} m^{\frac{m}{n}} k^{\frac{2k}{n}} \\ &\geqslant \frac{cm^{\frac{m}{n}} k^{\frac{2k}{n}}}{2^{k/m}}. \end{aligned} \tag{5.13}$$

Now, let $\epsilon > 0$. We choose $m = \eta k$ for some $\eta \simeq \frac{1}{\log\log n}$. If $n \geqslant n_0(\epsilon)$, then this choice gives the lower bound

$$I_2^2(K_{(\ell)}) \geqslant cn^{2-\epsilon}, \tag{5.14}$$

and the result follows. $\square$

Remark. We know that if $K_{(w)}$ has minimal mean width then $R(K_{(w)}) \leqslant Cn\log n$, and hence $I_2(K_{(w)}) \leqslant Cn\log n$. The estimate $d_{\mathrm{tr}}(K_{(w)}, K_{(i)}) \leqslant C\sqrt[4]{n}L_K$ indicates that the quantity $I_2^{(w)}(n)$ might be close to $\sqrt{n}L_K$. However, we have an example which shows that $I_2^{(w)}(n) \geqslant c\sqrt{n\log n}$. The body $Q = a\overline{B_1^k} \times bC_m$ is in minimal mean width position if $a \simeq (\log k)^{-\frac{k}{2(k+m)}}$ and $b \simeq (\log k)^{\frac{m}{2(k+m)}}$ (see [15, Section 5] for similar computations). Choosing $k \simeq \frac{m}{\log m}$ we have $m \leqslant n \leqslant 2m$. Then, $a^2 \simeq 1$ and $b^2 \simeq \log m \simeq \log n$. It follows that

$$I_2^2(Q) \simeq b^2 m + a^2 k \simeq n\log n. \tag{5.15}$$

Question 5.5. *To determine the exact order of $I_2^{(w)}(n)$.*

5.2 Lower bounds for $D_{\rm tr}(K_{(x)}, K_{(i)})$

The next examples show that the trace distance between $K_{(i)}$ and $K_{(s)}$, $K_{(j)}$ or $K_{(\ell)}$ can be of the order of $\sqrt{n}$.

(i) From Theorem 1.3 we know that $d_{\rm tr}(K_{(i)}, K_{(s)}) \leqslant c\sqrt{n}/L_K$ and $d_{\rm tr}(K_{(s)}, K_{(i)}) \leqslant c\sqrt{n}L_K$. Therefore,

$$D_{\rm tr}(K_{(i)}, K_{(s)}) \leqslant C\sqrt{n}L_K. \tag{5.16}$$

Let $K_{(s)} = a\overline{B_1^k} \times bC_m$ with $m \simeq \frac{k}{\log k}$. Then, $a \simeq 1$ and $b \simeq \sqrt{n}$. On the other hand, $K_{(i)} \simeq \overline{B_1^k} \times C_m$. Therefore, if $K_{(s)} = T(K_{(i)})$, we easily check that

$$\frac{{\rm tr}(T)}{n} = \frac{ka+mb}{n} \simeq \frac{\sqrt{n}}{\log n}. \tag{5.17}$$

It follows that $\max\{D_{\rm tr}(K_{(i)}, K_{(s)}) : K \in \mathcal{SK}_{[n]}\} \geqslant c\sqrt{n}/\log n$.

(ii) From Theorem 1.3 we know that $d_{\rm tr}(K_{(i)}, K_{(j)}) \leqslant c\sqrt{n}/L_K$ and $d_{\rm tr}(K_{(j)}, K_{(i)}) \leqslant c\sqrt{n}$. Therefore,

$$D_{\rm tr}(K_{(i)}, K_{(s)}) \leqslant C\sqrt{n}. \tag{5.18}$$

We consider the body $K_{(j)} = \overline{B_2^m \times Q_k} = a^{-1}(B_2^m \times Q_k)$, where

$$a = |B_2^m \times Q_k|^{1/n} = |B_2^m|^{1/n}|Q_k|^{1/n} \simeq m^{-\frac{m}{2n}}. \tag{5.19}$$

Choosing $k \simeq \frac{n}{\log n}$, we have $a^{-1} \simeq \sqrt{n}$. Since $K_{(i)} \simeq \overline{B_2^m} \times Q_k$, we easily check that if $T(K_{(i)}) = K_{(j)}$ then

$$\frac{{\rm tr}(T)}{n} \simeq \frac{m+ka^{-1}}{n} \simeq \frac{\sqrt{n}}{\log n}. \tag{5.20}$$

It follows that $\max\{D_{\rm tr}(K_{(i)}, K_{(j)}) : K \in \mathcal{SK}_{[n]}\} \geqslant c\sqrt{n}/\log n$.

5.3 Remarks on $r(K_{(x)})$ and $R(K_{(x)})$

Minimal mean width position. We start with some remarks on the inradius and the circumradius of a body $K_{(w)}$ in minimal mean width position. From results of Figiel-Tomczak [3], Lewis [14] and Pisier [19] it follows that if $K = K_{(w)}$ then $w(K)w(K^\circ) \leqslant c\log[d(X_K, \ell_2^n) + 1]$, which implies the general upper bound

$$w(K)w(K^\circ) \leqslant c_1 \log n \tag{5.21}$$

where $c_1 > 0$ is an absolute constant. Urysohn's inequality shows that $w(K) \geqslant c_2\sqrt{n}$, and hence $w(K^\circ) \leqslant c_3 \log n/\sqrt{n}$. It follows that

$$R(K^\circ) \leqslant c_4\sqrt{n}w(K^\circ) \leqslant c_5 \log n. \tag{5.22}$$

Then,

$$r(K) = \frac{1}{R(K^\circ)} \geqslant \frac{c_6}{\log n}. \tag{5.23}$$

Note that, by (4.1), this implies that

$$\partial(K_{(w)}) \leqslant Cn \log n. \tag{5.24}$$

In the same way, we see that $R(K) \leqslant c_4\sqrt{n}w(K) \leqslant c_5 n \log n$. It follows that

$$\frac{R(K)}{r(K)} = R(K)R(K^\circ) \leqslant c_7 n w(K)w(K^\circ) \leqslant c_8 n \log n. \tag{5.25}$$

The next example shows that all these estimates are sharp up to the logarithmic terms.

Lemma 5.6. *There exists an unconditional convex body $K_{(w)}$ of volume 1 in $\mathbb{R}^n$ which is in minimal mean width position and satisfies*

$$\frac{R(K_{(w)})}{r(K_{(w)})} \geqslant \frac{cn}{\sqrt{\log n}}, \tag{5.26}$$

where $c > 0$ is an absolute constant.

Proof. We consider the body $Q = a\overline{B_1^k} \times bC_m$ with $k \sim m \sim n/2$, $a \simeq (\log n)^{-1/4}$ and $b \simeq (\log n)^{1/4}$, which is in minimal mean width position. Then, $R(K_{(w)}) \simeq \frac{n}{\sqrt[4]{\log n}}$ and $r(K_{(w)}) \simeq \sqrt[4]{\log n}$ and the result follows. □

Isotropic position. The well-known bounds for the inradius and the circumradius of an isotropic convex body $K_{(i)}$ in $\mathbb{R}^n$ are stated in Lemma 2.1: one has $r(K_{(i)}) \geqslant L_K$ and $R(K_{(i)}) \leqslant cnL_K$. The next example shows that there exist isotropic bodies for which both estimates are sharp.

Lemma 5.7. *There exists an unconditional isotropic convex body K in $\mathbb{R}^n$ such that $R(K) \geqslant cnr(K)$.*

Proof. We consider the body $K = a\overline{B}_1^k \times bC_m$ with $k+m=n$, $k \sim m \sim n/2$ and $a \simeq b \simeq 1$. $\square$

Minimal surface area position. The situation is not clear in this case. We have some simple bounds which rely on the following observations: if $K = K_{(s)}$ has minimal surface area then, for every $\theta \in S^{n-1}$ we have

$$\frac{\partial(K)}{2n} \leqslant |P_{\theta^\perp}(K)| \leqslant \frac{\partial(K)}{2\sqrt{n}}. \tag{5.27}$$

On the other hand, a classical inequality of Rogers and Shephard (see [21]) states that

$$1 \leqslant 2h_K(\theta)|K \cap \theta^\perp| \leqslant 2h_K(\theta)|P_{\theta^\perp}(K)| \tag{5.28}$$

for every $\theta \in S^{n-1}$. We also know that

$$h_K(\theta) \leqslant c_1\sqrt{n}w(K) \leqslant \frac{c_2n^2}{\partial(K)} \tag{5.29}$$

by [15, Theorem 7.1]. Combining the above, we get

$$\frac{\sqrt{n}}{\partial(K)} \leqslant h_K(\theta) \leqslant \frac{c_2n^2}{\partial(K)} \tag{5.30}$$

Since $c_3\sqrt{n} \leqslant \partial(K) \leqslant 2n$, it follows that

$$R(K_{(s)}) \leqslant c_4n^{3/2}r(K_{(s)}), \quad r(K_{(s)}) \geqslant \frac{c_5}{\sqrt{n}} \quad \text{and} \quad R(K_{(s)}) \leqslant Cn^{3/2}. \tag{5.31}$$

All these bounds are probably non-optimal:

Question 5.8. *Determine the exact order of* $r^{(s)}(n) = \min\{r(K_{(s)}) : K \in \mathcal{SK}_{[n]}\}$ *and* $R^{(s)}(n) = \max\{R(K_{(s)}) : K \in \mathcal{SK}_{[n]}\}$.

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