Variants of the Busemann-Petty problem and of the Shephard problem

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Abstract

We provide an affirmative answer to a variant of the Busemann-Petty problem, proposed by V. Milman: Let K be a convex body in $\mathbb{R}^n$ and let D be a compact subset of $\mathbb{R}^n$ such that, for some $1 \leqslant k \leqslant n-1$,

$$|P_F(K)| \leqslant |D \cap F|$$

for all $F \in G_{n,k}$, where $P_F(K)$ is the orthogonal projection of K onto F and $D \cap F$ is the intersection of D with F. Then,

$$|K| \leqslant |D|.$$

We also provide estimates for the lower dimensional Busemann-Petty and Shephard problems, and we prove separation in the original Busemann-Petty problem.

1 Introduction

The Busemann-Petty problem was posed in [8], first in a list of ten problems concerning central sections of symmetric convex bodies in $\mathbb{R}^n$ and coming from questions in Minkowski geometry. It was originally formulated as follows:

Assume that K and D are origin-symmetric convex bodies in $\mathbb{R}^n$ and satisfy

$$|K \cap \xi^\perp| \leqslant |D \cap \xi^\perp| \tag{1.1}$$

for all $\xi \in S^{n-1}$. Does it follow that $|K| \leqslant |D|$?

Here $\xi^\perp$ is the central hyperplane perpendicular to ξ, and $|A|$ denotes the volume (Lebesgue measure) of A in the appropriate dimension. The answer is affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$ (for the history and the solution to this problem, see the monographs [12] and [19]). The isomorphic version of the Busemann-Petty problem asks if there exists an absolute constant $C_1 > 0$ (i.e. a numerical constant which is independent from the dimension and the bodies involved) such that whenever K and D satisfy (1.1) we have $|K| \leqslant C_1|D|$.

Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and if its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \xi\rangle^2 dx = L_K^2 \tag{1.2}$$

for every ξ in the Euclidean unit sphere S^{n-1}. Every convex body has an affine image $\tilde{K}$, uniquely determined modulo orthogonal transformations, which is isotropic; thus, the constant $L_{\tilde{K}}$ may be viewed as an affine invariant of K. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C \tag{1.3}$$

for all $n \geqslant 1$. Bourgain proved in [5] that $L_n \leqslant c\sqrt[4]{n}\,\log n$, and Klartag [17] improved this bound to $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [18]. For more information on isotropic convex bodies and log-concave measures see [7]. It is known that the isomorphic version of the Busemann-Petty problem is equivalent to the hyperplane conjecture. More precisely, if K and D are two centered convex bodies in $\mathbb{R}^n$ such that (1.1) holds true for all $\xi \in S^{n-1}$, then

$$|K|^{\frac{n-1}{n}} \leqslant c_1 L_n \, |D|^{\frac{n-1}{n}}, \tag{1.4}$$

where $c_1 > 0$ is an absolute constant.

Shephard's problem (see [31]) is dual to the Busemann-Petty problem.

Let K and D be two centrally symmetric convex bodies in $\mathbb{R}^n$. Suppose that

$$|P_{\xi^\perp}(K)| \leqslant |P_{\xi^\perp}(D)| \tag{1.5}$$

for every $\xi \in S^{n-1}$, where $P_{\xi^\perp}(A)$ is the orthogonal projection of $A \subset \mathbb{R}^n$ onto $\xi^\perp$. Does it follow that $|K| \leqslant |D|$?

The answer is affirmative if $n = 2$, but shortly after it was posed, Shephard's question was answered in the negative for all $n \geqslant 3$. This was done independently by Petty in [27] who gave an explicit counterexample in $\mathbb{R}^3$, and by Schneider in [29] for all $n \geqslant 3$. After these counterexamples, one might try to relax the question, asking for the smallest constant C_n (or the order of growth of this constant C_n as $n \to \infty$) for which: if K, D are centrally symmetric convex bodies in $\mathbb{R}^n$ and $|P_{\xi^\perp}(K)| \leqslant |P_{\xi^\perp}(D)|$ for all $\xi \in S^{n-1}$ then $|K| \leqslant C_n |D|$.

Such a constant C_n does exist, and a simple argument, based on John's theorem, shows that $C_n \leqslant c\sqrt{n}$, where $c > 0$ is an absolute constant. On the other hand, K. Ball has proved in [3] that this simple estimate is optimal: one has $C_n \simeq \sqrt{n}$.

In the first part of this note we discuss a variant of the two problems, proposed by V. Milman at the Oberwolfach meeting on Convex Geometry and its Applications (December 2015):

Question 1.1 (V. Milman)**.** *Assume that K and D are origin-symmetric convex bodies in $\mathbb{R}^n$ and satisfy*

$$|P_{\xi^\perp}(K)| \leqslant |D \cap \xi^\perp| \tag{1.6}$$

for all $\xi \in S^{n-1}$. Does it follow that $|K| \leqslant |D|$?

In Section 2 we show that the answer to this question is affirmative. In fact, the lower dimensional analogue of the problem has an affirmative answer. Moreover, one can drop the symmetry assumptions and even the assumption of convexity for D.

Theorem 1.2. *Let K be a convex body in $\mathbb{R}^n$ and let D be a compact subset of $\mathbb{R}^n$ such that, for some $1 \leqslant k \leqslant n-1$,*

$$|P_F(K)| \leqslant |D \cap F| \tag{1.7}$$

for all $F \in G_{n,n-k}$. Then,

$$|K| \leqslant |D|. \tag{1.8}$$

We also prove stability and separation in Theorem 1.2. In the hyperplane case, and assuming that D is a centered convex body, we can provide a more precise answer in terms of the isotropic constant L_D of D.

Theorem 1.3. *Let K and D be two convex bodies in $\mathbb{R}^n$, such that D is centered and*

$$|P_{\xi^\perp}(K)| \leqslant |D \cap \xi^\perp| \tag{1.9}$$

for all $\xi \in S^{n-1}$. Then,

$$|K| \leqslant \frac{c}{L_D} \, |D|, \tag{1.10}$$

where $c > 0$ is an absolute constant.

This means that if the hyperplane conjecture is not true then one can even have "pathologically good" (with respect to Question 1.1) pairs of convex bodies; if L_D is "large" then for any convex body K that satisfies (1.9) we see that the volume of K is "significantly smaller" than the volume of D. The proof of Theorem 1.3 carries over to higher codimensions but the dependence on L_D becomes more complicated and we prefer not to include the full statement of this version.

In Section 3 we collect some estimates on the lower dimensional Busemann-Petty problem. Let $1 \leqslant k \leqslant n-1$ and let $\beta_{n,k}$ be the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in $\mathbb{R}^n$ that satisfy

$$|K \cap F| \leqslant |D \cap F| \tag{1.11}$$

for all $F \in G_{n,n-k}$, one has

$$|K|^{\frac{n-k}{n}} \leqslant \beta^k \, |D|^{\frac{n-k}{n}}. \tag{1.12}$$

The following question is open:

Question 1.4. *Is it true that there exists an absolute constant $C_2 > 0$ such that $\beta_{n,k} \leqslant C_2$ for all n and k?*

Bourgain and Zhang [6] showed that $\beta_{n,k} > 1$ if $n-k > 3$. It is not known whether $\beta_{n,k}$ has to be greater than 1 when $n \geq 5$ and $n-k=2$ or $n-k=3$. It was proved in [22] and by a different method in [10] that $\beta_{n,k} \leq C\sqrt{n/k}(\log(en/k))^{3/2}$, where C is an absolute constant. In this note, we observe that the answer to Question 1.4 is affirmative if the convex body K has bounded isotropic constant, as follows.

Theorem 1.5. *Let $1 \leqslant k \leqslant n-1$ and let K be a centered convex body in $\mathbb{R}^n$ and D a compact subset of $\mathbb{R}^n$ such that*

$$|K \cap F| \leqslant |D \cap F| \tag{1.13}$$

for all $F \in G_{n,n-k}$. Then,

$$|K|^{\frac{n-k}{n}} \leqslant (c_0 L_K)^k \, |D|^{\frac{n-k}{n}}. \tag{1.14}$$

where $c_0 > 0$ is an absolute constant.

Theorem 1.5 is a refinement of the estimate $\beta_{n,k} \leqslant cL_n$, which was shown in [10]. The proof is based on estimates from [11] and on Grinberg's inequality (see (2.4) in Section 2).

We also discuss the lower dimensional Shephard problem. Let $1 \leqslant k \leqslant n-1$ and let $S_{n,k}$ be the smallest constant $S > 0$ with the following property: For every pair of convex bodies K and D in $\mathbb{R}^n$ that satisfy

$$|P_F(K)| \leqslant |P_F(D)| \tag{1.15}$$

for all $F \in G_{n,n-k}$, one has

$$|K|^{\frac{1}{n}} \leqslant S \, |D|^{\frac{1}{n}}. \tag{1.16}$$

Question 1.6. *Is it true that there exists an absolute constant $C_3 > 0$ such that $S_{n,k} \leqslant C_3$ for all n and k?*

Goodey and Zhang [15] proved that $S_{n,k} > 1$ if $n-k > 1$. In Section 4 we prove the following result.

Theorem 1.7. *Let K and D be two convex bodies in $\mathbb{R}^n$ such that*

$$|P_F(K)| \leqslant |P_F(D)| \tag{1.17}$$

for every $F \in G_{n,n-k}$. Then,

$$|K|^{\frac{1}{n}} \leqslant c_1 \sqrt{\frac{n}{n-k}} \log\left(\frac{en}{n-k}\right) |D|^{\frac{1}{n}}, \tag{1.18}$$

where $c_1 > 0$ is an absolute constant. It follows that $S_{n,k}$ is bounded by an absolute constant if $\frac{k}{n-k}$ is bounded.

We also prove a general estimate, which is logarithmic in n and valid for all k. The proof is based on estimates from [26].

Theorem 1.8. *Let K and D be two convex bodies in $\mathbb{R}^n$ such that*

$$|P_F(K)| \leqslant |P_F(D)| \tag{1.19}$$

for every $F \in G_{n,n-k}$. Then,

$$|K|^{\frac{1}{n}} \leqslant \frac{c_1 \min w(\tilde{D})}{\sqrt{n}} |D|^{\frac{1}{n}} \leqslant c_2(\log n)|D|^{\frac{1}{n}}, \tag{1.20}$$

where $c_1, c_2 > 0$ are absolute constants, $w(A)$ is the average of the support function of A on S^{n-1}, and the minimum is over all linear images $\tilde{D}$ of D that have volume 1.

The second inequality in (1.20) follows from the fact that if $\tilde{D}$ is a convex body of volume 1 in $\mathbb{R}^n$ which is in the minimal mean width position (i.e. $w(\tilde{D}) \leqslant w(T(\tilde{D}))$ for all $T \in SL(n)$) then $w(\tilde{D}) \leqslant c\sqrt{n}(\log n)$ for some absolute constant $c > 0$. This is a consequence of well-known results of Lewis, Figiel and Tomczak-Jaegermann, Pisier (see [1, Chapter 6] for a complete discussion).

Lutwak [25] proved that the answer to the Busemann-Petty problem is affirmative if the body K with smaller sections belongs to a special class of intersection bodies; see definition below. In Section 5 we prove separation in the Busemann-Petty problem, which can be considered as a refinement of Lutwak's result.

Theorem 1.9. *Suppose that $\varepsilon > 0$, K and D are origin-symmetric star bodies in $\mathbb{R}^n$, K is an intersection body. If*

$$|K \cap \xi^\perp| \leqslant |D \cap \xi^\perp| - \varepsilon, \tag{1.21}$$

for every $\xi \in S^{n-1}$, then

$$|K|^{\frac{n-1}{n}} \leqslant |D|^{\frac{n-1}{n}} - c\varepsilon \frac{1}{\sqrt{n}M(\overline{K})},$$

where $M(A)$ is the average of the Minkowski functional of A on S^{n-1}, $c > 0$ is an absolute constant and $\overline{K} = |K|^{-\frac{1}{n}}K$.

It was proved in [14] that if $\overline{K}$ is convex isotropic then

$$\frac{1}{M(\overline{K})} \geqslant c_1 \frac{n^{1/10}L_K}{\log^{2/5}(e+n)} \geqslant c_2 \frac{n^{1/10}}{\log^{2/5}(e+n)}. \tag{1.22}$$

It is also known (see [1, Chapter 6]) that if $\overline{K}$ is convex and is in the minimal mean width position then we have

$$\frac{1}{M(\overline{K})} \geqslant c_3 \frac{\sqrt{n}}{\log(e+n)}, \tag{1.23}$$

so, for these special positions of a convex body, the constant in Theorem 1.9 depends only on the dimension. This is an improvement of a previously known result from [20]. Also note that stability in the Busemann-Petty problem is easier and was proved in [20], as follows. If K is an intersection body in $\mathbb{R}^n$, D is an origin-symmetric star body in $\mathbb{R}^n$ and $\varepsilon > 0$ so that

$$|K \cap \xi^\perp| \leqslant |D \cap \xi^\perp| + \varepsilon \tag{1.24}$$

for every $\xi \in S^{n-1}$, then

$$|K|^{\frac{n-1}{n}} \leqslant |L|^{\frac{n-1}{n}} + c_n\varepsilon, \tag{1.25}$$

where $c_n = |B_2^{n-1}|/|B_2^n|^{\frac{n-1}{n}} < 1$. The constant is optimal. For more results on stability and separation in volume comparison problems and for applications of such results, see [21].

2 Milman's variant of the two problems

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,m}$ of m-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,m}$. Let $1\leqslant m\leqslant n-1$ and $F\in G_{n,m}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F=B_2^n\cap F$ and $S_F=S^{n-1}\cap F$.

The letters c,c',c_1,c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a\simeq b$, we mean that there exist absolute constants $c_1,c_2>0$ such that $c_1a\leqslant b\leqslant c_2a$. Also if $K,L\subseteq\mathbb{R}^n$ we will write $K\simeq L$ if there exist absolute constants $c_1,c_2>0$ such that $c_1K\subseteq L\subseteq c_2K$.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with nonempty interior. We say that K is origin-symmetric if $K=-K$. We say that K is centered if the center of mass of K is at the origin, i.e. $\int_K\langle x,\theta\rangle\,dx=0$ for every $\theta\in S^{n-1}$. We denote by $\mathcal{K}_n$ the class of centered convex bodies in $\mathbb{R}^n$. The support function of K is defined by $h_K(y):=\max\big\{\langle x,y\rangle : x\in K\big\}$, and the mean width of K is the average

$$w(K):=\int_{S^{n-1}}h_K(\theta)\,d\sigma(\theta) \tag{2.1}$$

of h_K on S^{n-1} (in the literature sometimes the right hand side of (2.1) corresponds to half of the mean width of K but this factor of 2 does not affect our estimates). For basic facts from the Brunn-Minkowski theory and the asymptotic theory of convex bodies we refer to the books [30] and [1] respectively.

The proof of Theorem 1.2 is based on two classical results:

1. *Aleksandrov's inequalities.* If K is a convex body in $\mathbb{R}^n$ then the sequence

$$Q_k(K)=\left(\frac{1}{\omega_k}\int_{G_{n,k}}|P_F(K)|\,d\nu_{n,k}(F)\right)^{1/k} \tag{2.2}$$

is decreasing in k. This is a consequence of the Aleksandrov-Fenchel inequality (see [9, Sections 20.1-20.2] and [30, Section 6.4]). In particular, for every $1\leqslant k\leqslant n-1$ we have

$$\left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}}\leqslant\left(\frac{1}{\omega_k}\int_{G_{n,k}}|P_F(K)|\,d\nu_{n,k}(F)\right)^{\frac{1}{k}}\leqslant w(K). \tag{2.3}$$

2. *Grinberg's inequality.* If D is a compact set in $\mathbb{R}^n$ then, for any $1\leqslant k\leqslant n-1$,

$$\tilde{R}_k(D):=\frac{1}{|D|^{n-k}}\int_{G_{n,n-k}}|D\cap F|^n\,d\nu_{n,n-k}(F)\leqslant\frac{1}{|B_2^n|^{n-k}}\int_{G_{n,n-k}}|B_2^n\cap F|^n\,d\nu_{n,n-k}(F), \tag{2.4}$$

where B_2^m is the Euclidean ball in $\mathbb{R}^m$ and $\omega_m=|B_2^m|$. This fact was proved by Grinberg in [16]. It is stated for convex bodies D but the proof applies to bounded Borel sets (see also [13]). It is useful to note that

$$\tilde{R}_k(B_2^n):=\frac{\omega_{n-k}^n}{\omega_n^{n-k}}\leqslant e^{\frac{kn}{2}}. \tag{2.5}$$

Moreover, Grinberg proved that the quantity $\tilde{R}_k(D)$ on the left hand side of (2.4) is invariant under $T\in GL(n)$: one has

$$\tilde{R}_k(T(D))=\tilde{R}_k(D) \tag{2.6}$$

for every $T\in GL(n)$.

Proof of Theorem 1.2. Let K be a convex body in $\mathbb{R}^n$ and D be a compact subset of $\mathbb{R}^n$. Assume that for some $1 \leqslant k \leqslant n-1$ we have

$$|P_F(K)| \leqslant |D \cap F| \tag{2.7}$$

for all $F \in G_{n,n-k}$. From (2.3) we get

$$\left(\frac{|K|}{\omega_n}\right)^{\frac{n-k}{n}} \leqslant \frac{1}{\omega_{n-k}} \int_{G_{n,n-k}} |P_F(K)| \, d\nu_{n,n-k}(F). \tag{2.8}$$

Our assumption, Hölder's inequality and Grinberg's inequality give

$$\begin{aligned} \frac{1}{\omega_{n-k}} \int_{G_{n,n-k}} |P_F(K)| \, d\nu_{n,n-k}(F) &\leqslant \frac{1}{\omega_{n-k}} \int_{G_{n,n-k}} |D \cap F| \, d\nu_{n,n-k}(F) \\ &\leqslant \frac{1}{\omega_{n-k}} \left(\int_{G_{n,n-k}} |D \cap F|^n \, d\nu_{n,n-k}(F) \right)^{\frac{1}{n}} \\ &\leqslant \frac{1}{\omega_{n-k}} \frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}} |D|^{\frac{n-k}{n}} = \left(\frac{|D|}{\omega_n}\right)^{\frac{n-k}{n}}. \end{aligned} \tag{2.9}$$

Therefore, $|K| \leqslant |D|$. □

Remark 2.1. Assume that $1 \leqslant k \leqslant n-2$, and let K and D be two convex bodies in $\mathbb{R}^n$ such that $|P_F(K)| \leqslant |D \cap F|$ for all $F \in G_{n,n-k}$ and $|K| = |D|$. Then, (2.8) must be an equality, which implies that K is a ball. We must also have equality in Grinberg's inequality for D, which implies that D is an ellipsoid. Finally, we must have $|P_F(K)| = |D \cap F|$ for all $F \in G_{n,n-k}$, which means that all $(n-k)$-dimensional sections of the ellipsoid D have the same volume, and hence D is also a ball.

Remark 2.2. Slightly modifying the proof of Theorem 1.2 one can get stability and separation results, as follows. Let $\varepsilon > 0$, and let K and D be as in Theorem 1.2. Suppose that for every $F \in G_{n,n-k}$

$$|P_F(K)| \leq |D \cap F| \pm \varepsilon.$$

Then

$$|K|^{\frac{n-k}{n}} \leq |D|^{\frac{n-k}{n}} \pm \gamma_{n,k}\varepsilon,$$

where $\gamma_{n,k} = \frac{\omega_n^{\frac{n-k}{n}}}{\omega_{n-k}} \in (e^{-k/2}, 1)$. The plus sign corresponds to stability, minus - to separation. Assuming that $\varepsilon = \max_F(|P_F(K) - |D \cap F|)$ in the stability result, we get

$$|K|^{\frac{n-k}{n}} - |D|^{\frac{n-k}{n}} \leq \gamma_{n,k} \max_F(|P_F(K) - |D \cap F|).$$

On the other hand, if $\varepsilon = \min_F(|D \cap F| - |P_F(K)|)$ in the separation result, then

$$|D|^{\frac{n-k}{n}} - |K|^{\frac{n-k}{n}} \geq \gamma_{n,k} \min_F(|D \cap F| - |P_F(K)|).$$

3 Estimates for the lower dimensional Busemann-Petty problem

In this section we provide some estimates for the lower dimensional Busemann-Petty problem. We need the next lemma, in which we collect known estimates about the quantities

$$G_{n,k}(A) := \left(\int_{G_{n,n-k}} |A \cap F|^n \, d\nu_{n,n-k}(F) \right)^{\frac{1}{kn}}, \tag{3.1}$$

where A is a centered convex body in $\mathbb{R}^n$. The proof of (3.2) can be found in [11], while (3.3) follows from (2.4) and (2.5).

Lemma 3.1. *Let A be a centered convex body in $\mathbb{R}^n$. Then,*

$$\frac{c_1}{L_A}|A|^{\frac{n-k}{kn}} \leqslant G_{n,k}(A) \leqslant \frac{c_2 L_k}{L_A}|A|^{\frac{n-k}{kn}} \leqslant \frac{c_3\sqrt[4]{k}}{L_A}|A|^{\frac{n-k}{kn}}. \tag{3.2}$$

Moreover, for every compact subset D of $\mathbb{R}^n$ we have

$$G_{n,k}(D) \leqslant \sqrt{e}|D|^{\frac{n-k}{kn}}. \tag{3.3}$$

Using Lemma 3.1 we show that the lower dimensional Busemann-Petty problem (Question 1.4) has an affirmative answer if the body K has bounded isotropic constant.

Proof of Theorem 1.5. Since $|K\cap F| \leqslant |D\cap F|$ for all $F \in G_{n,n-k}$, we know that

$$G_{n,k}(K) \leqslant G_{n,k}(D). \tag{3.4}$$

Using (3.2) and (3.3) we write

$$\frac{c_1}{L_K}|K|^{\frac{n-k}{kn}} \leqslant G_{n,k}(K) \leqslant G_{n,k}(D) \leqslant \sqrt{e}|D|^{\frac{n-k}{kn}}, \tag{3.5}$$

and the result follows. □.

Remark 3.2. Theorem 1.5 shows that if K belongs to the class

$$\mathcal{K}_n(\alpha) := \{K \in \mathcal{K}_n : L_K \leqslant \alpha\} \tag{3.6}$$

for some $\alpha > 0$, then for every compact set D in $\mathbb{R}^n$ which satisfies $|K\cap F| \leqslant |D\cap F|$ for all $F \in G_{n,n-k}$ we have

$$|K|^{\frac{n-k}{n}} \leqslant (c_0\alpha)^k\, |D|^{\frac{n-k}{n}}. \tag{3.7}$$

Classes of convex bodies with uniformly bounded isotropic constant include: unconditional convex bodies, convex bodies whose polar bodies contain large affine cubes, the unit balls of 2-convex spaces with a given constant α, bodies with small diameter (in particular, the class of zonoids) and the unit balls of the Schatten classes (see [7, Chapter 4]).

Example. K. Ball has proved in [2] that for every $1 \leqslant k \leqslant n-1$ and every $F \in G_{n,n-k}$ we have

$$|Q_n \cap F| \leqslant 2^{\frac{k}{2}}, \tag{3.8}$$

where Q_n is the cube of volume 1 in $\mathbb{R}^n$. Consider the ball $B_{n,k} = r_{n,k}B_2^n$, where

$$\omega_{n-k} r_{n,k}^{n-k} = 2^{\frac{k}{2}}. \tag{3.9}$$

Then, for every $F \in G_{n,n-k}$ we have

$$|Q_n \cap F| \leqslant |B_{n,k} \cap F|. \tag{3.10}$$

Therefore,

$$1 = |Q_n| \leqslant \beta_{n,k}^k |B_{n,k}|^{\frac{n-k}{n}} = \beta_{n,k}^k \omega_n^{\frac{n-k}{n}} r_{n,k}^{n-k} = 2^{\frac{k}{2}}\beta_{n,k}^k \frac{\omega_n^{\frac{n-k}{n}}}{\omega_{n-k}}. \tag{3.11}$$

This proves that

$$\beta_{n,k} \geqslant \frac{1}{\sqrt{2}}\left(\frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}}\right)^{\frac{1}{k}} \sim \frac{1}{\sqrt{2}}\left(\frac{n}{n-k}\right)^{\frac{n-k+1}{2k}} \tag{3.12}$$

as $n,k \to \infty$. Fix $d \geqslant 2$ and consider n and k that satisfy $n = (d+1)k$. Then, we have the following:

Proposition 3.3. *For every $d \geqslant 2$ there exists $k(d) \in \mathbb{N}$ such that*

$$\beta_{(d+1)k,k} \geqslant \frac{1}{\sqrt{2}}\left(1+\frac{1}{d}\right)^{\frac{d}{2}} > 1 \tag{3.13}$$

for all $k \geqslant k(d)$. □

A variant of the proof of Theorem 1.5 (based again on Lemma 3.1) establishes Theorem 1.3.

Proof of Theorem 1.3. Let K be a convex body in $\mathbb{R}^n$ and D be a centered convex body in $\mathbb{R}^n$ such that $|P_{\xi^\perp}(K)| \leqslant |D \cap \xi^\perp|$ for every $\xi \in S^{n-1}$. From Lemma 3.1 we know that

$$G_{n,1}(D) \leqslant \frac{c_1}{L_D}|D|^{\frac{n-1}{n}}, \tag{3.14}$$

where $c_1 > 0$ is an absolute constant. Then,

$$\begin{aligned} \left(\frac{|K|}{\omega_n}\right)^{\frac{n-1}{n}} &\leqslant \frac{1}{\omega_{n-1}}\int_{S^{n-1}}|P_{\xi^\perp}(K)|\,d\sigma(\xi) \leqslant \frac{1}{\omega_{n-1}}\int_{S^{n-1}}|D\cap\xi^\perp|\,d\sigma(\xi) \\ &= \frac{1}{\omega_{n-1}}\left(\int_{S^{n-1}}|D\cap\xi^\perp|^n\,d\sigma(\xi)\right)^{\frac{1}{n}} \\ &= \frac{1}{\omega_{n-1}}G_{n,1}(D) \leqslant \frac{c_1}{\omega_{n-1}L_D}|D|^{\frac{n-1}{n}}, \end{aligned} \tag{3.15}$$

which implies that

$$|K| \leqslant \frac{c_2\omega_n}{(\omega_{n-1}L_D)^{\frac{n}{n-1}}}\,|D| \leqslant \frac{c_3}{L_D}\,|D|, \tag{3.16}$$

where $c_2, c_3 > 0$ are absolute constants. □

4 Estimates for the lower dimensional Shephard problem

In this section we discuss the lower dimensional Shephard problem. First, we recall some facts for the class of zonoids. A zonoid is a limit of Minkowski sums of line segments in the Hausdorff metric. Equivalently, a symmetric convex body Z is a zonoid if and only if its polar body is the unit ball of an n-dimensional subspace of an L_1-space; i.e. if there exists a positive measure μ (the supporting measure of Z) on S^{n-1} such that

$$h_Z(x) = \|x\|_{Z^\circ} = \frac{1}{2}\int_{S^{n-1}}|\langle x,y\rangle|d\mu(y).$$

The class of origin-symmetric zonoids coincides with the class of projection bodies. Recall that the projection body ΠK of a convex body K is the symmetric convex body whose support function is defined by

$$h_{\Pi K}(\xi) = |P_{\xi^\perp}(K)|, \qquad \xi \in S^{n-1}.$$

From Cauchy's formula

$$|P_{\xi^\perp}(K)| = \frac{1}{2}\int_{S^{n-1}}|\langle u,\xi\rangle|\,d\sigma_K(u),$$

where σ_K is the surface area measure of K, it follows that the projection body of K is a zonoid whose supporting measure is σ_K. Minkowski's existence theorem implies that, conversely, every zonoid is the projection body of some symmetric convex body in $\mathbb{R}^n$.

Zoonoids play a central role in the study of the original Shephard problem: suppose that K is a convex body in $\mathbb{R}^n$ and Z is a zonoid in $\mathbb{R}^n$, and that $|P_{\xi^\perp}(K)| \leqslant |P_{\xi^\perp}(Z)|$ for all $\xi \in S^{n-1}$. Then,

$$|K| \leqslant |Z|. \tag{4.1}$$

The proof involves writing $Z = \Pi D$ for some convex body D, using the identity $V_1(K, \Pi D) = V_1(D, \Pi K)$ (where $V_1(A, B)$ is the mixed volume $V(A, \ldots, A, B)$), the hypothesis in the form $\Pi(K) \subseteq \Pi(Z)$, and the monotonicity of $V_1(D, .)$, to write

$$|Z| = V_1(Z, Z) = V_1(Z, \Pi D) = V_1(D, \Pi Z) \geqslant V_1(D, \Pi(K)) = V_1(K, \Pi D) = V_1(K, Z) \geqslant |K|^{\frac{n-1}{n}} |Z|^{\frac{1}{n}},$$

where in the last step we also employ Minokowski's first inequality. This shows that $|Z| \geqslant |K|$.

Since any projection of a zonoid is a zonoid, using an inductive argument we can prove the following (for a detailed account on this topic, see [12, Chapter 4]).

Theorem 4.1. *Let K be a convex body and let Z be a zonoid in $\mathbb{R}^n$ such that*

$$|P_F(K)| \leqslant |P_F(Z)| \tag{4.2}$$

for every $F \in G_{n,n-k}$. Then,

$$|K| \leqslant |Z|. \tag{4.3}$$

Using Theorem 4.1 and the fact that every ellipsoid is a zonoid, we can give a simple bound for the constants $S_{n,k}$.

Proposition 4.2. *For all n and $1 \leqslant k \leqslant n-1$ we have $S_{n,k} \leqslant c_0\sqrt{n}$, where $c_0 > 0$ is an absolute constant.*

Proof. Let K and D be two convex bodies in $\mathbb{R}^n$ such that $|P_F(K)| \leqslant |P_F(D)|$ for every $F \in G_{n,n-k}$. There exists an ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ such that $D \subseteq \mathcal{E}$ and $|\mathcal{E}|^{\frac{1}{n}} \leqslant c_0\sqrt{n}\,|D|^{\frac{1}{n}}$, where $c_0 > 0$ is an absolute constant (for example, see [4] where a sharp estimate for c_0 is also given). Since $D \subseteq \mathcal{E}$, we have

$$|P_F(K)| \leqslant |P_F(D)| \leqslant |P_F(\mathcal{E})| \tag{4.4}$$

for all $F \in G_{n,n-k}$. Since $\mathcal{E}$ is a zonoid, Theorem 4.1 implies that

$$|K|^{\frac{1}{n}} \leqslant |\mathcal{E}|^{\frac{1}{n}} \leqslant c_0\sqrt{n}\,|D|^{\frac{1}{n}}. \tag{4.5}$$

This shows that $S_{n,k} \leqslant c_0\sqrt{n}$. $\square$

We can elaborate on this argument if we use Pisier's theorem from [28] on the existence of α-regular M-ellipsoids for symmetric convex bodies in $\mathbb{R}^n$ (see [7, Theorem 1.13.3]). Recall that the covering number $N(A, B)$ of a convex body A by a second convex body B is the least integer N for which there exist N translates of B whose union covers A.

Theorem 4.3 (Pisier)**.** *For every $0 < \alpha < 2$ and every symmetric convex body A in $\mathbb{R}^n$, there exists an ellipsoid $\mathcal{E}_\alpha$ such that*

$$\max\{N(A, t\mathcal{E}_\alpha), N(\mathcal{E}_\alpha, tA)\} \leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $t \geqslant 1$, where $c(\alpha)$ is a constant depending only on α and satisfies $c(\alpha) = O\big((2-\alpha)^{-\alpha/2}\big)$ as $\alpha \to 2$.

Theorem 4.4. *Let $1 \leqslant m \leqslant n-1$ and let K and D be two convex bodies in $\mathbb{R}^n$ such that*

$$|P_F(K)| \leqslant |P_F(D)| \tag{4.6}$$

for every $F \in G_{n,m}$. Then,

$$|K|^{\frac{1}{n}} \leqslant c_1\sqrt{\frac{n}{m}}\log\left(\frac{en}{m}\right)|D|^{\frac{1}{n}}, \tag{4.7}$$

where $c_1 > 0$ is an absolute constant.

Proof. Consider the difference body $D-D$ of D, and the ellipsoid $\mathcal{E}_\alpha$ from Theorem 4.3, where $\alpha\in(0,2)$ will be chosen in the end, that corresponds to $A=D-D$. Note that

$$N(\mathcal{E}_\alpha, c(\alpha)^{1/\alpha}(D-D))\leqslant e^n, \tag{4.8}$$

therefore

$$|\mathcal{E}_\alpha|^{\frac{1}{n}}\leqslant ec(\alpha)^{1/\alpha}|D-D|^{\frac{1}{n}}. \tag{4.9}$$

Since

$$N(P_F(D-D),P_F(t\mathcal{E}_\alpha))\leqslant N(D-D,t\mathcal{E}_\alpha)\leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right) \tag{4.10}$$

for every $F\in G_{n,m}$, we have

$$|P_F(D-D)|\leqslant \exp\left(\frac{c(\alpha)n}{t_{n,m,\alpha}^\alpha}\right)|P_F(t_{n,m,\alpha}\mathcal{E}_\alpha)|=|P_F(et_{n,m,\alpha}\mathcal{E}_\alpha)| \tag{4.11}$$

if we choose

$$t_{n,m,\alpha}=\left(\frac{c(\alpha)n}{m}\right)^{\frac{1}{\alpha}}. \tag{4.12}$$

Now, if we set $\mathcal{E}:=et_{n,m,\alpha}\mathcal{E}_\alpha$, we have

$$|P_F(K)|\leqslant |P_F(D)|\leqslant |P_F(D-D)|\leqslant |P_F(\mathcal{E})| \tag{4.13}$$

for every $F\in G_{n,m}$, and since $\mathcal{E}$ is a zonoid, Theorem 4.1 shows that $|K|\leqslant|\mathcal{E}|$. Using also (4.9) and the fact that $c(\alpha)=O\big((2-\alpha)^{-\alpha/2}\big)$, we get

$$|K|^{\frac{1}{n}}\leqslant et_{n,m,\alpha}|\mathcal{E}_\alpha|^{\frac{1}{n}}\leqslant e^2t_{n,m,\alpha}c(\alpha)^{1/\alpha}|D-D|^{\frac{1}{n}}\leqslant \frac{c_1}{2-\alpha}\left(\frac{n}{m}\right)^{\frac{1}{\alpha}}|D|^{\frac{1}{n}}, \tag{4.14}$$

where $c_1>0$ is an absolute constant (we have also used the fact that $|D-D|^{\frac{1}{n}}\leqslant 4|D|^{\frac{1}{n}}$ by the Rogers-Shephard inequality). Choosing $\alpha=2-\frac{1}{\log\left(\frac{en}{m}\right)}$ we get the result. □

Remark 4.5. The lower dimensional Shephard problem is related to Lutwak's conjectures about the affine quermassintegrals: for every convex body K in $\mathbb{R}^n$ and every $1\leqslant m\leqslant n-1$, the quantities

$$\Phi_{n-m}(K)=\frac{\omega_n}{\omega_m}\left(\int_{G_{n,m}}|P_F(K)|^{-n}d\nu_{n,m}(F)\right)^{-1/n}, \tag{4.15}$$

were introduced by Lutwak in [23] (and Grinberg proved in [16] that these quantities are invariant under volume preserving affine transformations). Lutwak conjectured in [24] that the affine quermassintegrals satisfy the inequalities

$$\omega_n^j\Phi_i(K)^{n-j}\leqslant \omega_n^i\Phi_j(K)^{n-i} \tag{4.16}$$

for all $0\leqslant i<j<n$, where we agree that $\Phi_0(K)=|K|$ and $\Phi_n(K)=\omega_n$. Most of the conjectures about the affine quermassintegrals remain open (see [12, Chapter 9] for more details and references). If true, they would imply the following (see also [11]): there exist absolute constants $c_1,c_2>0$ such that for every convex body K in $\mathbb{R}^n$ and every $1\leqslant m\leqslant n-1$,

$$c_1\sqrt{n/m}\,|K|^{\frac{1}{n}}\leqslant\left(\int_{G_{n,m}}|P_F(K)|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}}\leqslant c_2\sqrt{n/m}\,|K|^{\frac{1}{n}}. \tag{4.17}$$

Assuming (4.17) we can give an affirmative answer to Question 1.6. Indeed, let K and D be two convex bodies in $\mathbb{R}^n$ such that $|P_F(K)| \leqslant |P_F(D)|$ for every $F \in G_{n,n-k}$. We write

$$
\begin{aligned}
(4.18)\qquad c_1\sqrt{n/(n-k)}\,|K|^{\frac{1}{n}} &\leqslant \left(\int_{G_{n,n-k}} |P_F(K)|^{-n}\,d\nu_{n,n-k}(F)\right)^{-\frac{1}{(n-k)n}} \\
&\leqslant \left(\int_{G_{n,n-k}} |P_F(D)|^{-n}\,d\nu_{n,n-k}(F)\right)^{-\frac{1}{(n-k)n}} \leqslant c_2\sqrt{n/(n-k)}\,|D|^{\frac{1}{n}},
\end{aligned}
$$

and this shows that $|K|^{\frac{1}{n}} \leqslant (c_2/c_1)\,|D|^{\frac{1}{n}}$.

The left hand side of (4.17) was proved by Paouris and Pivovarov in [26]:

Theorem 4.6 (Paouris-Pivovarov)**.** *Let A be a convex body in $\mathbb{R}^n$. Then,*

$$
(4.19)\qquad \left(\int_{G_{n,m}} |P_F(A)|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}} \geqslant c\sqrt{n/m}|A|^{\frac{1}{n}}.
$$

Using this fact one can obtain the following.

Proposition 4.7. *Let $1 \leqslant m \leqslant n-1$ and let K and D be two convex bodies in $\mathbb{R}^n$ such that*

$$
(4.20)\qquad |P_F(K)| \leqslant |P_F(D)|
$$

for every $F \in G_{n,m}$. Then,

$$
(4.21)\qquad |K|^{\frac{1}{n}} \leqslant \frac{c\,\min w(\tilde{D})}{\sqrt{n}}\,|D|^{\frac{1}{n}},
$$

where $c > 0$ is an absolute constant and the minimum is over all linear images $\tilde{D}$ of D that have volume 1.

Proof. Our assumption implies that

$$
(4.22)\qquad \left(\int_{G_{n,m}} |P_F(K)|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}} \leqslant \left(\int_{G_{n,m}} |P_F(D)|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}}
$$

By the linear invariance of $\Phi_{n-m}(D)$, for any $\tilde{D} = T(D)$ where $T \in GL(n)$ and $|\tilde{D}| = 1$, we have

$$
(4.23)\qquad \left(\int_{G_{n,m}} |P_F(D)|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}} = |D|^{\frac{1}{n}}\left(\int_{G_{n,m}} |P_F(\tilde{D})|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}}.
$$

Now, using Hölder's inequality we write

$$
(4.24)\qquad \left(\int_{G_{n,m}} |P_F(\tilde{D})|^{-n}\,d\nu_{n,m}(F)\right)^{-\frac{1}{mn}} \leqslant \left(\int_{G_{n,m}} |P_F(\tilde{D})|\,d\nu_{n,m}(F)\right)^{\frac{1}{m}}
$$

From Aleksandrov's inequalites we have

$$
(4.25)\qquad \left(\int_{G_{n,m}} |P_F(\tilde{D})|\,d\nu_{n,m}(F)\right)^{\frac{1}{m}} \leqslant \omega_m^{\frac{1}{m}}\,w(\tilde{D}) \leqslant c_2\sqrt{n/m}\frac{w(\tilde{D})}{\sqrt{n}}.
$$

Taking into account Theorem 4.6 we get

$$c\sqrt{n/m}|K|^{\frac{1}{n}} \leqslant \left(\int_{G_{n,m}} |P_F(K)|^{-\frac{n}{m}}\, d\nu_{n,m}(F)\right)^{-\frac{1}{n}} \leqslant \left(\int_{G_{n,m}} |P_F(D)|\, d\nu_{n,m}(F)\right)^{\frac{1}{m}} \tag{4.26}$$

$$= \left(\int_{G_{n,m}} |P_F(\tilde{D})|\, d\nu_{n,m}(F)\right)^{\frac{1}{m}} |D|^{\frac{1}{n}} \leqslant c_2\sqrt{n/m}\frac{w(\tilde{D})}{\sqrt{n}}|D|^{\frac{1}{n}},$$

and the result follows. □

As a corollary we have:

Theorem 4.8. *Let $1 \leqslant m \leqslant n-1$ and let K and D be two convex bodies in $\mathbb{R}^n$ such that*

$$|P_F(K)| \leqslant |P_F(D)| \tag{4.27}$$

for every $F \in G_{n,m}$. Then,

$$|K|^{\frac{1}{n}} \leqslant c(\log n)\,|D|^{\frac{1}{n}}, \tag{4.28}$$

where $c > 0$ is an absolute constant.

Proof. If D is in the minimal mean width position, we have (see [1, Chapter 6])

$$w(\overline{D}) \leqslant c_1\sqrt{n}(\log n). \tag{4.29}$$

The result follows from Proposition 4.7. □

5 Separation in the Busemann-Petty problem

For the proof of Theorem 1.9 we need several definitions from convex geometry. A closed bounded set K in $\mathbb{R}^n$ is called a star body if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geqslant 0 :\ x \in aK\} \tag{5.1}$$

is a continuous function on $\mathbb{R}^n$.

The radial function of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n,\ x \neq 0. \tag{5.2}$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x.

We use the polar formula for the volume of a star body:

$$|K| = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta, \tag{5.3}$$

where $d\theta$ stands for the uniform measure on the sphere with density 1.

The class of intersection bodies was introduced by Lutwak in [25]. Let K, D be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of D and write $K = ID$ if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = |D \cap \xi^\perp| = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \|\theta\|_D^{-n+1} d\theta \tag{5.4}$$

$$= \frac{1}{n-1} R\left(\|\cdot\|_D^{-n+1}\right)(\xi),$$

where $R : C(S^{n-1}) \to C(S^{n-1})$ is the spherical Radon transform

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)dx, \qquad \text{for all } f \in C(S^{n-1}). \tag{5.5}$$

All bodies K that appear as intersection bodies of different star bodies form the class of intersection bodies of star bodies. A more general class of intersection bodies is defined as follows. If μ is a finite Borel measure on S^{n-1}, then the spherical Radon transform $R\mu$ of μ is defined as a functional on $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x)d\mu(x), \qquad \text{for all } f \in C(S^{n-1}). \tag{5.6}$$

A star body K in $\mathbb{R}^n$ is called an *intersection body* if $\|\cdot\|_K^{-1} = R\mu$ for some measure μ, as functionals on $C(S^{n-1})$, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)dx = \int_{S^{n-1}} Rf(x)d\mu(x), \qquad \text{for all } f \in C(S^{n-1}). \tag{5.7}$$

Intersection bodies played the key role in the solution of the Busemann-Petty problem. The reader will find more information on the Radon transform and intersection bodies in the book [19].

Recall that $d\sigma(x) = dx/|S^{n-1}|$ is the normalized uniform measure on the sphere, and that

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma(x).$$

is the average of the norm $\|\cdot\|_K$ on S^{n-1}.

Proof of Theorem 1.9. By (5.4), the condition (1.21) can be written as

$$R(\|\cdot\|_K^{-n+1})(\xi) \leqslant R(\|\cdot\|_D^{-n+1})(\xi) - (n-1)\varepsilon, \text{ for all } \xi \in S^{n-1}. \tag{5.8}$$

Since K is an intersection body, there exists a finite Borel measure μ on S^{n-1} such that $\|\cdot\|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$. Together with (5.3), (5.8) and the definition of $R\mu$ (more precisely (5.7)), the latter implies that

$$n|K| = \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_K^{-n+1}\, dx = \int_{S^{n-1}} R\left(\|\cdot\|_K^{-n+1}\right)(\xi)\, d\mu(\xi) \tag{5.9}$$

$$\leqslant \int_{S^{n-1}} R\left(\|\cdot\|_D^{-n+1}\right)(\xi)\, d\mu(\xi) - (n-1)\varepsilon \int_{S^{n-1}} d\mu(\xi)$$

$$= \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_D^{-n+1}\, dx - (n-1)\varepsilon \int_{S^{n-1}} d\mu(x). \tag{5.10}$$

We estimate the first term in (5.10) using Hölder's inequality:

$$\int_{S^{n-1}} \|x\|_K^{-1}\|x\|_D^{-n+1}\, dx \leqslant \left(\int_{S^{n-1}} \|x\|_K^{-n}\, dx\right)^{\frac{1}{n}} \left(\int_{S^{n-1}} \|x\|_D^{-n}\, dx\right)^{\frac{n-1}{n}} = n|K|^{\frac{1}{n}}|D|^{\frac{n-1}{n}}. \tag{5.11}$$

We now estimate the second term in (5.10) adding the Radon transform of the unit constant function under the integral ($R\mathbf{1}(x) = \left|S^{n-2}\right|$ for every $x \in S^{n-1}$), and using again the fact that $\|\cdot\|_K^{-1} = R\mu$:

$$(n-1)\varepsilon \int_{S^{n-1}} d\mu(x) = \frac{(n-1)\varepsilon}{|S^{n-2}|} \int_{S^{n-1}} R1(x)\, d\mu(x) = \frac{(n-1)\varepsilon}{|S^{n-2}|} \int_{S^{n-1}} \|x\|_K^{-1}\, dx \tag{5.12}$$

$$\geqslant c_1\varepsilon \frac{(n-1)|S^{n-1}|}{|S^{n-2}|} \frac{1}{M(\overline{K})}|K|^{\frac{1}{n}}$$

$$\geqslant c_2\varepsilon\sqrt{n}\frac{1}{M(\overline{K})}|K|^{\frac{1}{n}}, \tag{5.13}$$

since

$$\int_{S^{n-1}} \|x\|_K^{-1}\, d\sigma(x) \geq \left(\int_{S^{n-1}} \|x\|_K d\sigma(x)\right)^{-1} = \frac{1}{M(\overline{K})}|K|^{\frac{1}{n}}, \tag{5.14}$$

by Jensen's inequality, homogeneity, and the formulas

$$\left|S^{n-2}\right| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \qquad \text{and} \qquad \left|S^{n-1}\right| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{5.15}$$

Combining (5.13) with (5.10) and (5.11), we get

$$n|K| \leqslant n|K|^{\frac{1}{n}}|L|^{\frac{n-1}{n}} - c_2\varepsilon\sqrt{n}\frac{1}{M(\overline{K})}|K|^{\frac{1}{n}} \tag{5.16}$$

and, after dividing by $n|K|^{1/n}$, the proof is complete. □

Separation implies a volume difference inequality.

Corollary 5.1. *Let D be any origin-symmetric star body in $\mathbb{R}^n$, and let K be an intersection body, which is a dilate of an isotropic body. Suppose that*

$$\min_{\xi\in S^{n-1}} \left(|D\cap\xi^\perp| - |K\cap\xi^\perp|\right) > 0.$$

Then,

$$|D|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \geqslant c_2\frac{1}{\sqrt{n}M(\overline{K})}\min_{\xi\in S^{n-1}} \left(|D\cap\xi^\perp| - |K\cap\xi^\perp|\right).$$

Remark 5.2. It was proved in [14] that there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and any origin-symmetric isotropic convex body K in $\mathbb{R}^n$

$$\frac{1}{M(K)} \geqslant\ c_1\frac{n^{1/10}L_K}{\log^{2/5}(e+n)} \geqslant c_2\frac{n^{1/10}}{\log^{2/5}(e+n)}. \tag{5.17}$$

Also, if K is convex, has volume 1 and is in the minimal mean width position then we have

$$\frac{1}{M(K)} \geqslant\ c_3\frac{\sqrt{n}}{\log(e+n)}. \tag{5.18}$$

Inserting these estimates into Theorem 1.9 and Corollary 5.1 we obtain estimates independent from the bodies.

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