On the volume ratio of two convex bodies

A. Giannopoulos and M. Hartzoulaki

Abstract

Let K and L be two convex bodies in $\mathbb{R}^n$. The volume ratio $\mathrm{vr}(K,L)$ of K and L is defined by $\mathrm{vr}(K,L)=\inf(|K|/|T(L)|)^{1/n}$, where the infimum is over all affine transformations T of $\mathbb{R}^n$ for which $T(L)\subseteq K$. We show that $\mathrm{vr}(K,L)\leq c\sqrt{n}\log n$, where $c>0$ is an absolute constant. This is optimal up to the logarithmic term.

1 Introduction

Let K and L be two convex bodies in $\mathbb{R}^n$. The *volume ratio* of K and L is the quantity

$$\mathrm{vr}(K,L):=\inf\left(\frac{|K|}{|T(L)|}\right)^{1/n},$$

where the infimum is taken over all affine transformations T of $\mathbb{R}^n$ for which $T(L)\subseteq K$ (by $|\cdot|$ we denote n-dimensional volume).

Let B_n denote the Euclidean unit ball in $\mathbb{R}^n$. Using the Brascamp-Lieb inequality, Ball [1] proved that $\mathrm{vr}(K,B_n)$ is maximal when K is the simplex S_n. A consequence of Barthe's reverse Brascamp-Lieb inequality [2] is that $\mathrm{vr}(B_n,L)$ is also maximal when $L=S_n$. It follows that

$$\mathrm{vr}(K,L)\leq \mathrm{vr}(K,B_n)\mathrm{vr}(B_n,L)\leq \mathrm{vr}(S_n,B_n)\mathrm{vr}(B_n,S_n)=n$$

for every pair of convex bodies K and L. A direct proof of the same fact was given in [5] where the "maximal volume position" of L inside K is studied.

The purpose of this note is to prove the following general estimate.

Theorem. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then,*

$$\mathrm{vr}(K,L)\leq c\sqrt{n}\log n$$

where $c>0$ is an absolute constant.

[1]2000 Mathematics Subject Classification: Primary 52A40, 46B07; Secondary 52A21, 52A20.

The example of the ball and the simplex shows that this estimate is optimal up to the logarithmic term. The proof of the theorem is based on the method of random orthogonal factorizations; actually, an essentially direct application of this method gives the estimate $\mathrm{vr}(K,L) = O(\sqrt{n}\log^2 n)$. We can remove one logarithmic term using an idea of Rudelson [11] who started with the same method to obtain the estimate $O(n^{4/3}\log^\beta n)$ for the Banach-Mazur distance of two convex bodies K and L in $\mathbb{R}^n$.

2 Proof of the theorem

We assume that $\mathbb{R}^n$ is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$ and denote the corresponding Euclidean norm by $|\cdot|$. B_n is the Euclidean unit ball and S^{n-1} is the unit sphere. We also write $|\cdot|$ for the volume (Lebesgue measure) in $\mathbb{R}^n$, σ for the rotationally invariant probability measure on S^{n-1}, and μ for the Haar probability measure on the orthogonal group $O(n)$. The letters c, c_1, c_2 etc. denote absolute positive constants which may change from line to line.

Let W be a symmetric convex body in $\mathbb{R}^n$. Then, the function $\|x\|_W = \inf\{\lambda \geq 0 : x \in \lambda K\}$ is a norm on $\mathbb{R}^n$ and W is the unit ball of the normed space $(\mathbb{R}^n, \|\cdot\|_W)$. We write ℓ_2^n for the Euclidean space $(\mathbb{R}^n, |\cdot|)$. The polar body of W is defined by

$$W^\circ = \{y \in \mathbb{R}^n : |\langle x,y\rangle| \leq 1 \text{ for all } x \in W\}.$$

In other words, $\|y\|_{W^\circ} = \max_{x\in W}|\langle x,y\rangle|$. Note that $X_{W^\circ} = X_W^*$; W° is the unit ball of the dual space of X_W. Note also that $(TW)^\circ = (T^{-1})^*(W^\circ)$ for every $T \in GL(n)$.

If X_{W_1} and X_{W_2} are two n-dimensional normed spaces as above, their Banach-Mazur distance $d(X_{W_1}, X_{W_2})$ is defined by

$$d(X_{W_1}, X_{W_2}) = \inf_{T\in GL(n)} \|T : X_{W_1} \to X_{W_2}\| \cdot \|T^{-1} : X_{W_2} \to X_{W_1}\|.$$

We write d_W for the Banach-Mazur distance $d(X_W, \ell_2^n)$.

For every origin symmetric convex body W in $\mathbb{R}^n$ we define the mean width $b(W)$ of W by

$$b(W) = \int_{S^{n-1}} \max_{y\in W}|\langle\theta,y\rangle|\sigma(d\theta) = \int_{S^{n-1}} \|\theta\|_{W^\circ}\sigma(d\theta).$$

Then, Urysohn's inequality (see [10], pp. 6) states that

$$\left(\frac{|W|}{|B_n|}\right)^{1/n} \leq b(W) \tag{1}$$

with equality if and only if W is a ball.

Let $L(\ell_2^n, X_W)$ denote the space of all linear operators from ℓ_2^n to X_W. The ℓ-norm of an operator $T \in L(\ell_2^n, X_W)$ is defined by

$$\ell(T) = \left(\int_{\mathbb{R}^n} \|T(x)\|_W^2 \gamma_n(dx)\right)^{1/2}$$

where γ_n is the canonical Gaussian probability measure on $\mathbb{R}^n$.

Figiel and Tomczak-Jaegermann [4] introduced the ℓ-norm and, using a general result of Lewis [7] about trace dual norms of operators, they proved that for every W there exists $T \in L(\ell_2^n, X_W)$ such that

$$\ell(T)\ell((T^{-1})^*) \leq nK(X_W), \tag{2}$$

where $K(X_W)$ is the K-convexity constant of X_W (see [10], pp. 20). On the other hand, an important inequality of Pisier [9] (see also [10], Chapter 2) states that

$$K(X_W) \leq c_1 \log(d_W + 1) \tag{3}$$

for every W, where $c_1 > 0$ is an absolute constant.

We will alternatively write $\ell(T^{-1}(W))$ instead of $\ell(T)$. With this notation, our first tool will be the following immediate consequence of (2) and (3):

Lemma 1 *Let W be a symmetric convex body in $\mathbb{R}^n$. There exists $T \in GL(n)$ such that*

$$\ell(TW)\ell((TW)^\circ) \leq c_1 n \log(d_W + 1),$$

where $c_1 > 0$ is an absolute constant. $\square$

We will also use some simple facts about the ℓ-functional $W \mapsto \ell(W)$.

Lemma 2 *Let W be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\sqrt{n} b(W) \leq c\ell(W^\circ)$$

(in fact, the two quantities are equivalent up to absolute constants). Also, if I denotes the identity operator, then $\ell((I+S)(W)) \leq \ell(W)$ for every positive linear operator S on $\mathbb{R}^n$.

Proof: For the proof of the first assertion we apply the Cauchy-Schwarz inequality to get

$$\ell(W^\circ) \geq \int_{\mathbb{R}^n} \|x\|_{W^\circ} \gamma_n(dx)$$

and then use polar integration. For the second assertion, observe that

$$\ell((I+S)(W)) \leq \|(I+S)^{-1} : \ell_2^n \to \ell_2^n\| \cdot \ell(W)$$

(see also [10], pp. 35) and use the fact that S is positive. $\square$

Our second tool will be Chevet's inequality which will be used in the spirit of Benyamini and Gordon (see [3] and [13, pp. 325]).

Lemma 3 *Let K and L be two symmetric convex bodies in $\mathbb{R}^n$. Then,*

$$\int_{O(n)} \|U : X_L \to X_K\| \, \mu(dU) \leq \frac{c_2}{\sqrt{n}} \big(\|I : \ell_2^n \to X_L^*\| \ell(K) + \|I : \ell_2^n \to X_K\| \ell(L^\circ) \big),$$

where $c_2 > 0$ is an absolute constant. $\square$

Proposition. *Let K and L be two symmetric convex bodies in $\mathbb{R}^n$. Then,*

$$\mathrm{vr}(K,L) \leq c_3\big(d_L \log(d_K+1) + d_K \log(d_L+1)\big)$$

where $c_3 > 0$ is an absolute constant.

Proof: By Lemma 1 we may assume that K and L° satisfy

$$\ell(K) \leq \sqrt{n}\ ,\ \ell(K^\circ) \leq c_1\sqrt{n}\log(d_K+1)\ ,\ \ell(L^\circ) \leq \sqrt{n} \text{ and } \ell(L) \leq c_1\sqrt{n}\log(d_L+1).$$

Let E_K be a distance ellipsoid of K and let $S \in GL(n)$ be a positive linear operator such that $S(E_K) = B_n$. Then, $B_n \subseteq S(K) \subseteq d_K B_n$; therefore,

$$\ell(S(K)) \leq c\sqrt{n} \text{ and } \ell((S(K))^\circ) \leq c\sqrt{n}d_K.$$

If $T = I + aS$, $a = \log(d_K+1)/d_K$, we have

$$\begin{aligned} \|I : \ell_2^n \to X_{TK}\| &= \|(I+aS)^{-1} : \ell_2^n \to X_K\| \\ &\leq \|((aS)^{-1}+I)^{-1} : \ell_2^n \to \ell_2^n\| \cdot \|(aS)^{-1} : \ell_2^n \to X_K\| \\ &\leq \|(aS)^{-1} : \ell_2^n \to X_K\| = \|I : \ell_2^n \to X_{aSK}\| \\ &\leq 1/a = d_K/\log(d_K+1) \end{aligned}$$

and, by the second assertion of Lemma 2,

$$\ell(TK) \leq \ell(K) \leq \sqrt{n}.$$

Also,

$$\begin{aligned} \ell((TK)^\circ) &\leq \ell(K^\circ) + a\ell((SK)^\circ) \leq c_1\sqrt{n}\log(d_K+1) + a\sqrt{n}d_K \\ &\leq c\sqrt{n}\log(d_K+1). \end{aligned}$$

Working in the same way with L° we can find an operator R such that

$$\|I : \ell_2^n \to X_{RL^\circ}\| \leq d_L/\log(d_L+1)$$

and $\ell(RL^\circ) \leq \sqrt{n}$, while $\ell((R^{-1})^*(L)) \leq c\sqrt{n}\log(d_L+1)$.

Let $K_1 = TK$ and $L_1 = (R^{-1})^*(L)$. Applying Lemma 3 for X_{K_1} and X_{L_1} we get

$$\int_{O(n)} \|U : X_{L_1} \to X_{K_1}\|\mu(dU) \leq c\left(\frac{d_K}{\log(d_K+1)} + \frac{d_L}{\log(d_L+1)}\right).$$

This shows that there exists $U_0 \in O(n)$ such that

$$U_0(L_1) \subseteq c\left(\frac{d_K}{\log(d_K+1)} + \frac{d_L}{\log(d_L+1)}\right) K_1.$$

Therefore,

$$\mathrm{vr}(K,L) \leq c\left(\frac{d_K}{\log(d_K+1)} + \frac{d_L}{\log(d_L+1)}\right)\left(\frac{|K_1|}{|B_n|}\right)^{1/n}\left(\frac{|B_n|}{|U_0(L_1)|}\right)^{1/n}.$$

Urysohn's inequality (1) and the first assertion of Lemma 2 give

$$\left(\frac{|K_1|}{|B_n|}\right)^{1/n} \leq b(K_1) \leq \frac{c\ell(K_1^\circ)}{\sqrt{n}} \leq c' \log(d_K + 1)$$

while Hölder's inequality implies that

$$\begin{aligned} \left(\frac{|B_n|}{|U_0(L_1)|}\right)^{1/n} &= \left(\frac{|B_n|}{|L_1|}\right)^{1/n} = \left(\int_{S^{n-1}} \|x\|^{-n} \sigma(dx)\right)^{-1/n} \\ &\leq b(L_1^\circ) \leq \frac{c\ell(L_1)}{\sqrt{n}} \leq c' \log(d_L + 1). \end{aligned}$$

Combining the above, we get

$$\mathrm{vr}(K, L) \leq c_3\big(d_L \log(d_K + 1) + d_K \log(d_L + 1)\big). \qquad \square$$

Remark. John's theorem [6] states that $d_W \leq \sqrt{n}$ for every symmetric convex body W in $\mathbb{R}^n$. It follows from the Proposition that

$$\mathrm{vr}(K, L) \leq c\sqrt{n} \log n$$

for every pair of symmetric convex bodies in $\mathbb{R}^n$. For the general case we are using the following standard argument:

Proof of the Theorem: Let K and L be two convex bodies in $\mathbb{R}^n$. We may assume that their centre of gravity is at the origin. An inequality of Rogers and Shephard [12] shows that

$$|L - L|^{1/n} \leq 4|L|^{1/n}.$$

On the other hand, Milman and Pajor [8] proved that

$$|K|^{1/n} \leq 2|K \cap (-K)|^{1/n}.$$

By the Proposition, there exists $T \in GL(n)$ such that $T(L - L) \subseteq K \cap (-K)$ and $|K \cap (-K)|^{1/n} \leq c\sqrt{n} \log n |T(L - L)|^{1/n}$. We obviously have $T(L) \subseteq K$ and

$$\begin{aligned} \mathrm{vr}(K, L) &\leq \left(\frac{|K|}{|K \cap (-K)|} \, \frac{|K \cap (-K)|}{|T(L - L)|} \, \frac{|T(L - L)|}{|T(L)|}\right)^{1/n} \\ &\leq 8c\sqrt{n} \log n. \qquad \square \end{aligned}$$

References

[1] K.M. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. (2) **44** (1991), 351-359.

[2] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. **134** (1998), 335-361.

[3] Y. Benyamini and Y. Gordon, *Random factorization of operators between Banach spaces*, J. d'Analyse Math. **39** (1981), 45-74.

[4] T. Figiel and N. Tomczak-Jaegermann, *Projections onto Hilbertian subspaces of Banach spaces*, Israel J. Math. **33** (1979), 155-171.

[5] A.A. Giannopoulos, I. Perissinaki and A. Tsolomitis, *John's theorem for an arbitrary pair of convex bodies*, Geom. Dedicata **84** (2001), 63-79.

[6] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, New York, 1948, pp. 187-204.

[7] D.R. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika **26** (1979), 18-29.

[8] V.D. Milman and A. Pajor, *Entropy and asymptotic geometry of non-symmetric convex bodies*, Adv. Math. **152** (2000), 314-335.

[9] G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. of Math. **115** (1982), 375-392.

[10] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics **94** (1989).

[11] M. Rudelson, *Distances between non-symmetric convex bodies and the MM^*-estimate*, Positivity **4** (2000), 161-178.

[12] C.A. Rogers and G. Shephard, *The difference body of a convex body*, Arch. Math. **8** (1957), 220-233.

[13] N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite Dimensional Operator Ideals*, Pitman Monographs **38** (1989), Pitman, London.

A. Giannopoulos: Department of Mathematics, University of Crete, Iraklion, Greece. *E-mail*: giannop@fourier.math.uoc.gr

M. Hartzoulaki: Department of Mathematics, University of Crete, Iraklion, Greece. *E-mail*: hmarian@math.uch.gr