Enriched duality in double categories

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Outline

1. Background
2. Enriched duality in monoidal double categories
3. Oplax monoidal double structure
4. Further directions

The category Alg of k-algebras is enriched in Coalg of k-coalgebras.

Monoids and comonoids

Suppose $(\mathcal{V}, \otimes, I)$ is monoidal category.

- A *monoid* is an object A together with maps $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$ which are associative and unital

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \\ {\scriptstyle 1\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} I \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A & \xleftarrow{1 \otimes \eta} & A \otimes I \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & A & & \end{array}$$

- Dually, a *comonoid* is an object C together with maps $\delta\colon C \to C \otimes C$ and $\epsilon\colon C \to I$ which are coassociative and counital.

$\star$ These form categories Mon and Comon, with maps preserving structure.

When $(\mathcal{V}, \otimes, I, \sigma)$ is symmetric, Mon and Comon are monoidal with

$$C \otimes D \xrightarrow{\delta \otimes \delta} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \sigma \otimes 1} C \otimes D \otimes C \otimes D$$

■ If $\mathcal{V}$ is monoidal closed, induced $[\text{-},\text{-}]\colon \mathsf{Comon}(\mathcal{V})^{\mathrm{op}}\times\mathsf{Mon}(\mathcal{V})\to\mathsf{Mon}(\mathcal{V})$ makes $[C,A]$ into a monoid via *convolution*

$$[C,A]\otimes[C,A]\otimes C \xrightarrow{1\otimes\delta} [C,A]\otimes[C,A]\otimes C\otimes C \xrightarrow{1\otimes\sigma\otimes 1} [C,A]\otimes C\otimes[C,A]\otimes C \xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} A\otimes A \xrightarrow{\mu} A$$

$$(f*g)(c)=\sum_{(c)} f(c_1)g(c_2)$$

⋆ In Vect_k, linear dual $C^* = \mathrm{Hom}_k(C,k)$ for a k-coalgebra is a k-algebra – while A^* for a k-algebra is a k-coalgebra only if it is finite dimensional.

Sweedler dual 'fixes' that: $A^o=\{f\in A^* \mid \ker f \text{ contains cofinite ideal}\}$ is a k-coalgebra, and $\mathsf{Alg}(A,C^*)\cong\mathsf{Coalg}(C,A^o)$.

More generally, there exists *universal measuring k-coalgebra* with $\mathsf{Alg}(A,\mathrm{Hom}_k(C,B))\cong\mathsf{Coalg}(C,P(A,B))$ – so $A^o=P(A,k)$.

Moving to general context of symmetric monoidal closed categories, local presentability gives passage from Vect_k to (d)gVect_k, Mod_R etc.

Suppose $\mathcal{V}$ is a symmetric monoidal closed and locally presentable category. There is a parameterized adjunction between

$$[\text{-},\text{-}]\colon \mathsf{Comon}^{\mathrm{op}} \times \mathsf{Mon} \to \mathsf{Mon} \quad \text{convolution}$$
$$P(\text{-},\text{-})\colon \mathsf{Mon}^{\mathrm{op}} \times \mathsf{Mon} \to \mathsf{Comon} \quad \text{universal measuring}$$

• In Set, $P(A,B)$ is $\mathsf{Mon}(A,B)$; in Vect_k, it contains k-algebra maps as grouplike elements; in dgVect_k, it relates to bar-cobar adjunction.

⋆ Convolution [-, -] is an *action* of the monoidal $\mathsf{Comon}^{(\mathrm{op})}$ on Mon.

An adjoint of an action $\bullet\colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ gives rise to a $\mathcal{V}$-enriched structure on $\mathcal{C}$.

The category Mon is enriched in the monoidal Comon.

From monoidal to double categories

One to many objects: generalize from monoids in $\mathcal{V}$, to $\mathcal{V}$-categories.

What about comonoids? Opcategories=$\mathcal{V}^{\mathrm{op}}$-categories not as convenient, formally... identify common framework!

- A double category $\mathbb{D}$ has object category $\mathbb{D}_0$ (0-cells & vertical 1-cells), arrow category $\mathbb{D}_1$ (horizontal 1-cells & 2-maps)

$$\begin{array}{ccc} X & \overset{A}{\longrightarrow\!\!\!\!\!\!\!+\;\,} & Y \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle g} \\ Z & \underset{B}{\longrightarrow\!\!\!\!\!\!\!+\;\,} & W \end{array}$$

and $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$ + coherent isos.

0-cells, horizontal 1-cells, *globular* 2-maps
make horizontal bicategory $\mathcal{H}(\mathbb{D})$.

- A monad in a double category $\mathbb{D}$ is $A\colon X \nrightarrow X$ with associative, unital

$$\begin{array}{ccccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \| & & \Downarrow\mu & & \| \\ X & & \xrightarrow[A]{\hspace{3em}} & & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \| & \Downarrow\eta & \| \\ X & \xrightarrow[A]{} & X \end{array}$$

Dually, comonad $C\colon X \nrightarrow X$. These form categories $\mathsf{Mnd}(\mathbb{D})$ & $\mathsf{Cmd}(\mathbb{D})$.

$\star$ Morphisms are different than those for (co)monads in bicategories.

For $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ of sets, functions and $\mathcal{V}$-matrices $S\colon X \nrightarrow Y$ i.e. $\{S(x,y)\} \in \mathcal{V}$ with $(S \circ T)(x,z) = \sum_y T(x,y) \otimes S(y,z)$, $\mathsf{Mnd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathsf{Cat}$ and $\mathsf{Cmd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathsf{Cocat}$.

A $\mathcal{V}$-cocategory comes with cocomposition $C(x,z) \to \sum_y C(x,y) \otimes C(y,z)$ and coidentities $C(x,x) \to I$, coassociative and counital.

For $\mathbb{D}$ a $(\dots)$ double category, $\mathsf{Mnd}(\mathbb{D})$ is enriched in $\mathsf{Cmd}(\mathbb{D})$.

■ Fibrant: vertical 1-cells f turn to horizontal, companion $\hat{f}$ & conjoint $\check{f}$.

In $\mathcal{V}$-$\mathbb{M}$at, $f\colon X \to Y$ gives matrices $\hat{f}(x,y) = \check{f}(y,x) = \begin{cases} I \text{ if } fx = y \\ 0 \text{ if } fx \neq y \end{cases}$

$\mathsf{Mnd}(\mathbb{D}) \to \mathbb{D}_0$ is a fibration; reindexing $\check{f} \circ \text{-} \circ \hat{f} \colon \mathsf{Mnd}(\mathbb{D})_Y \to \mathsf{Mnd}(\mathbb{D})_X$.
Dually, $\mathsf{Cmd}(\mathbb{D}) \to \mathbb{D}_0$ is an opfibration.

■ Monoidal: $\mathbb{D}_0$ & $\mathbb{D}_1$ monoidal, $(N \circ M) \otimes (N' \circ M') \cong (N \otimes N') \circ (M \otimes M')$.

In $\mathcal{V}$-$\mathbb{M}$at, $(X \otimes Y) = X \times Y$ & $(S \otimes T)((x,y),(z,w)) = S(x,z) \otimes T(y,w)$.

$\mathsf{Mnd}(\mathbb{D})$ and $\mathsf{Cmd}(\mathbb{D})$ are monoidal, with $C \otimes D$ comonad via
$C \otimes D \to (C \circ C) \otimes (D \circ D) \cong (C \otimes D) \circ (C \otimes D)$.

■ Monoidal closed : lax double functor $H\colon \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ such that $\mathbb{D}_0$ & $\mathbb{D}_1$ monoidal closed, $\mathfrak{s}, \mathfrak{t}$ maps of adjunctions.

In $\mathcal{V}$-$\mathbb{M}$at, $[X, Y] = Y^X$ and $H(S,T)(f,g) = \prod_{x,y}[S(x,y), T(fx, gy)]$.

H induces functor $\mathrm{Cmd}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mnd}(\mathbb{D}) \to \mathrm{Mnd}(\mathbb{D})$, an action. *convolution*

■ Locally presentable : $\mathbb{D}_0$ & $\mathbb{D}_1$ locally presentable, $\mathfrak{s}, \mathfrak{t}$ cocontinuous right adjoints, - $\circ$ - accessible in each variable.

In $\mathcal{V}$-$\mathbb{M}$at, Set is l. p. & $\mathcal{V}$-$\mathbb{M}\mathrm{at}_1$ is too as pullback

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbb{M}\mathrm{at}_1 & \to & \mathrm{Fam}(\mathcal{V}) \\ {\scriptstyle (\mathfrak{s},\mathfrak{t})}\downarrow & & \downarrow \\ \mathrm{Set}^2 & \xrightarrow{\times} & \mathrm{Set} \end{array}$$

by the Limit Theorem.

Induced H has adjoint $\mathrm{Mnd}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mnd}(\mathbb{D}) \to \mathrm{Cmd}(\mathbb{D})$. *univ. measuring*

◇ Obtain enrichment of $\mathrm{Mnd}(\mathbb{D})$ in $\mathrm{Cmd}(\mathbb{D})$!

Moving to other contexts

$\star$ For $\mathcal{V}$ symmetric monoidal closed & locally presentable, $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ is a fibrant, monoidal closed & locally presentable double category.

- Enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories

Goal: employ/*extend* theory to apply to $\mathbb{D} = \mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$

operads

- objects are sets $X, Y, \ldots$
- vertical 1-cells are functions $f, g, \ldots$
- horizontal 1-cells are coloured symmetric sequences $M\colon SX^{\mathrm{op}} \times Y \to \mathcal{V}$
- 2-maps are $M(x_1, \ldots, x_n; y) \to N(fx_1, \ldots, fx_n; gy)$

Horizontal composition is generalization of substitution for species.

■ $\mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$ is fibrant; but not monoidal double anymore!

Oplax monoidal double structure

- A double category is *oplax* monoidal with comparison maps

$$(N \circ M) \otimes (N' \circ M') \to (N \otimes N') \circ (M \otimes M'), \quad 1_X \otimes 1_{X'} \to 1_{X \otimes X'}$$
$$I_1 \to I_1 \circ I_1, \quad I_1 \to 1_I$$

satisfying coherence axioms.

An oplax monoidal double category with a single object and vertical arrow is precisely a duoidal category.

End result: $\mathcal{V}$-$\mathbb{S}$ym is an oplax monoidal double category. How? As a Kleisli-type structure on $\mathcal{V}$-$\mathbb{P}$rof with induced monoidality.

Kleisli double category

- A *(vertical) double monad* is a double functor $T\colon \mathbb{D} \to \mathbb{D}$ with transformations $m\colon TT \Rightarrow T$, $e\colon 1 \Rightarrow T$ with

$$\begin{array}{ccc} TTX & \xrightarrow{TTM} & TTY \\ {\scriptstyle m_X}\downarrow & \Downarrow m_M & \downarrow {\scriptstyle m_Y} \\ TX & \xrightarrow[TM]{} & TY \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{M} & Y \\ {\scriptstyle e_X}\downarrow & \Downarrow e_M & \downarrow {\scriptstyle e_Y} \\ TX & \xrightarrow[TM]{} & TY \end{array}$$

- It is *special* when $\hat{m}_X, \hat{e}_X$ exist, and transposes of m_M, e_M are invertible.

■ Each special double monad $T\colon \mathbb{D} \to \mathbb{D}$ gives double category $\mathbb{K}\mathrm{l}(T)$

- $\mathbb{K}\mathrm{l}(T)_0$ is $\mathbb{D}_0$
- $M\colon X \rightsquigarrow Y$ are horizontal $M\colon X \nrightarrow TY$ in $\mathbb{D}$
- 2-maps $\begin{array}{ccc} X & \overset{M}{\rightsquigarrow} & Y \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow {\scriptstyle g} \\ Z & \underset{N}{\rightsquigarrow} & W \end{array}$ are $\begin{array}{ccc} X & \overset{M}{\nrightarrow} & TY \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow {\scriptstyle Tg} \\ Z & \underset{N}{\nrightarrow} & TW \end{array}$ in $\mathbb{D}$
- horizontal composition is $X \xrightarrow{M} TY \xrightarrow{TN} TTZ \xrightarrow{\hat{m}_Z} TZ$

Monoidal structure

- A double monad T on a monoidal double category $\mathbb{D}$ is *pseudomonoidal* when T lax monoidal and m, e pseudomonoidal

$$\text{e.g.} \quad \begin{array}{ccc} TTX_1 \otimes TTX_2 & \xrightarrow{1} & TTX_1 \otimes TTX_2 \\ \downarrow & & \downarrow{\scriptstyle m\otimes m} \\ TT(X_1 \otimes X_2) & \Downarrow & TX_1 \otimes TX_2 \\ {\scriptstyle m}\downarrow & & \downarrow \\ T(X_1 \otimes X_2) & \xrightarrow{1} & T(X_1 \otimes X_2) \end{array} \quad \text{is invertible}$$

For T a pseudomonoidal special double monad, if lax structure maps $I \to TI$, $TX \otimes TY \xrightarrow{\tau} T(X \otimes Y)$ have companions, $\mathbb{K}\mathsf{l}(T)$ is an oplax monoidal double category.

$\star$ Induced tensor is $M \boxtimes N = X \otimes Z \xrightarrow{M \otimes N} TY \otimes TW \xrightarrow{\hat{\tau}} T(Y \otimes W)$.

Coloured symmetric sequences

⋆ 'Free symmetric strict monoidal category 2-monad' $S\colon \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$

$$\text{with } S_n(C)((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{\sigma} \prod_{1\leq i\leq n} C(x_{\sigma(i)}, y_i)$$

extends to such a double monad on the monoidal double category $\mathcal{V}\text{-}\mathbb{P}\mathrm{rof}$ (for $\mathcal{V}$ cartesian monoidal).

Kleisli double category is $\mathcal{V}\text{-}\mathbb{C}\mathrm{at}\mathbb{S}\mathrm{ym}$ of *categorical symmetric sequences* $M\colon SX^{\mathrm{op}}\times Y \to \mathcal{V}$, with 'discrete' case $\mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$.

▶ Oplax monoidal structure is many-object *arithmetic product* of species

$$(M \boxtimes N)(\vec{a},(x,z)) = \int^{\vec{y},\vec{w}} S(Y\times W)(\vec{a}, \vec{y}\boxtimes\vec{w}) \times M(\vec{y},x) \times N(\vec{w},z)$$

Some future directions

- Extend previous results from monoidal double to oplax monoidal double categories : do we still obtain enrichment of monads in comonads?

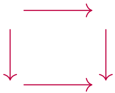
- Further explore $\mathcal{V}$-$\mathbb{S}$ym: is it monoidal closed and locally presentable as a double category?

[One-object case] If $\mathcal{V}$ is symmetric monoidal closed and loc presentable,

- positive operads are enriched in positive cooperads, if $\mathcal{V}$ has biproducts;
- symmetric operads are enriched in symmetric cooperads, if $\mathcal{V}$ is cartesian.

- Extend full story $\begin{array}{ccc} \mathsf{Mod}(\mathbb{D}) & \overset{\text{enriched}}{\dashrightarrow} & \mathsf{Comod}(\mathbb{D}) \\ {\scriptstyle\text{fibered}}\downarrow & & \downarrow{\scriptstyle\text{opfibered}} \\ \mathsf{Mnd}(\mathbb{D}) & \underset{\text{enriched}}{\dashrightarrow} & \mathsf{Cmd}(\mathbb{D}) \end{array}$ to oplax monoidal $\mathbb{D}$.

Thank you for your attention!



- *Aravantinos-Sotiropoulos, Vasilakopoulou*, "Enriched duality in double categories II: modules and comodules", arXiv:2408.03180
- *Gambino, Garner, Vasilakopoulou*, "Monoidal Kleisli bicategories and the arithmetic product of symmetric sequences", Documenta Mathematica (2024)