Oplax Hopf Algebras

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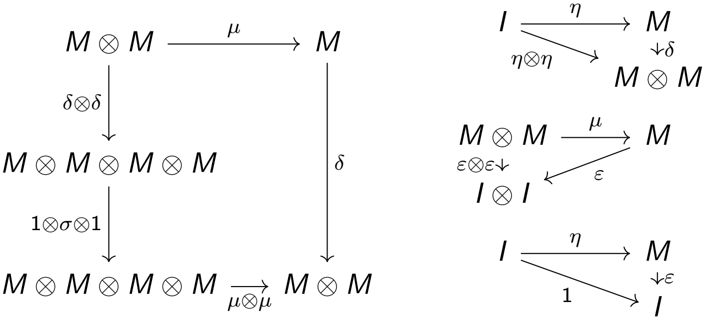
Outline

1. Some preliminaries
2. Oplax bimonoids and oplax Hopf monoids
3. The symmetric monoidal bicategory Span$|\mathcal{V}$
4. Hopf categories as oplax Hopf algebras in Span$|\mathcal{V}$

Bimonoids in braided monoidal categories

Suppose $(\mathcal{V}, \otimes, I, \sigma)$ is braided monoidal category.

- A *bimonoid* is an object M with a monoid (M, μ, η) and a comonoid (M, δ, ε) structure which are compatible, in that



A *bimonoid morphism* is an arrow $f\colon M \to N$ that is a monoid and comonoid morphism$\rightsquigarrow$ obtain a category $\mathrm{Bimon}(\mathcal{V})$.

Hopf monoids in braided monoidal categories

- A *Hopf monoid* is a bimonoid with an *antipode*, i.e. $s\colon M \to M$ with

$$\begin{array}{ccccc} & M \otimes M & \xrightarrow{1 \otimes s} & M \otimes M & \\ {}^{\delta}\nearrow & & & & \searrow^{\mu} \\ M & \xrightarrow{\quad\varepsilon\quad} & I & \xrightarrow{\quad\eta\quad} & M \\ {}_{\delta}\searrow & & & & \nearrow_{\mu} \\ & M \otimes M & \xrightarrow[s \otimes 1]{} & M \otimes M & \end{array}$$

$\star$ Equivalently, s is the inverse of 1_M under convolution in $\mathcal{V}(M, M)$:

$$f \odot g := M \xrightarrow{\delta} M \otimes M \xrightarrow{f \otimes g} M \otimes M \xrightarrow{\mu} M, \quad I_{\odot} \colon M \xrightarrow{\varepsilon} I \xrightarrow{\eta} M$$

These form a full subcategory $\mathrm{Hopf}(\mathcal{V})$ of $\mathrm{Bimon}(\mathcal{V})$.

Hopf categories

Idea: like a category can be thought as a many-object monoid, a Hopf category is the many-object generalization of a Hopf monoid.

Comon($\mathcal{V}$) is monoidal when $\mathcal{V}$ is braided monoidal:

$$C \otimes D \xrightarrow{\delta\otimes\delta} C \otimes C \otimes D \otimes D \xrightarrow{1\otimes\sigma\otimes 1} C \otimes D \otimes C \otimes D$$

- A *semi-Hopf $\mathcal{V}$-category H* is a Comon($\mathcal{V}$)-enriched category:

$$\mu_{xyz}\colon H_{x,y} \otimes H_{y,z} \to H_{x,z} \qquad \eta_x\colon I \to H_{x,x} \qquad \textit{global multiplication}$$

$$lcomlt_{ab}\colon H_{a,b} \to H_{a,b} \otimes H_{a,b} \qquad \varepsilon_{ab}\colon H_{a,b} \to I \qquad \textit{local comultiplication}$$

$$\begin{array}{ccc} H_{x,y} \otimes H_{y,z} & \xrightarrow{\mu_{xyz}} & H_{x,z} \\ \downarrow{\scriptstyle \delta_{xy}\otimes\delta_{yz}} & & \\ H_{x,y} \otimes H_{x,y} \otimes H_{y,z} \otimes H_{y,z} & & \downarrow{\scriptstyle \delta_{xz}} \\ \downarrow{\scriptstyle 1\otimes\sigma\otimes 1} & & \\ H_{x,y} \otimes H_{y,z} \otimes H_{x,y} \otimes H_{y,z} & \xrightarrow[\mu_{xyz}\otimes\mu_{xyz}]{} & H_{x,z} \otimes H_{x,z} \end{array}$$

+3 other axioms

- A *Hopf category* is a semi-Hopf category with $s_{xy} \colon H_{x,y} \to H_{y,x}$ so that

$$\begin{array}{ccccc} & H_{x,y} \otimes H_{x,y} & \xrightarrow{1 \otimes s_{xy}} & H_{x,y} \otimes H_{y,x} & \\ {}^{\delta_{xy}}\nearrow & & & & \searrow^{\mu_{xyx}} \\ H_{x,y} & \xrightarrow{\quad\varepsilon_{xy}\quad} & I & \xrightarrow{\quad\eta_x\quad} & H_{x,x} \end{array}$$

With Comon($\mathcal{V}$)-functors, obtain categories Hopf-$\mathcal{V}$-Cat $\subseteq$ sHopf-$\mathcal{V}$-Cat.

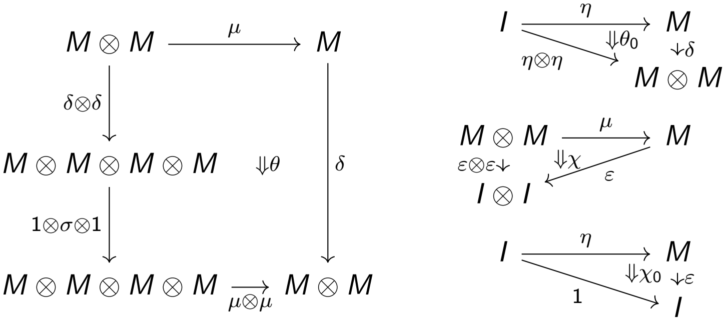
Examples

- A one-object semi-Hopf/Hopf $\mathcal{V}$-category is a bimonoid/Hopf monoid in $\mathcal{V}$.
- If $\mathcal{V}$ is cartesian monoidal, any $\mathcal{V}$-category is semi-Hopf (since Comon($\mathcal{V}$) $\cong \mathcal{V}$).
- When $\mathcal{V}$ is also locally presentable and closed, Mon($\mathcal{V}$) is a semi-Hopf $\mathcal{V}$-category.
- A Hopf (Set-)category is a groupoid.

Idea: 'relax' the monoid/comonoid compatibility in a 2-categorical setting.

Suppose $(\mathcal{K}, \otimes, I, \sigma)$ is braided monoidal bicategory.

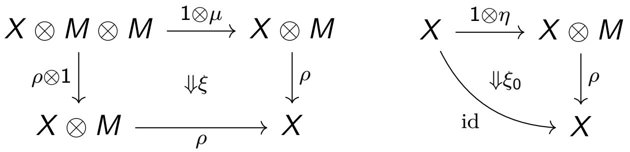
- A *pseudo(co)monoid* is an object with (co)multiplication, (co)unit that are associative and unital up to iso. Can have lax or oplax maps!
- An *oplax bimonoid* is pseudomonoid & pseudocomonoid $(M, \mu, \eta, \delta, \varepsilon)$ along with 2-cells (satisfying axioms...)



These form a bicategory $\mathsf{OplBimon}(\mathcal{V}) = \mathsf{PsComon}_{\mathsf{opl}}(\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K}))$.

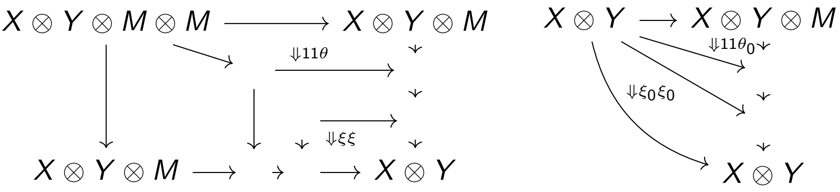
⋆ For ordinary bimonoid M, the forgetful $\mathrm{Mod}_\mathsf{M} \to \mathcal{V}$ is strict monoidal.

- An *oplax M-module* X for pseudomon (M, μ, η) has $X \otimes M \xrightarrow{\rho} M$ with



For an oplax bimonoid M, the bicategory $\mathrm{OplMod}^\mathsf{M}_{\mathrm{opl}}$ has a monoidal structure such that $\mathrm{OplMod}^\mathsf{M}_{\mathrm{opl}} \to \mathcal{K}$ is strict monoidal.

$X \otimes Y \otimes M \xrightarrow{11\delta} X \otimes Y \otimes M \otimes M \xrightarrow{1\sigma 1} X \otimes M \otimes Y \otimes M \xrightarrow{\rho\rho} X \otimes Y$ with

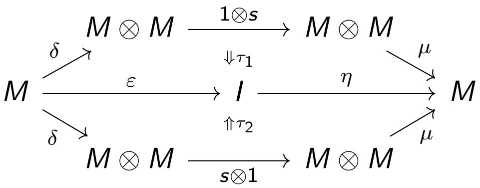


Oplax Hopf monoids

An *oplax inverse* for X in a monoidal $\mathcal{V}$ is Y with $X\otimes Y \xrightarrow{\tau_1} I \xleftarrow{\tau_2} Y\otimes X$ such that $1\otimes\tau_1 = \tau_2\otimes 1 : Y\otimes X\otimes Y \xrightarrow{\sim} Y$, $\tau_1\otimes 1 = 1\otimes\tau_2 : X\otimes Y\otimes X \xrightarrow{\sim} X$.

⋆ The above is a special case of a *firm Morita context* in a bicategory, which also makes Y unique up to isomorphism.

- An *oplax antipode* for an oplax bimonoid M is an oplax inverse of 1_M in $\mathcal{K}(M, M)$ with convolution.



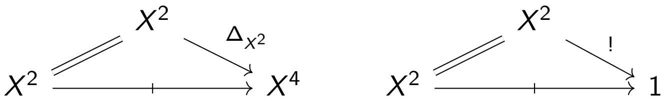
s.t. $1_s \odot \tau_1 = \tau_2 \odot 1_s$,

$\tau_1 \odot 1_{1_M} = 1_{1_M} \odot \tau_2$

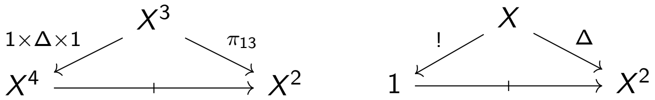
- An *oplax Hopf monoid* is an oplax bimonoid with an oplax antipode. With maps that preserve the oplax antipode, get bicategory $\mathrm{OplHopf}(\mathcal{K})$.

Example: X^2 in $(\mathsf{Span}, \times, \mathbf{1})$

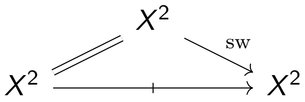
- Every set has a (strict) comonoid structure in Span; e.g. for X^2



- For every set X, codiscrete groupoid X^2 gives pseudomonoid



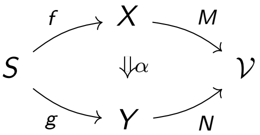
- There is a (unique!) oplax bimonoid structure on X^2 with the above structures; it is an oplax Hopf monoid with oplax antipode



$\star$ Choosing the reverse structures, X^2 is again an oplax Hopf monoid.

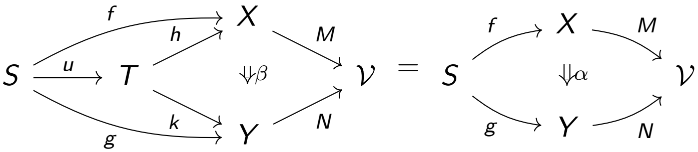
There is a bicategory Span$|\mathcal{V}$ with

- 0-cells pairs M_X with X a set, $M\colon X \to \mathcal{V}$ a functor i.e. $\{M_x\}_{x\in X} \in \mathcal{V}$
- 1-cells $M_X \to N_Y$ are spans $X \xleftarrow{f} S \xrightarrow{g} Y$ with natural transf



with components $\alpha_s\colon M_{fs} \to N_{gs}$

- 2-cells are maps of spans that satisfy



- If $\mathcal{V}$ is monoidal, $M_X \otimes N_Y := X \times Y \xrightarrow{M\times N} \mathcal{V}\times\mathcal{V} \xrightarrow{\otimes} \mathcal{V}$

$\star$ Span$|\mathcal{K}$ was introduced by Böhm for arbitrary monoidal bicategory $\mathcal{K}$, here for a monoidal 1-category $\mathcal{V}$ with trivial 2-cells.

⋆ There is a (strict) monoidal functor of bicategories $U\colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ which forgets all data associated to $\mathcal{V}$.

If $\mathcal{V}$ has colimits, the forgetful $U\colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ is a 2-opfibration.

Sketch: Build a pseudofunctor $\mathsf{Span} \to \mathsf{Cat} \hookrightarrow 2\text{-}\mathsf{Cat}$ via $X \mapsto [X, \mathcal{V}]$ and

$$X \xleftarrow{f} S \xrightarrow{g} Y \mapsto \mathrm{Lan}_g(\text{-} \circ f)\colon [X, \mathcal{V}] \longrightarrow [Y, \mathcal{V}]$$

$$X \xrightarrow{M} \mathcal{V} \longmapsto \begin{array}{ccccc} S & \xrightarrow{f} & X & \xrightarrow{M} & \mathcal{V} \\ {\scriptstyle g}\downarrow & & & \nearrow{\scriptstyle \mathrm{Lan}_g(Mf)} & \\ Y & & & & \end{array}$$

Then 2-dimensional Grothendieck construction is isomorphic to U.

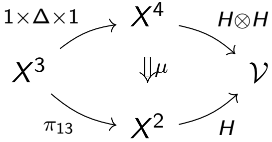
Any braided monoidal pseudofunctor of bicategories preserves pseudo(co)monoids, oplax bimonoids and oplax Hopf monoids.

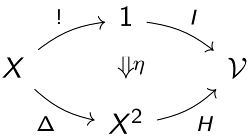
⋆ In particular, an oplax Hopf monoid in $\mathsf{Span}|\mathcal{V}$ will have as underlying set an oplax Hopf monoid in Span – think X^2 with earlier structure...

Hopf categories as oplax Hopf algebras

Idea: each piece of structure of a Hopf category $(H, \mu, \eta, \delta, \varepsilon, s)$ corresponds to that of an oplax Hopf monoid in Span$|\mathcal{V}$.

A pseudomonoid in Span$|\mathcal{V}$ over the pseudomonoid X^2 is a $\mathcal{V}$-category.

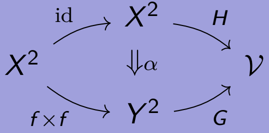




with components

$$H_{x,y} \otimes H_{y,z} \xrightarrow{\mu_{xyz}} H_{x,z}$$
$$I \xrightarrow{\eta_x} H_{x,x}$$

An oplax pseudomonoid map between H_{X^2} and G_{Y^2} in Span$|\mathcal{V}$ of the form



is a $\mathcal{V}$-functor between the resp. $\mathcal{V}$-categories.

⋆ We realize $\mathcal{V}$-Cat as a specific subcategory of $\text{PsMon}_{\text{opl}}(\text{Span}|\mathcal{V})$.

- A comonoid in Span$|\mathcal{V}$ over the comonoid X^2 is a Comon$(\mathcal{V})$-graph.
- An oplax bimonoid in Span$|\mathcal{V}$ over the oplax bimonoid X^2 is a semi-Hopf category.
- An oplax Hopf monoid in Span$|\mathcal{V}$ over the oplax Hopf monoid X^2 is a Hopf category.

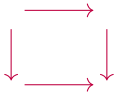
$\star$ Similar correspondences exist for morphisms. Thus Hopf-$\mathcal{V}$-Cat is a specific subcategory of OplHopf(Span$|\mathcal{V}$).

▶ *Frobenius $\mathcal{V}$-categories* are Frobenius pseudomonoids in Span$|\mathcal{V}$

$$\mu_{xyz}\colon H_{x,y}\otimes H_{y,z}\to H_{x,z} \qquad \eta_x\colon I\to H_{x,x} \qquad \textit{global multiplication}$$
$$d_{abc}\colon H_{a,c}\to H_{a,b}\otimes H_{b,c} \qquad e_a\colon H_{a,a}\to I \qquad \textit{global comultiplication}$$

A Hopf $\mathcal{V}$-category is Frobenius under certain assumptions...

Thank you for your attention!



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