Sweedler theory for double categories

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Outline

1. Motivation and background
2. Sweedler theory for monoidal double categories
3. Oplax monoidal double structure
4. Further directions

Fibered versus enriched categories

• A $\mathcal{V}$-enriched category has hom-objects that belong to $\mathcal{V}$, and composition rule is a morphism in $\mathcal{V}$, for a monoidal category $\mathcal{V}$

• A category fibered over a category $\mathcal{B}$ is an ordinary category, whose objects and morphisms lie over specified objects and morphisms in $\mathcal{B}$, with certain (cartesian) liftings

... a common pattern in what follows is

$$\begin{tikzcd}
\mathcal{C} \arrow[r, dotted, "\text{enriched}"] \arrow[d, "\text{fibered}"'] & \mathcal{D} \arrow[d, "\text{opfibered}"] \\
\mathcal{A} \arrow[r, dotted, "\text{enriched}"'] & \mathcal{B}
\end{tikzcd}$$

⋆ Bunge / Shulman: *enriched indexed categories*, over a fixed base. Above picture? Theory of *enriched fibrations*, over different bases.

Monoids and comonoids

Suppose $(\mathcal{V}, \otimes, I)$ is monoidal category.

- A *monoid* is an object A together with maps $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$ which are associative and unital:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} I \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr] & A \otimes A \arrow[d, "\mu"] & A \otimes I \arrow[l, "1\otimes\eta"'] \arrow[dl] \\ & A & \end{tikzcd}$$

- Dually, a *comonoid* is an object C together with maps $\delta\colon C \to C \otimes C$ and $\epsilon\colon C \to I$ which are coassociative and counital.

$\star$ These form categories Mon and Comon, with maps preserving structure.

When $(\mathcal{V}, \otimes, I, \sigma)$ is braided, Mon and Comon are monoidal, with I and

$$A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\sigma\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\mu\otimes\mu} A \otimes B$$

■ If $\mathcal{V}$ is also closed, induced $[\text{-},\text{-}]\colon \mathsf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathsf{Mon}(\mathcal{V}) \to \mathsf{Mon}(\mathcal{V})$ makes $[C,A]$ into a monoid via *convolution*

$$\begin{tikzcd}[C,A]\otimes[C,A]\otimes C \arrow[r, "1\otimes\delta"] \arrow[rrdd, dashed, bend right] & {[C,A]\otimes[C,A]\otimes C\otimes C} \arrow[r, "1\otimes\sigma\otimes 1"] & {[C,A]\otimes C\otimes[C,A]\otimes C} \arrow[d, "\mathrm{ev}\otimes\mathrm{ev}"] \\ & & A\otimes A \arrow[d, "\mu"] \\ & & A\end{tikzcd}$$

⋆ In Vect_k, linear dual $C^* = \mathrm{Hom}_k(C,k)$ for a k-coalgebra is a k-algebra – while A^* for a k-algebra is a k-coalgebra only if it is finite dimensional.

Sweedler dual 'fixes' that: $A^o = \{f \in A^* \mid \mathit{ker} f$ contains cofinite ideal$\}$ is a k-coalgebra, and $\mathsf{Alg}(A, C^*) \cong \mathsf{Coalg}(C, A^o)$ by adjunction

$$\begin{tikzcd}\mathsf{Coalg}^{\mathrm{op}} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathsf{Alg} \arrow[l, bend left, "((-)^o)^{\mathrm{op}}"]\end{tikzcd}$$

'Sweedler theory' for monoidal categories

- *Universal measuring k-coalgebra* $\mathrm{Alg}(A,[C,B])\cong\mathrm{Coalg}(C,P(A,B))$, algebraically as terminal object in category of k-coalgebras that *measure*.

$\star$ Moving to general context of braided monoidal closed categories, local presentability gives passage from Vect_k to (d)gVect_k, Mod_R &many more.

Suppose $\mathcal{V}$ is a braided monoidal closed and locally presentable category. There is a parameterized adjunction between

$$[\text{-},\text{-}]\colon \mathrm{Comon}^{\mathrm{op}}\times\mathrm{Mon}\to\mathrm{Mon} \quad \text{convolution}$$
$$P(\text{-},\text{-})\colon \mathrm{Mon}^{\mathrm{op}}\times\mathrm{Mon}\to\mathrm{Comon} \quad \text{universal measuring}$$

$\boxed{P(A,B)=\int^C \mathrm{Mon}(A,[C,B])\cdot C}$ in Set is $\mathrm{Mon}(A,B)$; in Vect_k contains k-algebra maps as grouplike; in dgVect_k, relates to bar-cobar adjunction.

- Convolution $[\text{-},\text{-}]$ is an *action* of the monoidal $\mathsf{Comon}^{(\mathrm{op})}$ on Mon.

Any parameterized adjoint of an action $\bullet\colon \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ gives rise to a $\mathcal{V}$-enriched structure on $\mathcal{C}$, and all tensored $\mathcal{V}$-categories arise this way.

If $\mathcal{V}$ further symmetric, Mon is enriched in symmetric monoidal Comon.

Digression: *semi-Hopf* $\mathcal{V}$-categories generalize bimonoids

· $H(x,y)\otimes H(y,z)\to H(z,x),\ I\to H(x,x)$ 'global' multipl

· $H(a,b)\to H(a,b)\otimes H(a,b),\ H(a,b)\to I$ 'local' comultipl

$$\begin{tikzcd}
H_{x,y}\otimes H_{y,z} \arrow[rr, "\delta_{xy}\otimes\delta_{yz}"] \arrow[d, "\mu_{xyz}"'] & & H_{x,y}\otimes H_{x,y}\otimes H_{y,z}\otimes H_{y,z} \arrow[d, "(\mu_{xyz}\otimes\mu_{xyz})\circ(1\otimes\sigma\otimes 1)"] \\
H_{x,z} \arrow[rr, "\delta_{xz}"] & & H_{x,z}\otimes H_{x,z}
\end{tikzcd}$$

The category of monoids in $\mathcal{V}$ is a semi-Hopf $\mathcal{V}$-category.

(Co)modules enter the picture

▶ For monoid A, an A-module M comes with associative and unital $\mu\colon A\otimes M\to M$, and dually a C-comodule X comes with $\chi\colon X\to C\otimes X$. With maps preserving (co)actions, categories ${}_A\mathrm{Mod}$ and ${}_C\mathrm{Comod}$.

■ 'Global' categories Mod, Comod of (co)modules for any (co)monoid, maps for Mod are $g\colon {}_AM\to {}_BN$ in $\mathcal{V}$ with $f\colon A\to B$ in Mon that

$$\begin{tikzcd}
A\otimes M \arrow[rr, "\mu"] \arrow[d, "1\otimes g"'] & & M \arrow[d, "g"] \\
A\otimes N \arrow[r, "f\otimes 1"] & B\otimes N \arrow[r, "\mu"] & N
\end{tikzcd}$$

⋆ Naturally form fibration Mod → Mon and opfibration Comod → Comon.

If $\mathcal{V}$ symmetric monoidal closed&locally presentable, adjunction between

$$[\text{-},\text{-}]\colon \mathrm{Comod}^{\mathrm{op}}\times\mathrm{Mod}\to\mathrm{Mod} \quad \text{convolution}$$

$$Q(\text{-},\text{-})\colon \mathrm{Mod}^{\mathrm{op}}\times\mathrm{Mod}\to\mathrm{Comod} \quad \text{universal measuring}$$

and Mod is enriched in the symmetric monoidal Comod.

From one to many objects

Goal: generalize from monoids in $\mathcal{V}$, to $\mathcal{V}$-categories!

$$\begin{array}{ccccccc} \mathsf{Mod} & & \mathsf{Comod} & & \mathcal{V}\text{-}\mathsf{Mod} & & \mathcal{V}\text{-}\mathsf{Comod} \\ \downarrow & \text{enriched in} & \downarrow & \rightsquigarrow & \downarrow & \text{enriched in} & \downarrow \\ \mathsf{Mon} & & \mathsf{Comon} & & \mathcal{V}\text{-}\mathsf{Cat} & & \mathcal{V}\text{-}\mathsf{Cocat} \end{array}$$

Method: work 'bottom-up' (technical but direct), or identify general framework and work 'top-down'

- A $\mathcal{V}$-module M consists of $\{M(x)\}_{x\in X}$ in $\mathcal{V}$ with an action $(\sum_y)A(x,y)\otimes M(y)\to M(x)$ for A some $\mathcal{V}$-category ($A \twoheadrightarrow \mathcal{I}$).
- For $\mathcal{V}$ with coproducts preserved by $\otimes$, a $\mathcal{V}$-cocategory C consists of $\{C(x,z)\}_X$ in $\mathcal{V}$ with $C(x,z)\to\sum_y C(x,y)\otimes C(y,z)$, $C(x,x)\to I$.
- For $\mathcal{V}$ with coproducts preserved by $\otimes$, a $\mathcal{V}$-comodule K consists of $\{K(x)\}_{x\in X}$ in $\mathcal{V}$ with a coaction $K(x)\to\sum_y C(x,y)\otimes K(y)$.

From monoidal to double categories

$\star$ Opcategories=$\mathcal{V}^{\mathrm{op}}$-categories? Not as convenient, formally!

- A double category $\mathbb{D}$ has $\mathbb{D}_0$ (0-cells&vertical 1-cells), $\mathbb{D}_1$ (horizontal 1-cells&2-maps) and $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$+coherent isos.
- A monad in a double category $\mathbb{D}$ is $A\colon X \to\!\!\!\!\!+ X$ with associative, unital

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, equal] & X \arrow[r, "A"] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[d, equal] \\ X \arrow[rr, "A"'] & & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & X \arrow[d, equal] \\ X \arrow[r, "A"'] & X \end{tikzcd}$$

Dually, comonad $C\colon X \to\!\!\!\!\!+ X$. These form categories $\mathsf{Mnd}(\mathbb{D})$ & $\mathsf{Cmd}(\mathbb{D})$.

$\star$ Morphisms are different than those for (co)monads in bicategories.

For $\mathbb{D} = \mathcal{V}$-$\mathbb{M}$at of sets, functions and $\mathcal{V}$-matrices $S\colon X \to\!\!\!\!\!+ Y$ i.e. $\{S(x,y)\} \in \mathcal{V}$ with $(S \circ T)(x,z) = \sum_y T(x,y) \otimes S(y,z)$, $\mathsf{Mnd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathsf{Cat}$ and $\mathsf{Cmd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathsf{Cocat}$.

- A module in $\mathbb{D}$ is $M\colon Z \nrightarrow X$ with an action of a monad $A\colon X \nrightarrow X$

$$\begin{tikzcd} Z \arrow[r, "M"] \arrow[d, equal] & X \arrow[r, "A"] \arrow[d, phantom, "\Downarrow"] & X \arrow[d, equal] \\ Z \arrow[rr, "M"'] & & X \end{tikzcd}$$

Dually, comodule $Q\colon W \nrightarrow X$ with globular $Q \Rightarrow C \circ Q$ for comonad $C\colon X \nrightarrow X$. These form categories $\mathrm{Mod}(\mathbb{D})$ & $\mathrm{Comod}(\mathbb{D})$, with maps

$$\begin{tikzcd} Z \arrow[r, "M_A"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & X \arrow[d, "f"] \\ T \arrow[r, "N_B"'] & Y \end{tikzcd} \text{ s.t. } \begin{tikzcd} Z \arrow[r, "M"] \arrow[d, equal] & X \arrow[r, "A"] \arrow[d, phantom, "\Downarrow"] & X \arrow[d, equal] \\ Z \arrow[rr, "M"] \arrow[d, "g"'] & \arrow[d, phantom, "\Downarrow\beta"] & X \arrow[d, "f"] \\ T \arrow[rr, "N"'] & & Y \end{tikzcd} = \begin{tikzcd} Z \arrow[r, "M"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & X \arrow[r, "A"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] \\ T \arrow[r, "N"'] \arrow[d, equal] & Y \arrow[r, "B"'] \arrow[d, phantom, "\Downarrow"] & Y \arrow[d, equal] \\ T \arrow[rr, "N"'] & & Y \end{tikzcd}$$

$\star$ Subcategories of interest: fixed-monad ${}_A\mathrm{Mod}(\mathbb{D})$, fixed-dom ${}^Z\mathrm{Mod}(\mathbb{D})$, bicat modules ${}^Z_A\mathrm{Mod}(\mathbb{D}) = \mathrm{Mod}(\mathcal{H}(\mathbb{D}))$...several monadicity results.

${}^{\{*\}}\mathrm{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathrm{Mod}$ and ${}^{\{*\}}\mathrm{Comod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathrm{Comod}$.
$\mathrm{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at})$ has $\{M(x,z)\}_{X\times Z}$ with $A(x,x') \otimes M(x',z) \to M(x,z)$.

For $\mathbb{D}$ a double category, $\mathrm{Mnd}(\mathbb{D})$ is enriched in $\mathrm{Cmd}(\mathbb{D})$ and $\mathrm{Mod}(\mathbb{D})$ is enriched in $\mathrm{Comod}(\mathbb{D})$, *under certain conditions*.

- Fibrant: vertical 1-cells f turn to horizontal, companion $\hat{f}$ & conjoint $\check{f}$.

In $\mathcal{V}$-$\mathbb{M}$at, $f\colon X \to Y$ gives matrices $\hat{f}(x,y) = \check{f}(y,x) = \begin{cases} I \text{ if } fx = y \\ 0 \text{ if } fx \neq y \end{cases}$

$\mathrm{Mod}(\mathbb{D}) \to \mathrm{Mnd}(\mathbb{D})$ is a fibration, with reindexing $\check{f} \circ \text{-}\colon {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$.
$\mathrm{Comod}(\mathbb{D}) \to \mathrm{Cmd}(\mathbb{D})$ is an opfibration, with $\hat{f} \circ \text{-}\colon {}_C\mathrm{Comod} \to {}_D\mathrm{Comod}$.

- Monoidal: $\mathbb{D}_0$ & $\mathbb{D}_1$ monoidal, $(M \otimes N)\circ(M' \otimes N') \cong (M \circ M')\otimes(N \circ N')$.

In $\mathcal{V}$-$\mathbb{M}$at, $(X \otimes Y) = X \times Y$ & $(S \otimes T)((x,y),(z,w)) = S(x,z) \otimes T(y,w)$.

$\mathrm{Cmd}(\mathbb{D})$ and $\mathrm{Comod}(\mathbb{D})$ are monoidal, with $Q_C \otimes P_D = (Q \otimes P)_{C \otimes D}$ via $Q \otimes P \to (C \circ Q) \otimes (D \circ P) \cong (C \otimes D) \circ (Q \otimes P)$. Symmetry is inherited.

■ Locally closed monoidal: lax double functor $H\colon \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ such that $\mathbb{D}_0$ & $\mathbb{D}_1$ monoidal closed, $\mathfrak{s}, \mathfrak{t}$ maps of adjunctions.

In $\mathcal{V}$-$\mathbb{M}$at, $[X, Y] = Y^X$ and $H(S,T)(f,g) = \prod_{x,y}[S(x,y), T(fx,gy)]$.

Lax double H induce functors $\mathrm{Cmd}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mnd}(\mathbb{D}) \to \mathrm{Mnd}(\mathbb{D})$ and $\mathrm{Comod}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mod}(\mathbb{D}) \to \mathrm{Mod}(\mathbb{D})$ which are actions. *convolution*

■ Locally presentable: $\mathbb{D}_0$ & $\mathbb{D}_1$ locally presentable, $\mathfrak{s}, \mathfrak{t}$ accessible right adjoints, $\mathbf{1}$ accessible, $- \circ -$ accessible in each variable.

In $\mathcal{V}$-$\mathbb{M}$at, Set is locally presentable & $\mathcal{V}$-$\mathbb{M}\mathrm{at}_1$ is too (Limit theorem. . .)

Induced functors H have adjoints $\mathrm{Mnd}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mnd}(\mathbb{D}) \to \mathrm{Cmd}(\mathbb{D})$ and $\mathrm{Mod}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mod}(\mathbb{D}) \to \mathrm{Comod}(\mathbb{D})$. *universal measuring*

◇ Obtain enrichment of $\mathrm{Mnd}(\mathbb{D})$ in $\mathrm{Cmd}(\mathbb{D})$ & of $\mathrm{Mod}(\mathbb{D})$ in $\mathrm{Comod}(\mathbb{D})$.

Moving to other contexts

$\star$ For $\mathcal{V}$ symmetric monoidal closed & locally presentable, $\mathbb{D} = \mathcal{V}$-$\mathbb{M}$at is a fibrant, locally closed monoidal & locally presentable double category.

- Enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories
- Enrichment of $\mathcal{V}$-modules in $\mathcal{V}$-comodules

Goal: employ/*extend* theory to apply to $\mathbb{D} = \mathcal{V}$-$\mathbb{S}$ym

operads

- objects are sets $X, Y, \ldots$
- vertical 1-cells are functions $f, g, \ldots$
- horizontal 1-cells are coloured symmetric sequences $M\colon SX \times Y \to \mathcal{V}$
- 2-maps are $M(x_1, \ldots, x_n; y) \to N(fx_1, \ldots, fx_n; gy)$

Horizontal composition is generalization of substitution for species...

■ $\mathcal{V}$-$\mathbb{S}$ym is fibrant; but not monoidal double anymore!

Oplax monoidal double structure

- A double category is *oplax* monoidal with comparison maps

$$(N \circ M) \otimes (N' \circ M') \to (M \otimes M') \circ (N \otimes N'), \quad 1_X \otimes 1_{X'} \to 1_{X \otimes X'}$$
$$I_1 \to I_1 \circ I_1, \quad I_1 \to 1_I$$

making $\otimes \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$, $I \colon \mathbf{1} \to \mathbb{D}$ into oplax double functors.

> An oplax monoidal double category with a single object and vertical arrow is precisely a duoidal category.

- *Normality* condition (I pseudo, $\otimes$ pseudo in each variable) reduces to normal duoidal structure & gives passage to 'oplax monoidal bicategories'.

Idea: show that $\mathcal{V}$-$\mathbb{S}$ym (in fact $\mathcal{V}$-$\mathbb{C}$at$\mathbb{S}$ym...) is normal oplax monoidal, by expressing it as a *Kleisli double category* of monoidal double $\mathcal{V}$-$\mathbb{P}$rof.

Arithmetic product of coloured symmetric sequences

Kleisli double category is $\mathcal{V}$-$\mathbb{C}\mathrm{at}\mathbb{S}\mathrm{ym}$ of *categorical symmetric sequences* $M\colon SY^{\mathrm{op}}\times X\to\mathcal{V}$, with 'discrete' case $\mathcal{V}$-$\mathbb{S}\mathrm{ym}$.

- Horizontal composition is many-object generalisation of substitution

$$(N\circ M)(\vec{z},x)=\int^{SZ,SY} SZ[\vec{z},\bigotimes_i \vec{w}^i]\times\prod N(\vec{w}^i,y_i)\times M(\vec{y},x)$$

- Oplax monoidal structure is many-object generalisation of 'arithmetic product' of species

$$(M\boxtimes N)(\vec{a},(x,z))=\int^{\vec{y},\vec{w}} S(Y\times W)(\vec{a},\vec{y}\boxtimes\vec{w})\times M(\vec{y},x)\times N(\vec{w},z)$$

Some future directions

- Extend 'Sweedler theory' from monoidal double to oplax monoidal double categories: do we still obtain an enrichment of monads in comonads, and of modules in comodules?

- Further explore structure of $\mathcal{V}$-$\mathbb{S}$ym: is it locally monoidal closed and locally presentable as a double category, for introduced definitions?

[One-object case] If $\mathcal{V}$ is symmetric monoidal closed and loc presentable,

- positive operads are enriched in positive cooperads, if $\mathcal{V}$ has biproducts;
- symmetric operads are enriched in symmetric cooperads, if $\mathcal{V}$ is cartesian.

- Boardman-Vogt tensor of bimodules of symmetric coloured operads and bimodules: abstract double categorical framework behind that?

Thank you for your attention!

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

- *Aravantinos-Sotiropoulos, Vasilakopoulou*, "Enriched duality in double categories II: $\mathcal{V}$-modules and $\mathcal{V}$-comodules", in preparation
- *Gambino, Garner, Vasilakopoulou*, "Monoidal Kleisli bicategories and the arithmetic product of symmetric sequences", arXiv:2206.06858