Colorings, Penrose Evaluations and MultiVirtual Knots & Links

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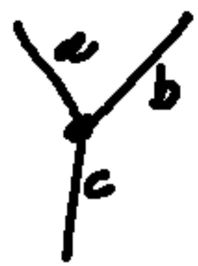
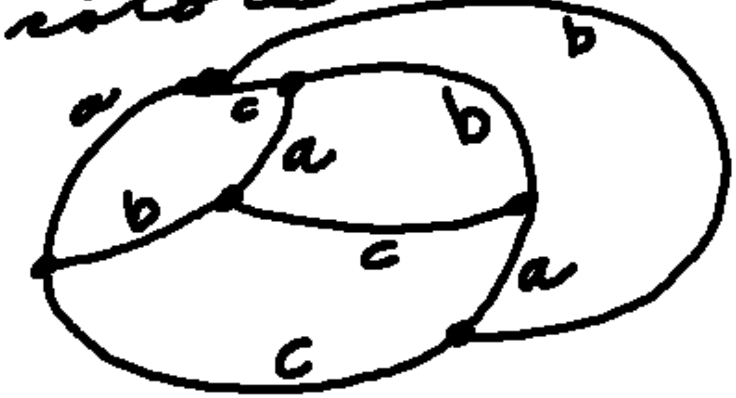
Recall problem of 3-coloring the edges of a cubic graph.

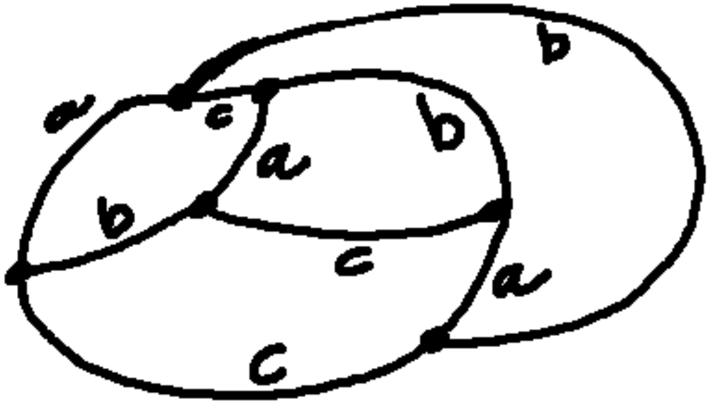


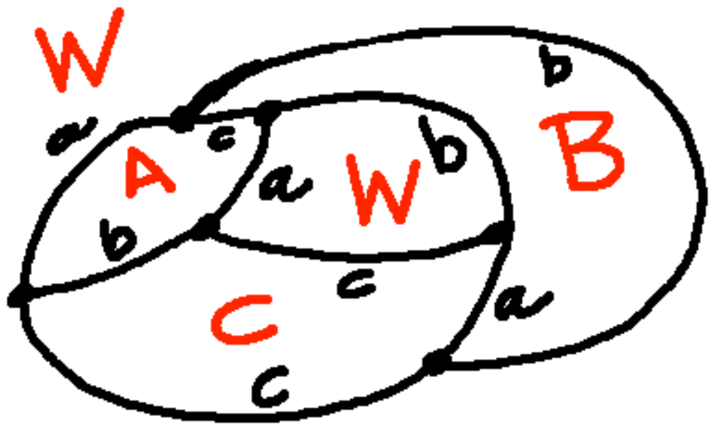
3 colors $\{a, b, c\}$

- all distinct
- require 3 distinct colors at each node.

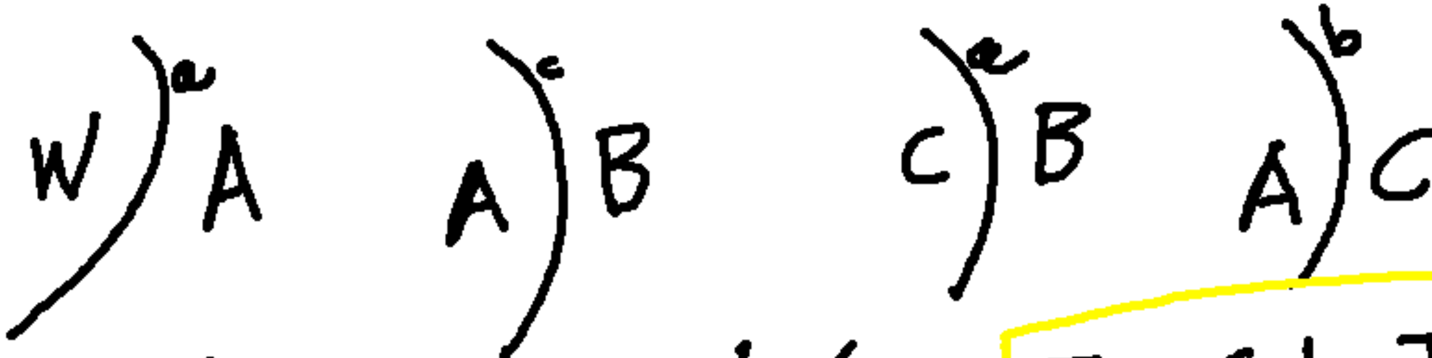
e.g.







$$G = \{W, A, B, C \mid W = id,\ AB = BA = C,\ A^2 = B^2 = C^2 = W\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$



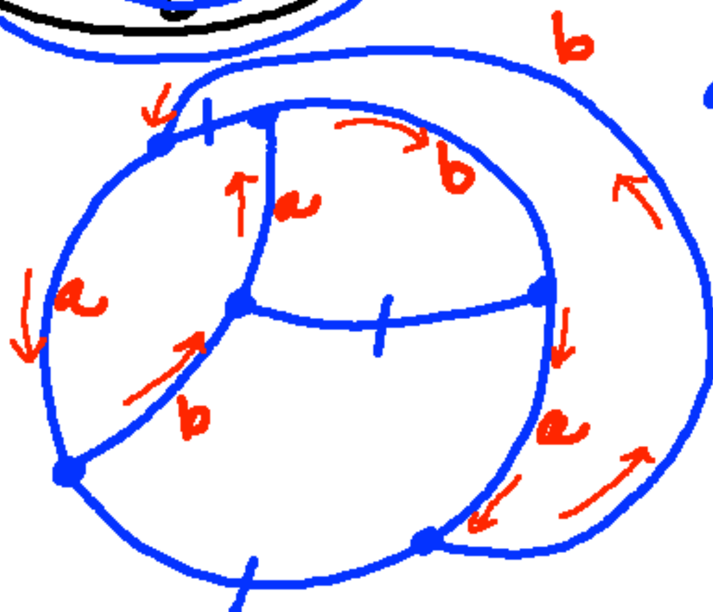
Four Color Thm

$\Longleftrightarrow$

cubic 1-connected planar are edge 3-colorable

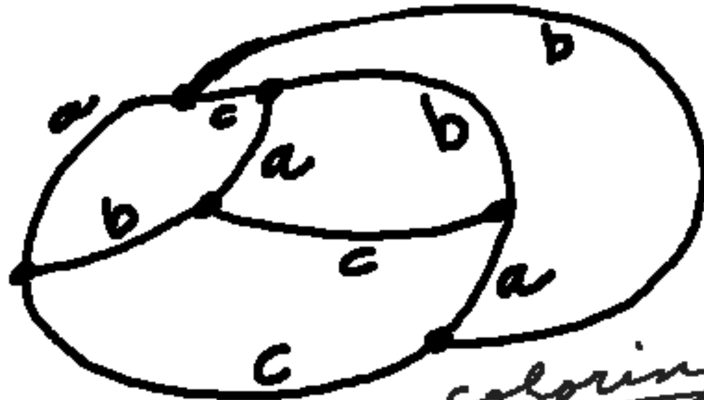
Choose one color (say c) and mark all c edges.

The result is an even perfect matching for the graph G. (Every node taken by the selected edges. Selected edges are disjoint.)



Even PM: Every cycle in (G − PMedges) is an even cycle.

Nota Bene: G is 3 colorable $\Leftrightarrow$ G has an even PM.



coloring

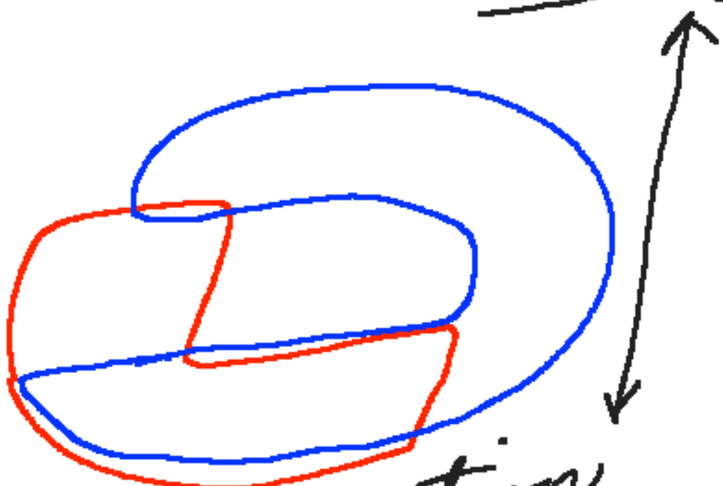
Let a = red

b = blue

c = purple

||

red/blue

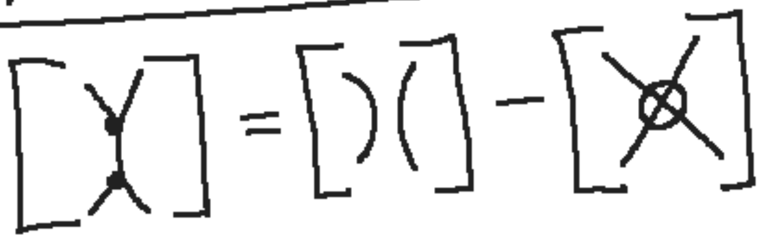


formation

How red meets blue.

One can directly construct infinitely many formations. 4CT ⇒ formations include all planar 3-conn cubic graphs.

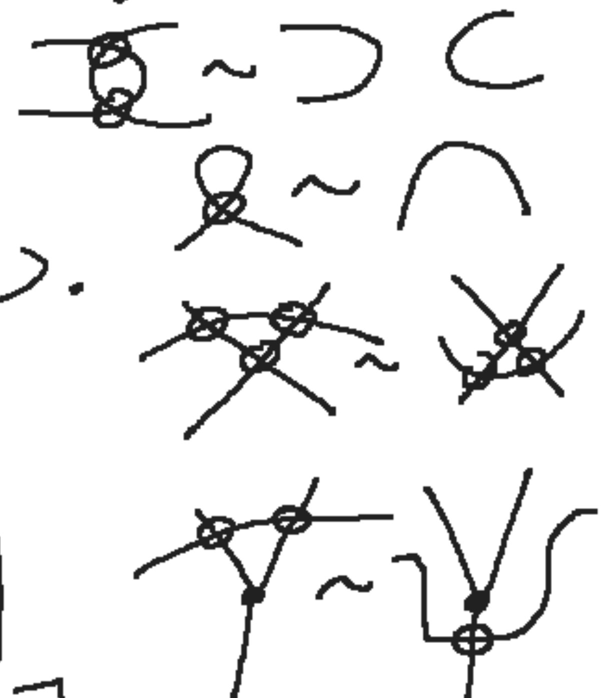
The Penrose Formulas



$[\bigcirc] = 3$

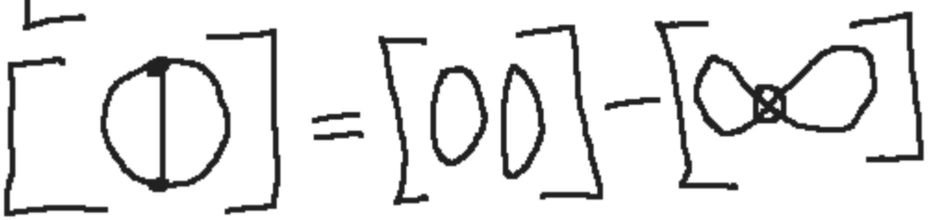
Compute recursively.

 is a virtual crossing

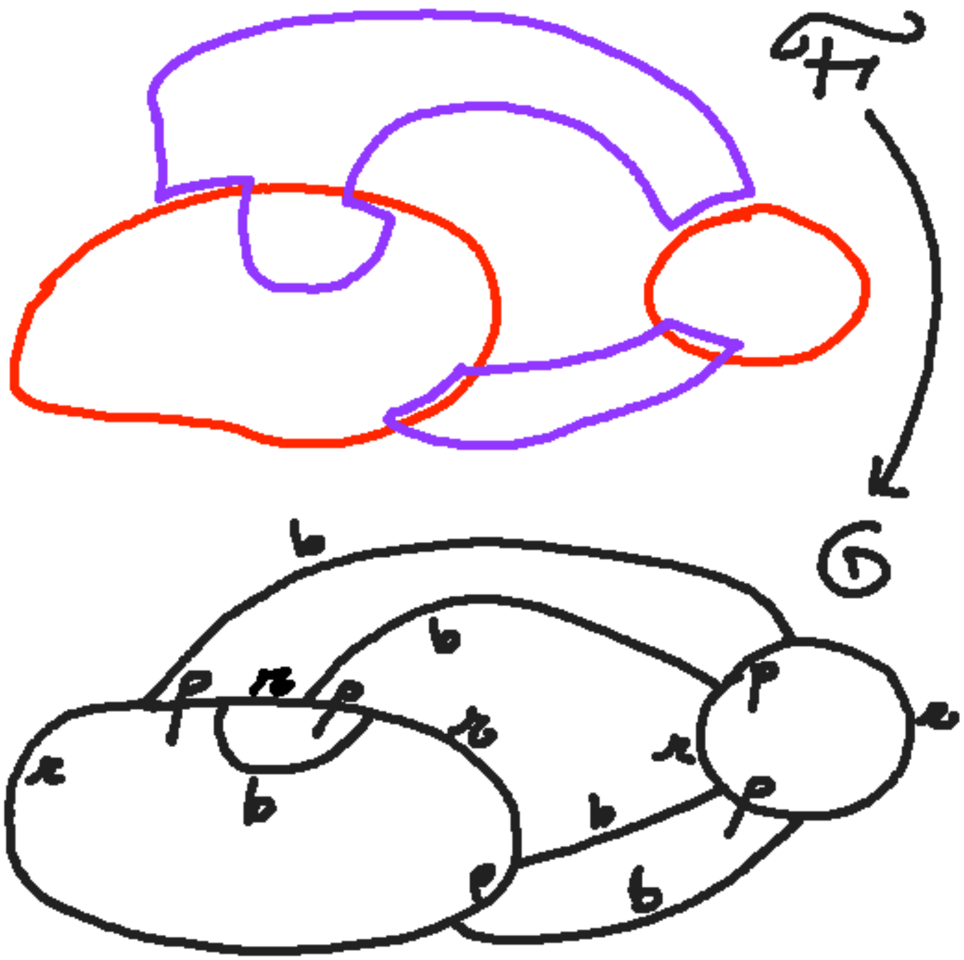


Penrose Theorem. G cubic plane graph $\Rightarrow [G] = \#$ of 3 colorings of G.

e.g.  $= \phi$.

 $= 3^2 - 3 = 6$.

Formation



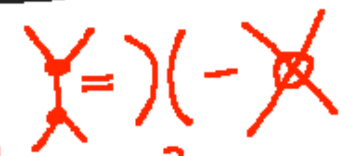
Four Color Theorem

$|||$

Every 1-connected planar G can be formatted

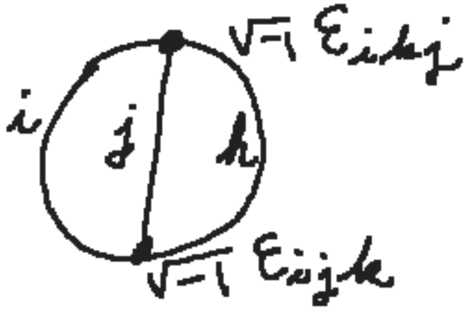
Thus 4CT $\iff$ $[G] \neq 0$ whenever G is plane cubic, 1conn.

Penrose Definition (The Ubiquitous Epsilon)



is a tensor identity

Convert G to a tensor net & contract.



$\longleftrightarrow \sqrt{-1}\,\varepsilon_{ijk}$

where $i,j,k \in \{1,2,3\}$

& $\varepsilon_{ijk} = \begin{cases} \operatorname{sgn}(ijk) & \text{if all distinct.} \\ 0 & \text{if not a permutation of } 123. \end{cases}$

$$[G] = \text{Contraction of Tensor Net}(G)$$

$$= \sum_{\sigma \in \text{colors of } G} \prod_{\text{nodes of } G} (\pm\sqrt{-1}) = \langle \sigma \rangle$$

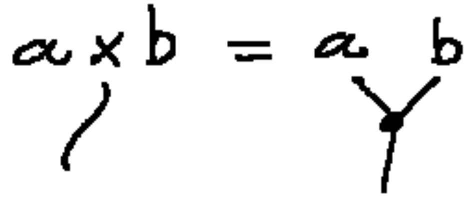
Show: $\langle \sigma \rangle = +1$ for each coloring σ.

Algebraic Remarks

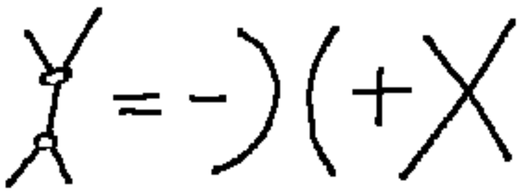
$$(a\times b)_k = \sum_{ij} \varepsilon_{ijk}\, a_i b_j$$


$= \varepsilon_{ijk}$

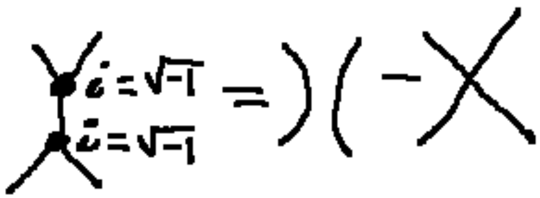

$= \sqrt{-1}\,\varepsilon_{ijk}$



Vector cross product



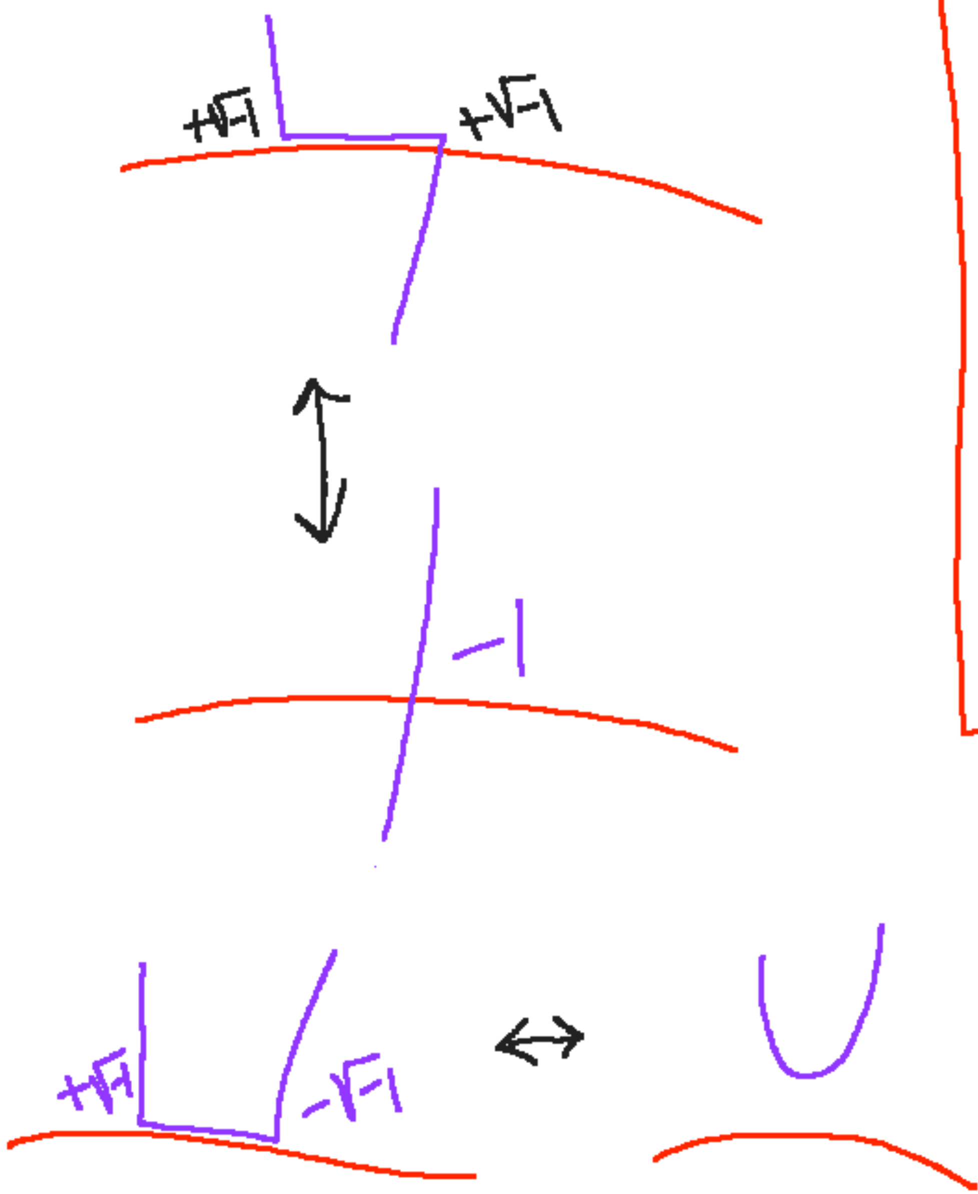
epsilon identity

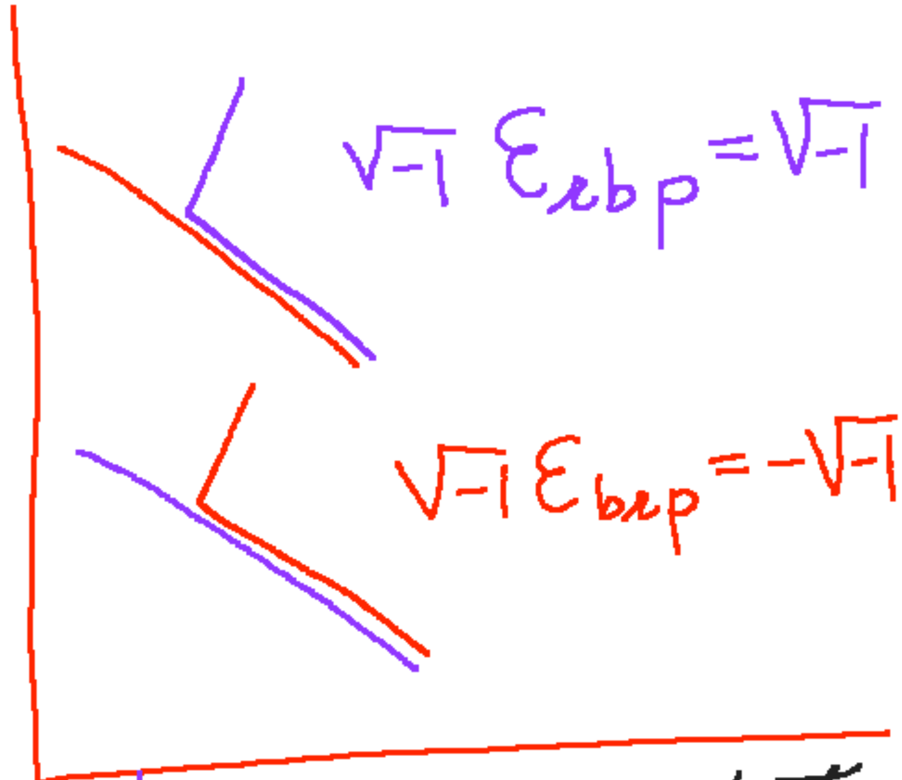


$\longleftrightarrow$ Penrose Identity

4CT $\longleftrightarrow$ solvability of equations $(\neq 0)$
of type $(a\times b)\times(c\times d) = (a\times(b\times c))\times d$
over vector cross product
algebra $\{I, J, K\}$.

$\longleftrightarrow$ non-zero products in quaternions.





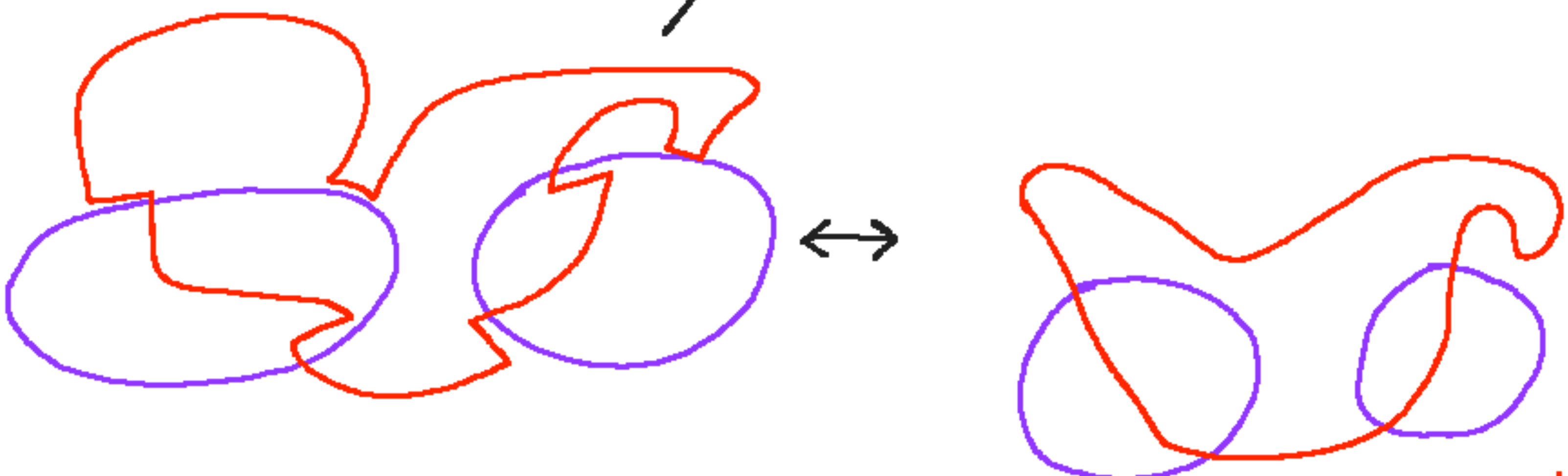
$\sqrt{-1}\,\varepsilon_{rbp} = \sqrt{-1}$

$\sqrt{-1}\,\varepsilon_{brp} = -\sqrt{-1}$

Thus the product of $\pm\sqrt{-1}$'s for a resolving $= (-1)^n$ where $n = \#$ crossings $=$ of "state curves".

But the number of crossings of red + blue curves is <u>even</u> due to the Jordan curve theorem.

$\therefore\ [G] = \#$ of colorings of G when G is a plane cubic graph.



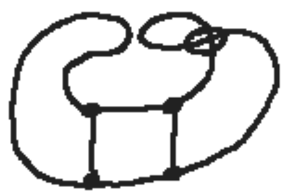
$n=4 \Rightarrow (-1)^n = +1.$

The Penrose formula does not work for nonplanar graphs (but we shall fix it).

ex. 

$G = K_{33}$

Exercise. # of 3 colorings of G is 12.

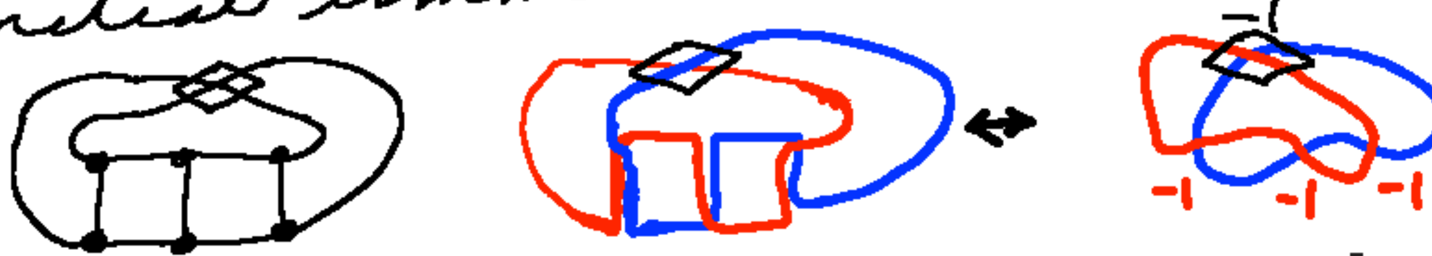
However $[G] =$  $-$  $\simeq$ 

$=$  $-$  $= \emptyset$.

Thus Penrose formula gives zero but we would like it to give 12!

The Fix

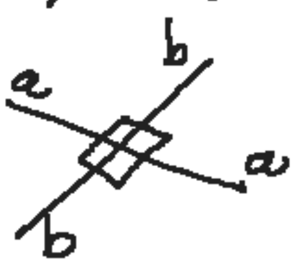
Add a new tensor at the initial immersion crossings.



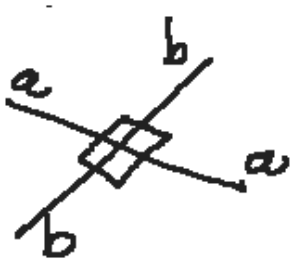
$(-1)^4 = 1$

We change this to a new tensor and a new virtual marker:  "innocuous"

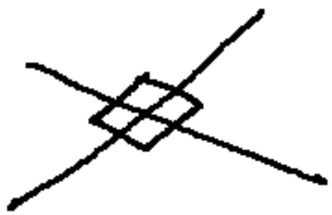
 $= -1$

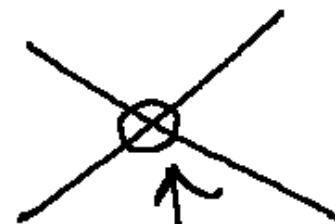
but 

$$= \begin{cases} 1 & \text{if } a = b \\ -1 & \text{if } a \neq b \end{cases}$$



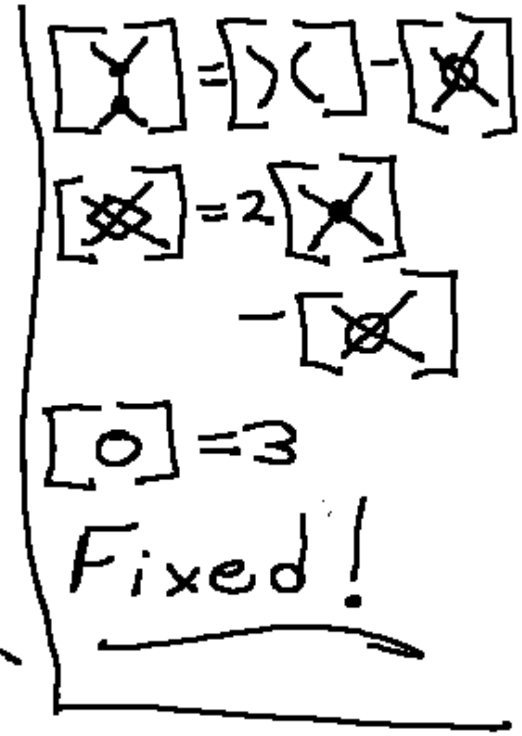
$$= \begin{cases} 1 & \text{if } a = b \\ -1 & \text{if } a \neq b \end{cases}$$



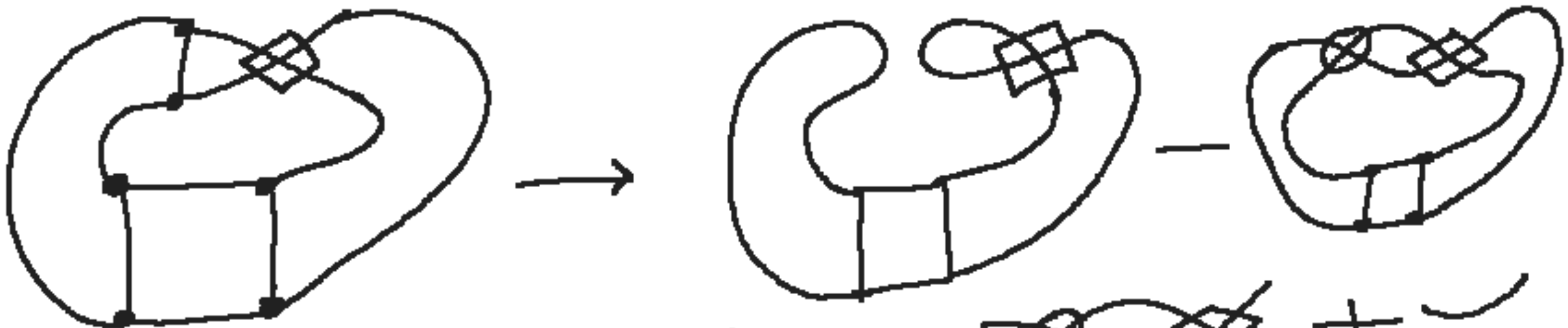
$= 2$  $-$ 

same color

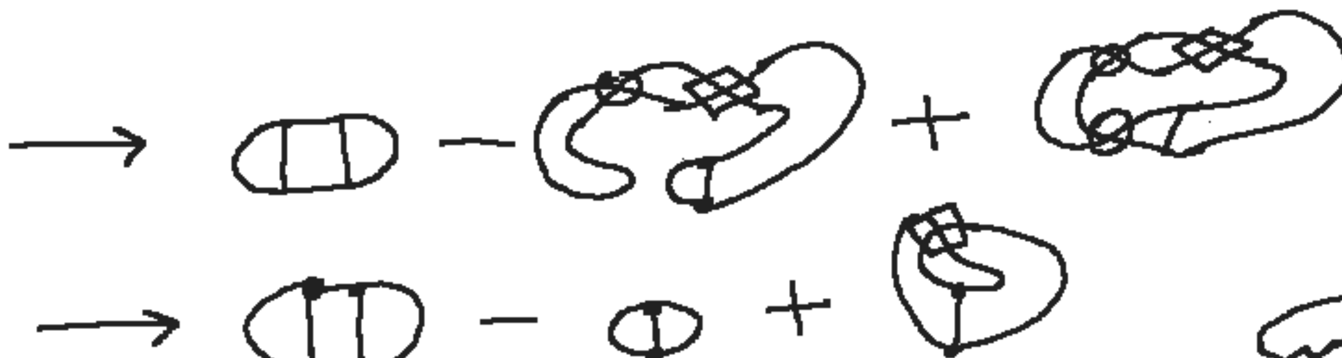
same or different color



We now have two virtual crossings  ,  .



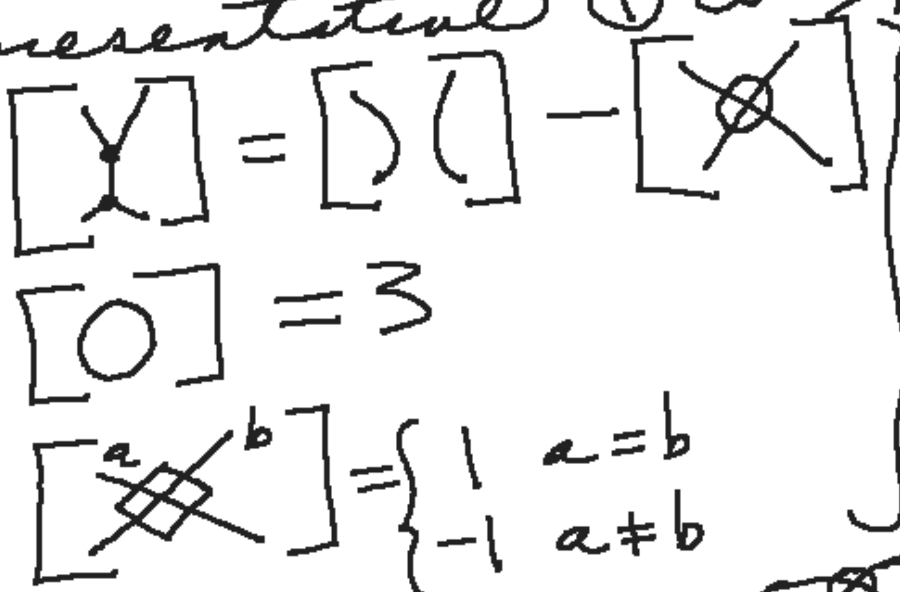
Nota Bene: 



$= 2 \times 2 \times 3 - 6 +$  $-$ 

$= 6 + 3 - [3 - 6] = 12$

Thus we can now formulate a general Penrose evaluation to count the number of colorings of arbitrary cubic graphs: Use an immersed representative $G \xrightarrow{f} \mathbb{R}^2$.



In context of a doubled virtual ~~crossing~~ context.

Note that structures like  have a separate chromatic computation.

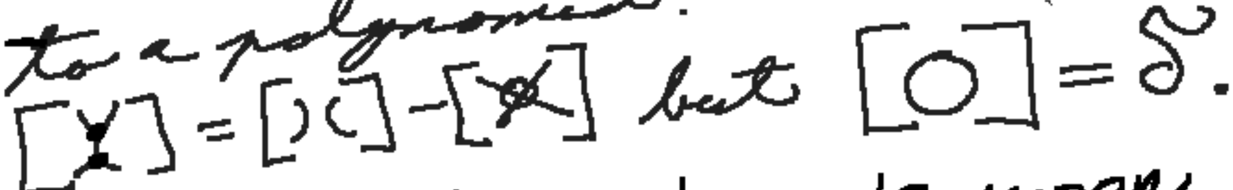
Onward

1. Generalized Penrose Polynomials for graphs with a perfect matching

2. Generalized doubled virtual knot theory.

Relations between 1) + 2)

Generalize Penrose evaluation to a polynomial.

Try 

Then evaluation depends upon choice of perfect matching.

So let G be given a perfect matching and define an expansion via

 $= \delta$

and we need to explain handling  : we want $\delta - \delta(\delta - 1) = 2\delta - \delta^2$

Let  mean "same" so that

 $= \delta$. Let  $= 2$  $-$ 

e.g.  $= 2$  $-$  $= 2\delta - \delta^2$.

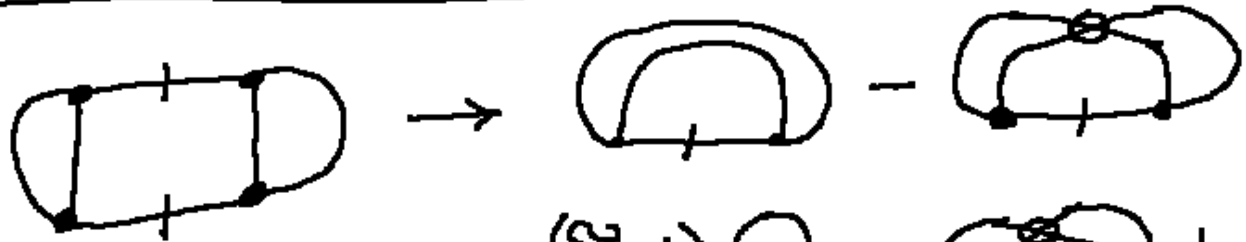
any = same + diff

 = same − diff

= 2 (same) − any

$= 2$  $-$ 

 $= (\delta - 1)$ 



$= (\delta - 1)$ 

$= (\delta - 1)\delta - \delta + \delta^2$

$= 2\delta^2 - 2\delta$

Perfect Matching Polynomials

 $(\delta - 1)^2$  $= (\delta - 1)^2 \delta$

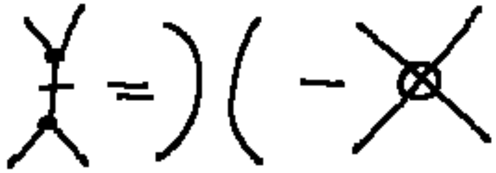
$= \delta^3 - 2\delta^2 + \delta$

N.B. $2 \cdot 3^2 - 2 \cdot 3 = 18 - 6 = 12$

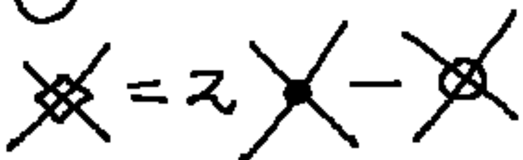
$27 - 18 + 3 = 12$

Agreement at $\delta = 3$.

Lets work with



$\bigcirc = \delta = n \in \{3, 4, 5, 6, 7, \dots\}$



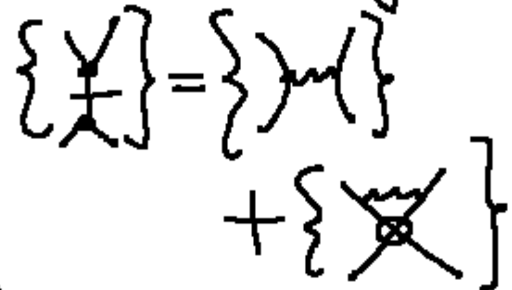
and call the poly in n, $[G]$.

(See paper Scott Baldridge, LK, Ben McCarty)

We relate $[G]$ to a homology theory and we interprect $[G]$ as a <u>coloring count</u>.

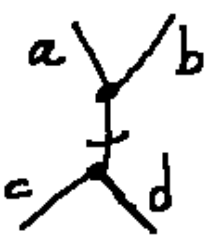
We discuss here the <u>counting</u>.

Tautology



where

$a)m(b : a \neq b$



$a \neq b$

$c \neq d$

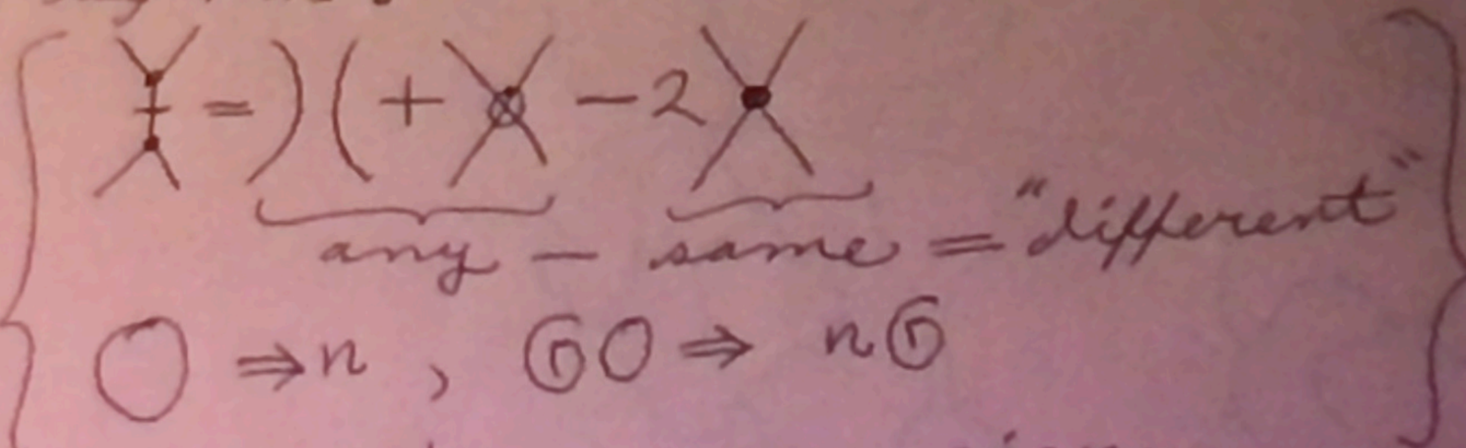
$\{a, b\} = \{c, d\}$

$a, b, c, d \in \{1, 2, \dots, n\}$

<u>n colors</u>

Color Condition for a given perfect matching.

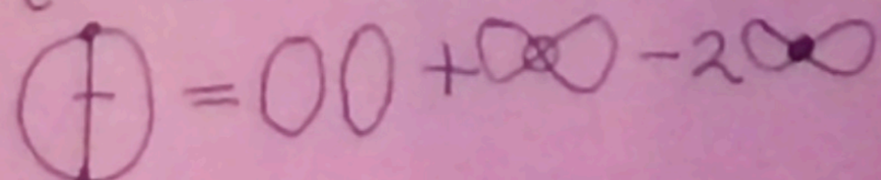
Remark: Thinking chromatically we can say this:



This is a Penrose type expansion and works for all cubic graphs.

$$= \delta^a_c \delta^b_d + \delta^a_d \delta^b_c - 2\eta^{ab}_{cd}$$

ex:



$$= q^2 + q - 2q$$
$$= q^2 - q \checkmark$$

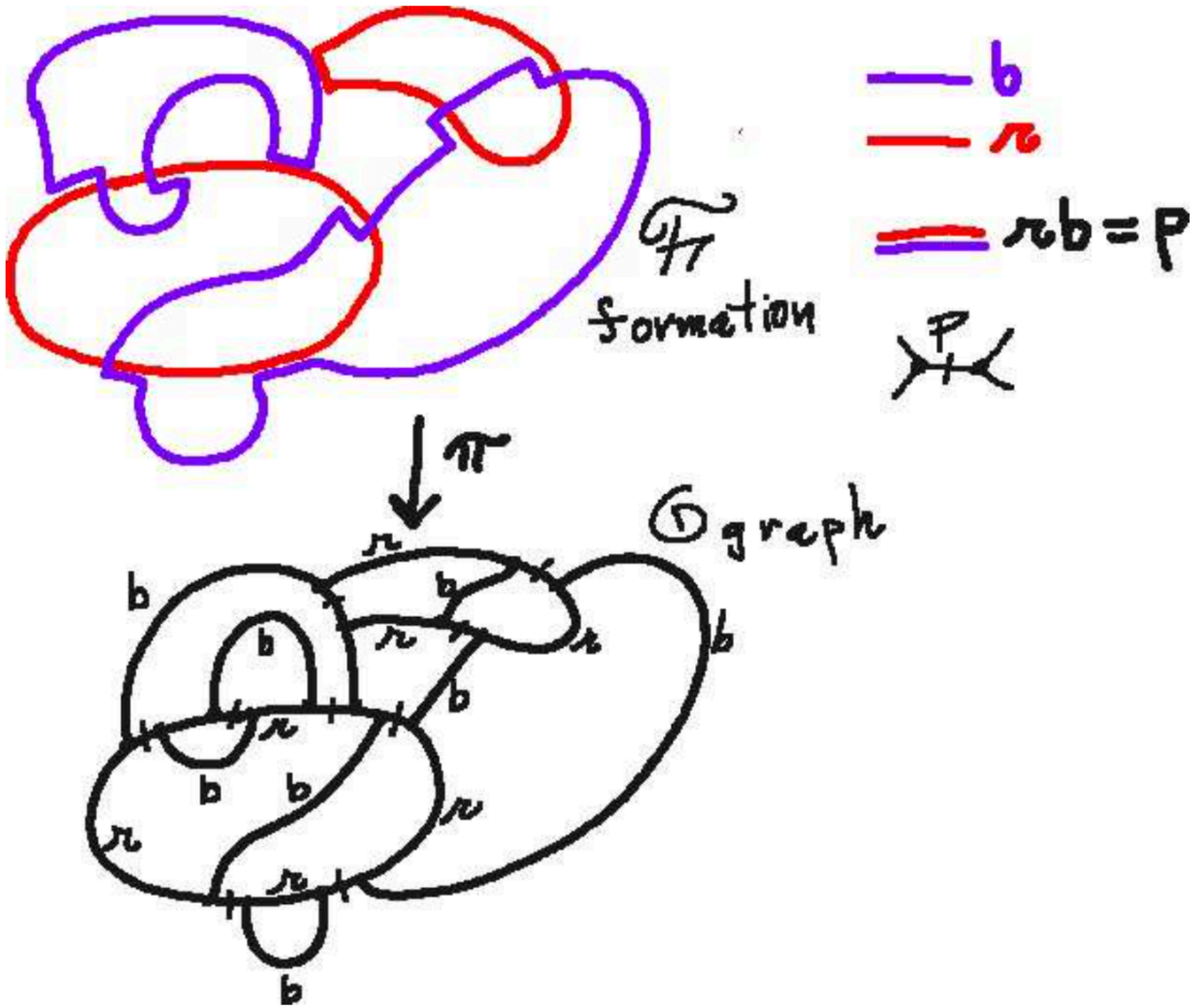


Figure 1: **Standard Formation and Graph**

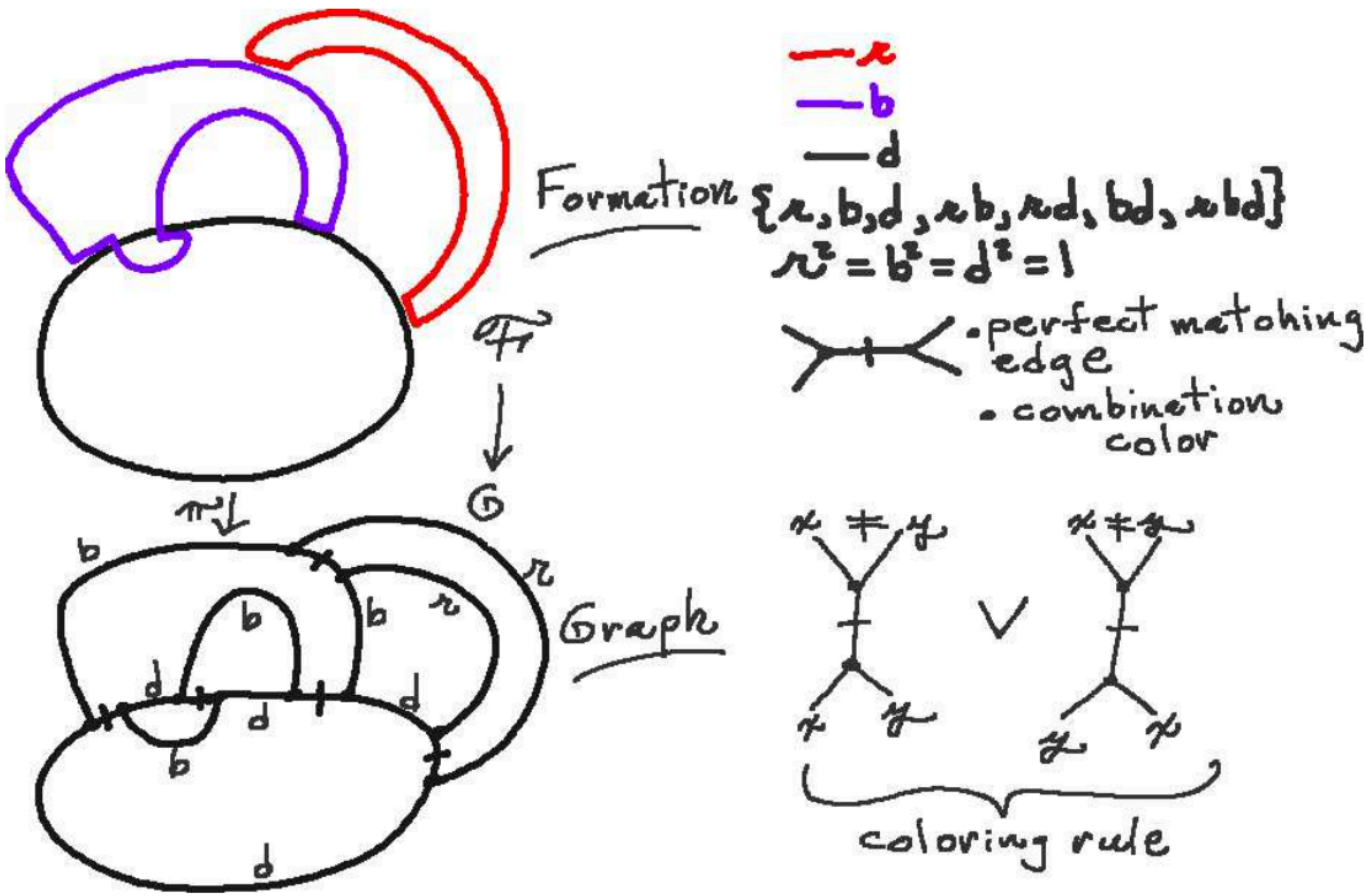
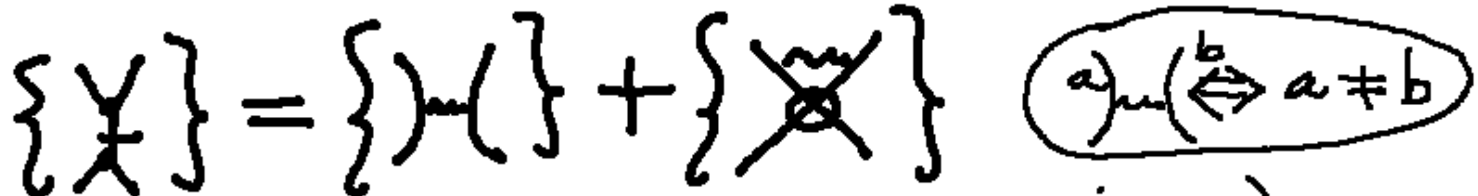


Figure 2: **Generalized Formation and Graph**

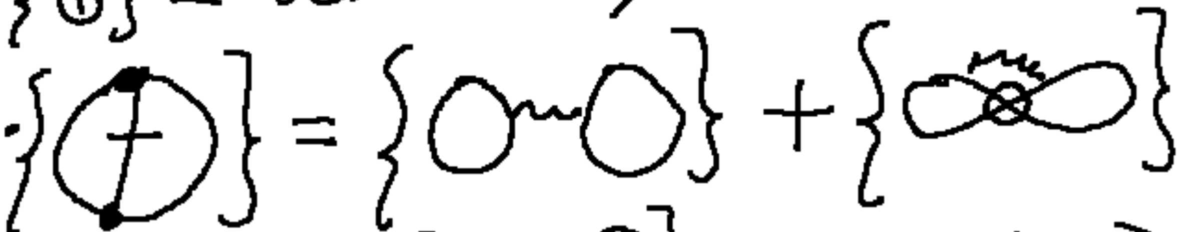
Tautological Expansion

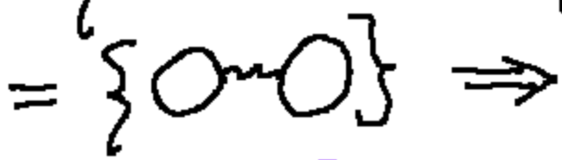
$\{a,b\} = \{c,d\}$
$a \neq b$:

or

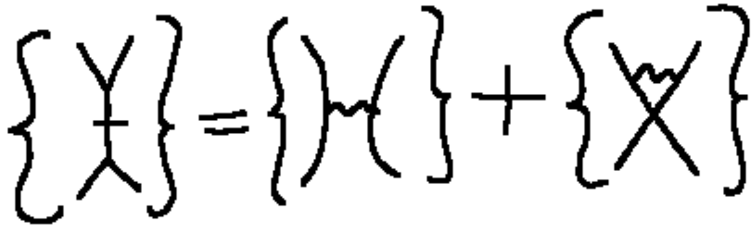


$\{G\}$ = Union of all colorings.

e.g. 

 $\Rightarrow$ $n(n-1)$ colorings

Compare :  $= n^2 - n$.

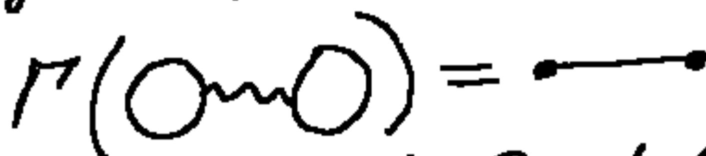


Matching Polynomial

Associate to a state S in this expansion a graph $\Gamma(S)$:

Loops(S) = Nodes$(\Gamma(S))$,

Wiggles(S) = Edges$(\Gamma(S))$.

e.g. 

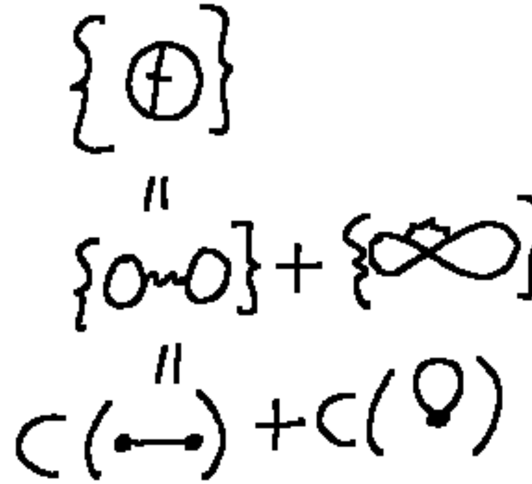
For each state S, define

$\{S\} = C(\Gamma(S))$ = chromatic poly of $\Gamma(S)$ where $C(\bullet) = n = \delta$.

Then $\{G, M\} = \sum_S C(\Gamma(S))$.

cubic graph

perfect matching on G.



$$n(n-1) + \phi$$

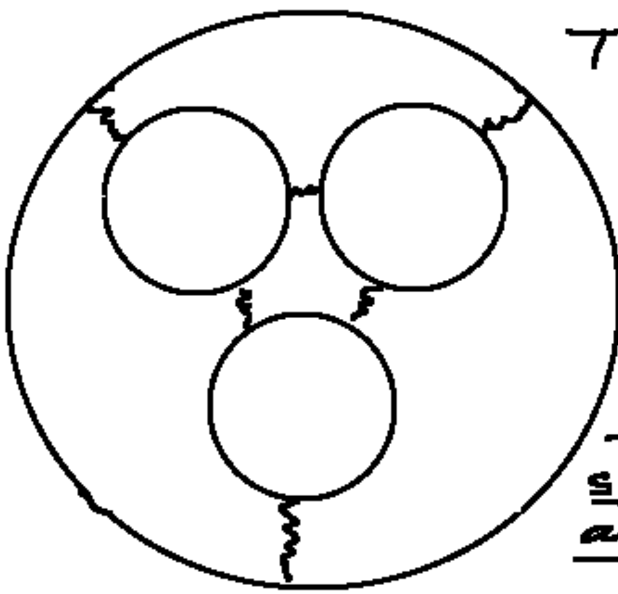
$$\|$$

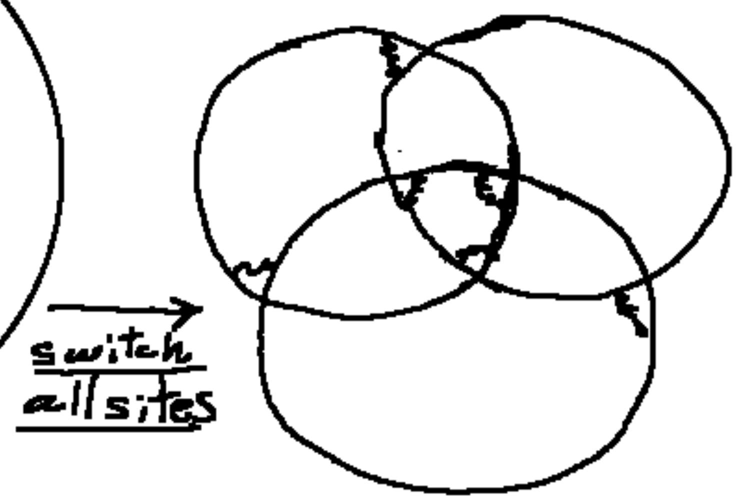
$$n(n-1)$$

From point of view
of tautological expansion,
start with cycles,)(local sites,
possibility to switch)(→ $\times$.

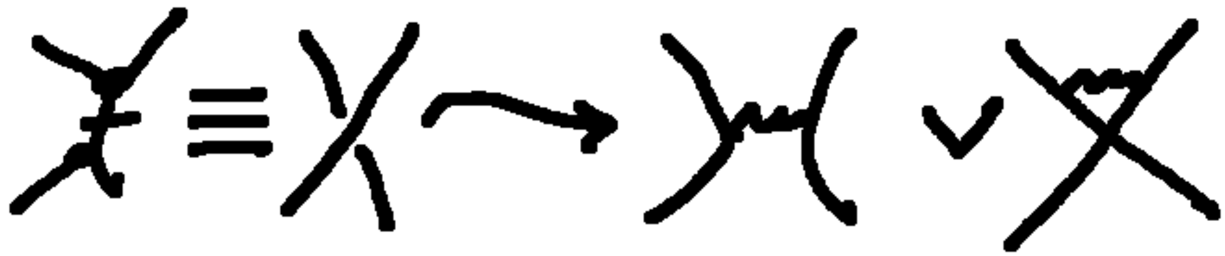
4GT ⟺ [planar states can be switched
to colorable states.]

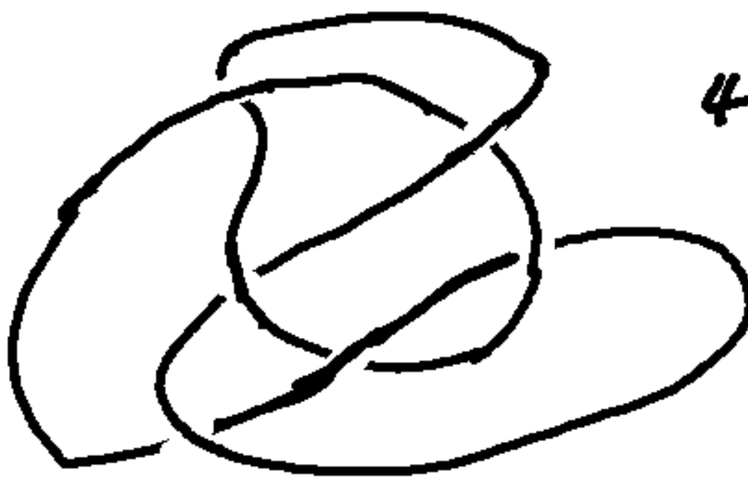
This state is n=3
uncolorable.



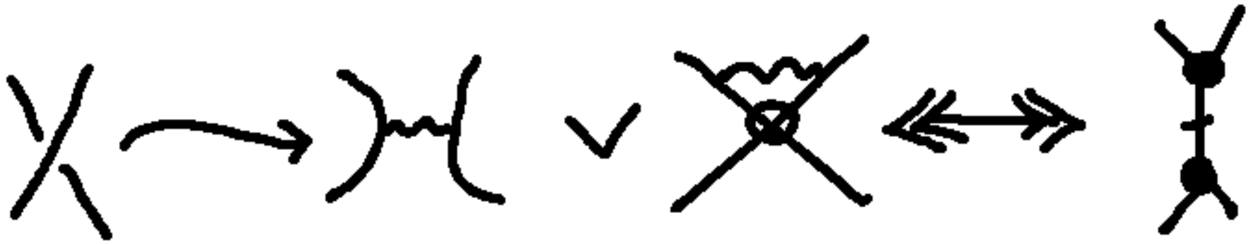


This means you can think in terms of knot/link diagrams.

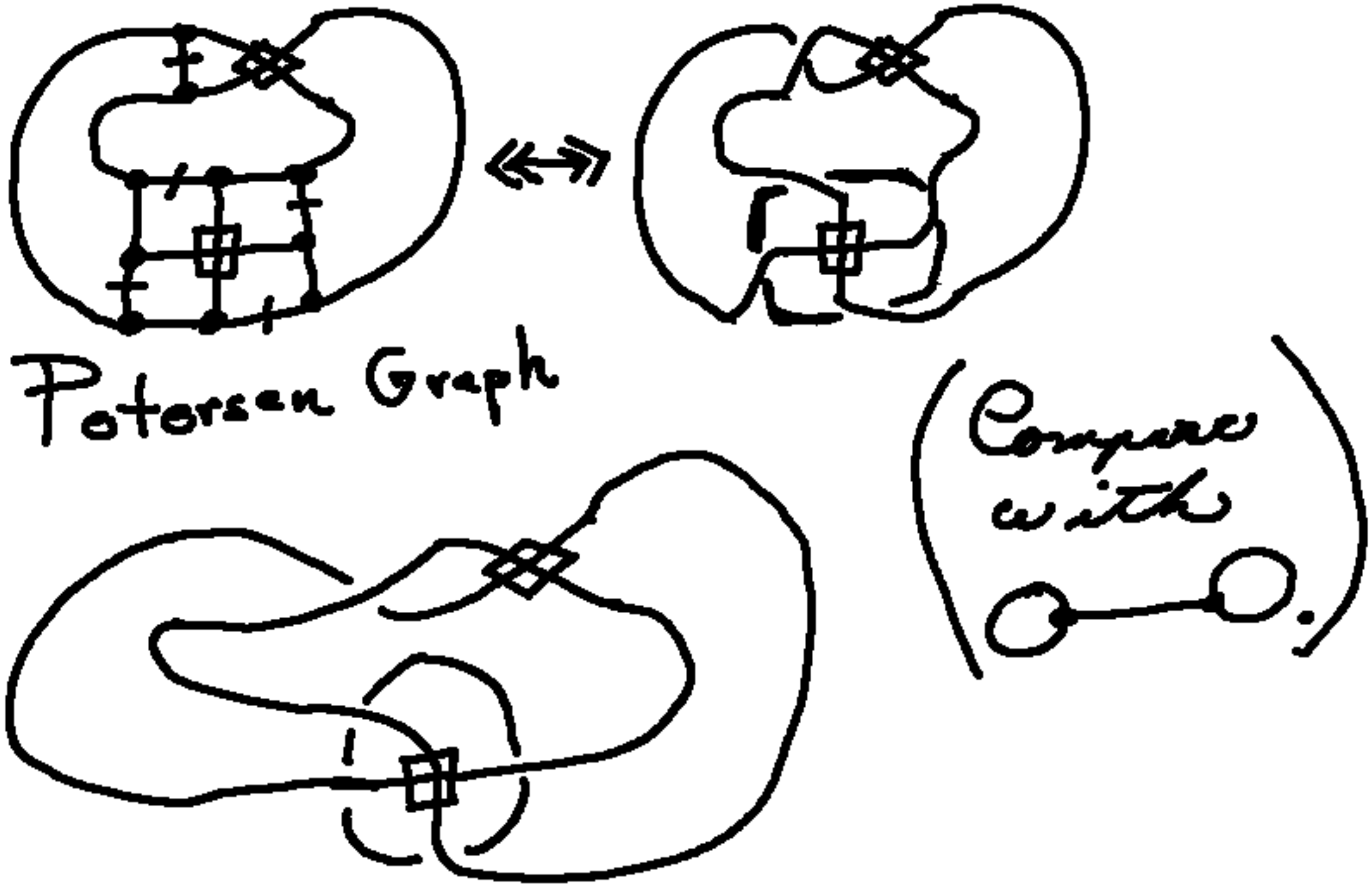




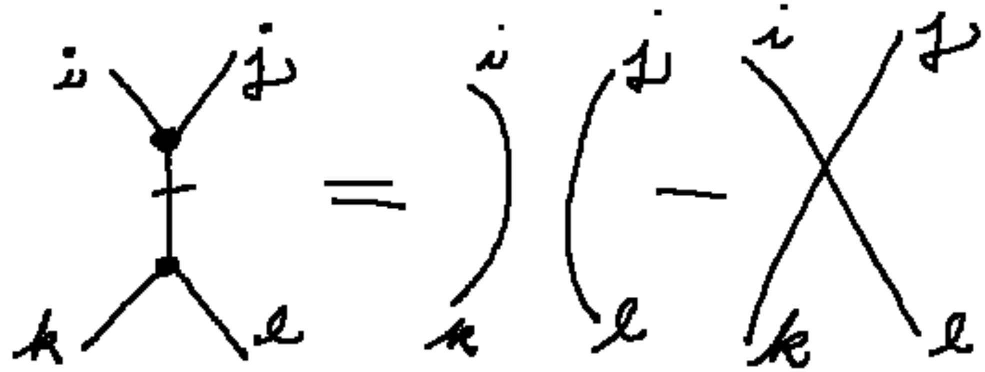
4CT says you can color planar diagrams with 3 colors.

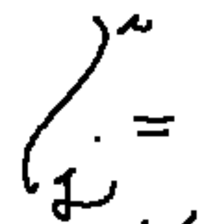


e.g.



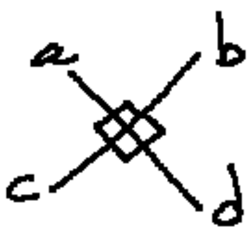
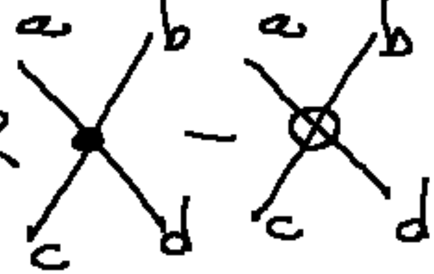
Tensors



 $= \delta^i_j$

Kronecker Delta

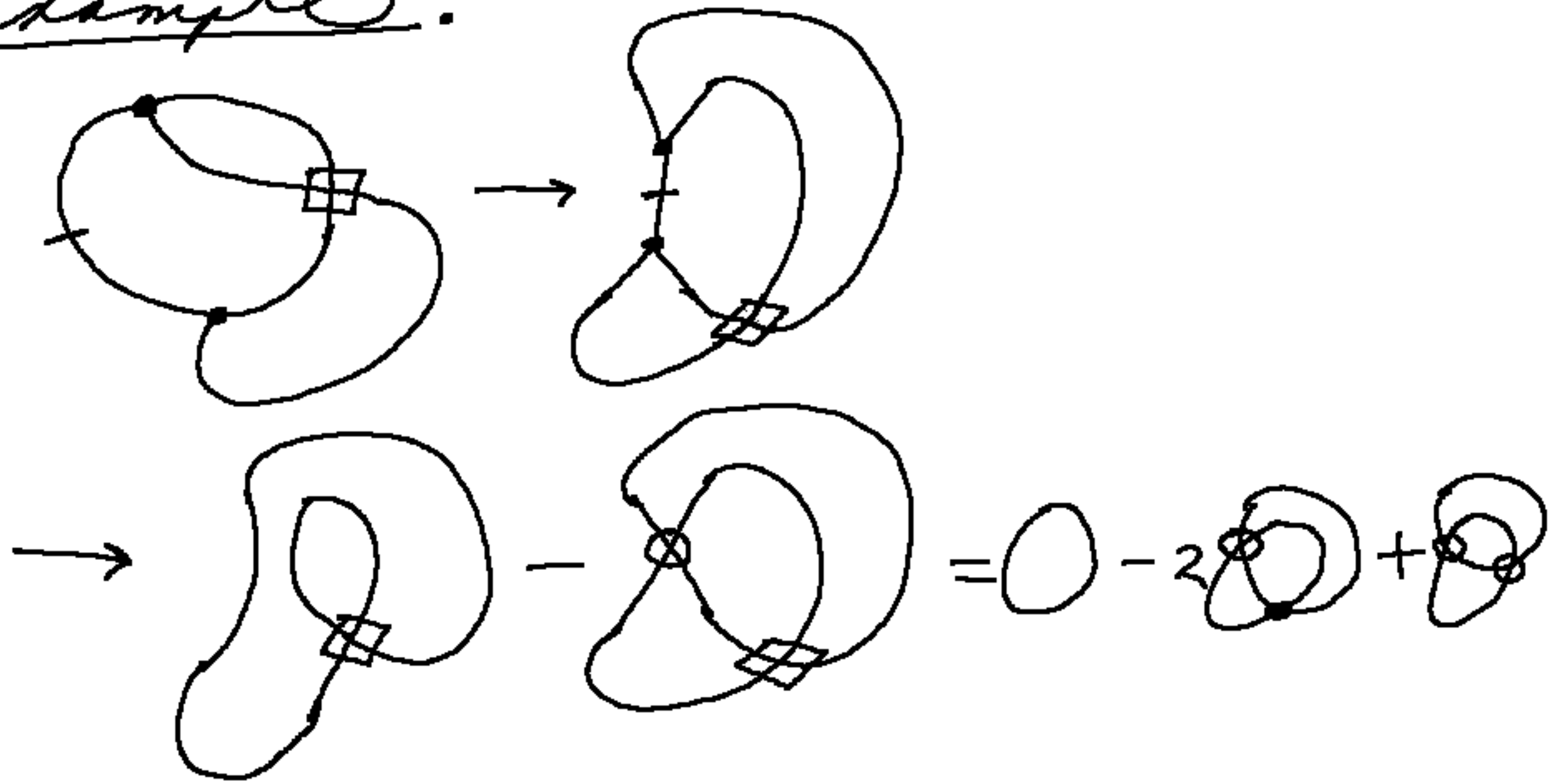
 $= n =$ trace of $n \times n$ identity matrix

 $= \begin{cases} 1 : a=d=b=c \\ -1 : a=d\neq b=c \\ 0 : \text{else} \end{cases} = 2$ 

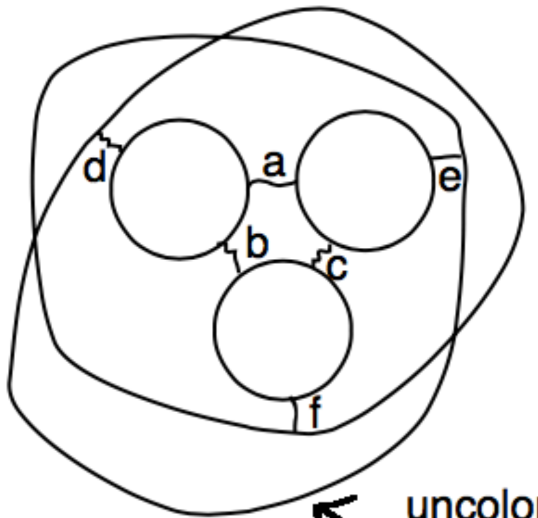
$\Rightarrow$ 

The same arguments as before show that 1) the $\neq 0$ tensor states are all the colorings. & 2) each contributes $+1$. //

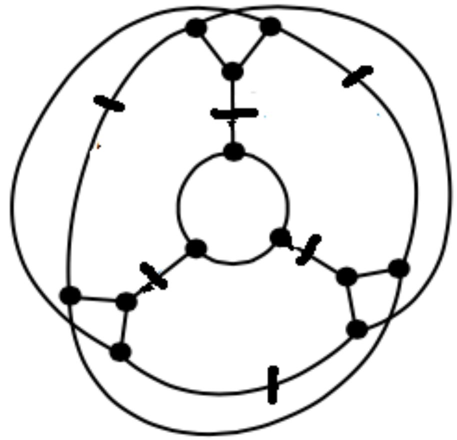
Example.



$$= n - 2n + n^2 = n(n-1).$$



uncolorable

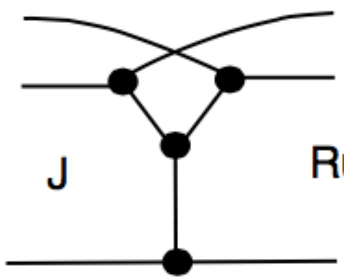


J_3

J_3 is not colorable in 3 colors.

But J_3 can be colored with 4 colors.

(give outer loop a 4th color)



J

Rufus Isaac's J Construction.

We now examine polynomials for snarks. Here $P(J_3, n)$

$$= n(-6 + 11n - 6n^2 + n^3)$$

$$= \left.\begin{array}{l} \phi,\ n=3 \\ 24,\ n=4 \end{array}\right\}$$

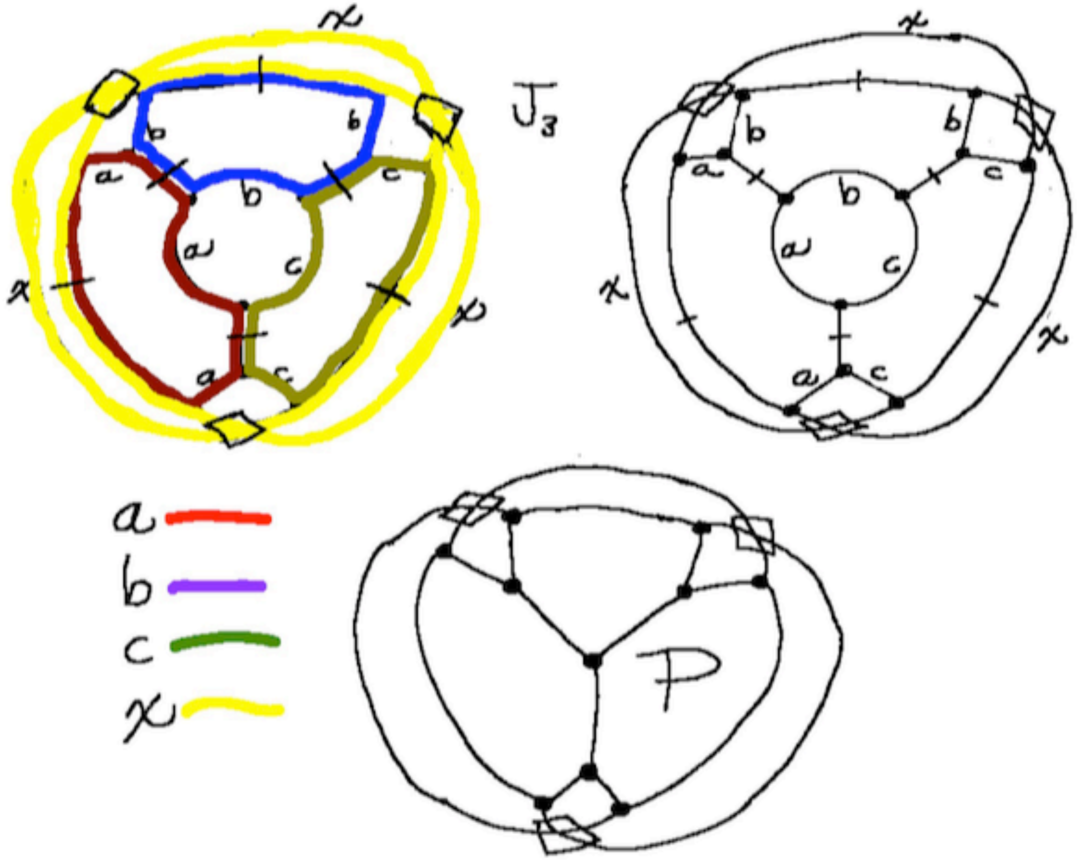
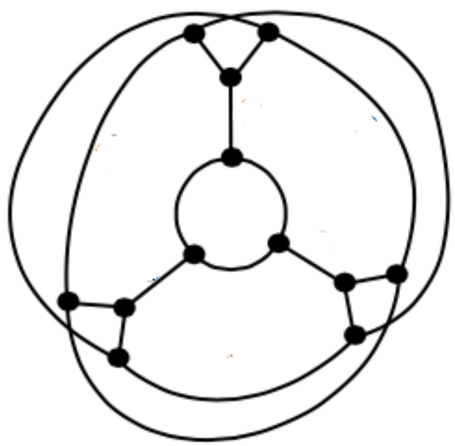
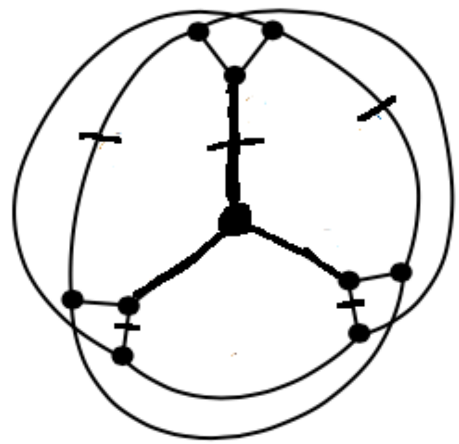


Figure 20: **Isaacs J_3 can be PM-colored with four colors (but not with three colors).**

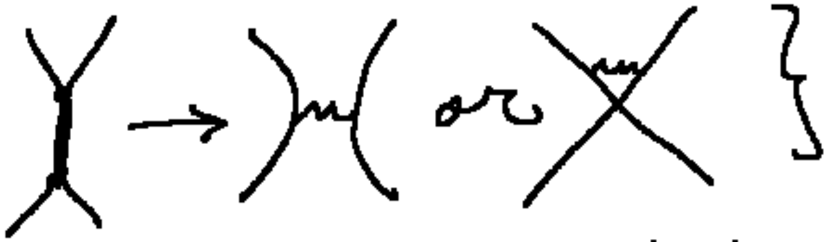


J_3



P

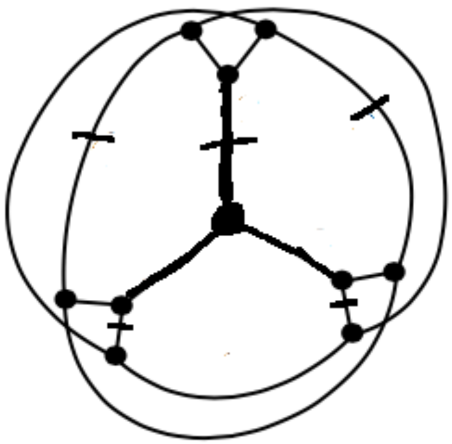
J_3 contracts to the minimal uncolorable Petersen Graph.



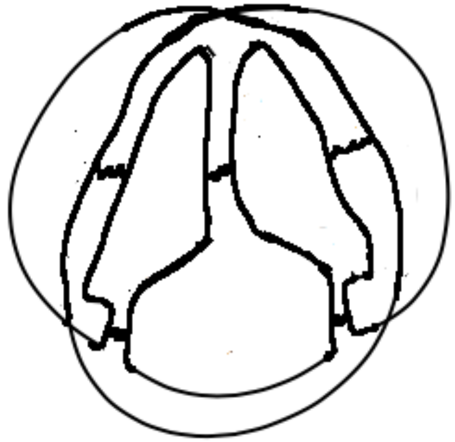
general coloring possibility using n colors.

<u>Fact</u>: P cannot be colored with n colors for any n. Call P <u>strongly uncolorable</u>.

<u>Conjecture</u>. If G trivalent is strongly uncolorable, then $G \supset P$ as a <u>substructure</u>.



P

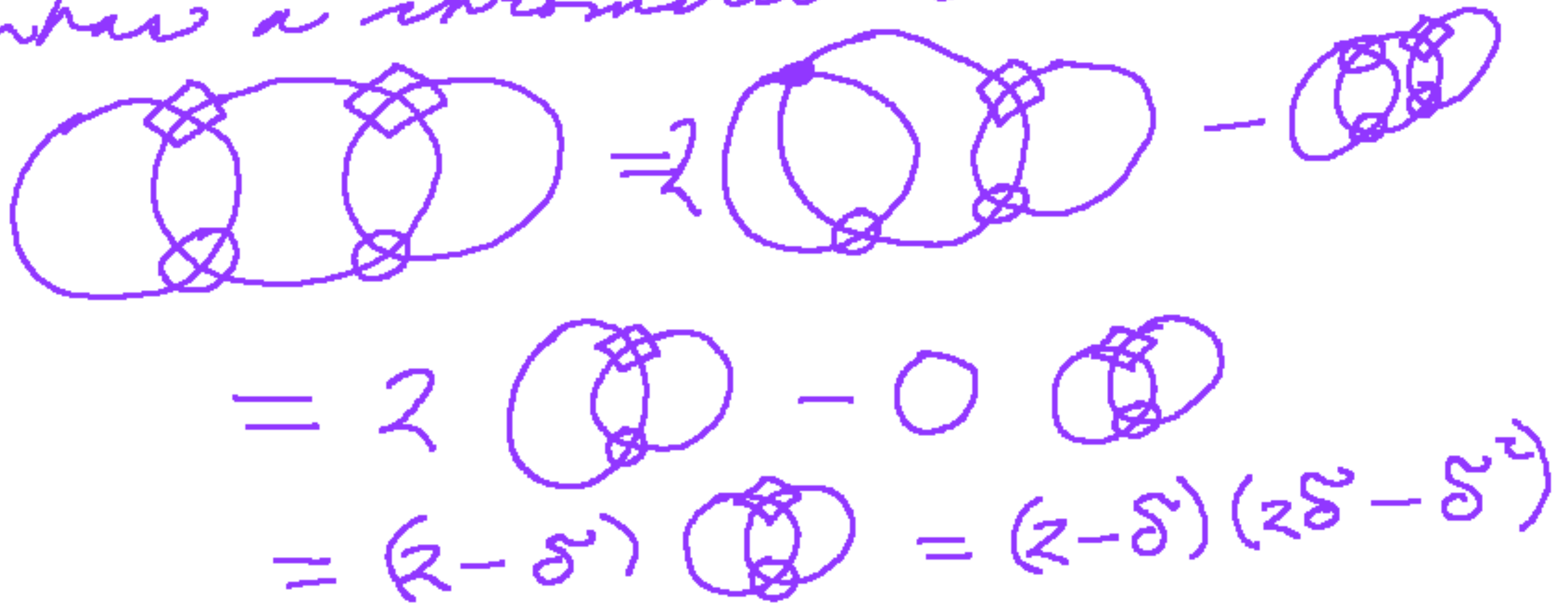


not colorable

multiple component states
but still uncolorable

Any form of loops with the two types of virtual crossing then has a chromatic evaluation.

e.g.

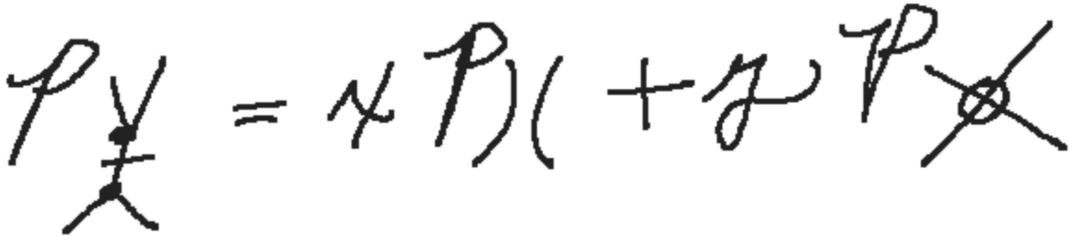


We can define this chromatic evaluation via nodal colors

by

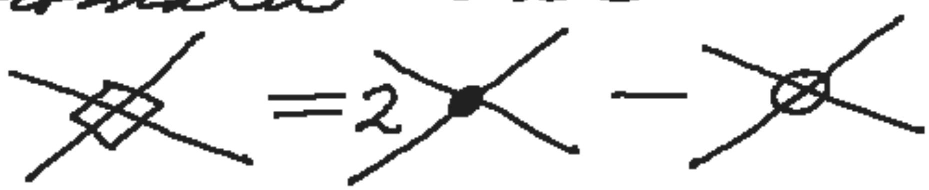
and so it is a contraction/deletion algorithm.

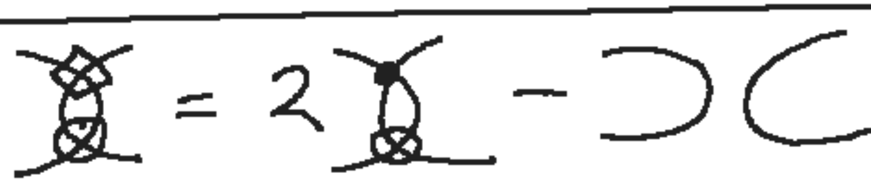
Now we have a generalized Penrose perfect matching polynomial.



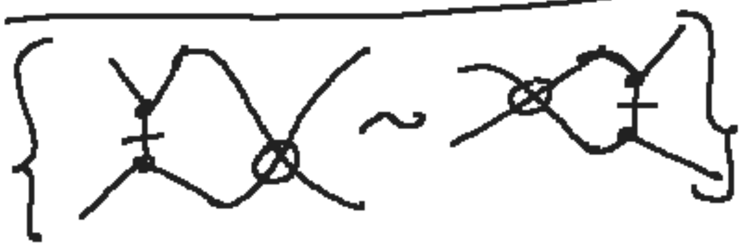
$$P_{\bigcirc} = \delta$$

In context of double virtual chromatic evaluation

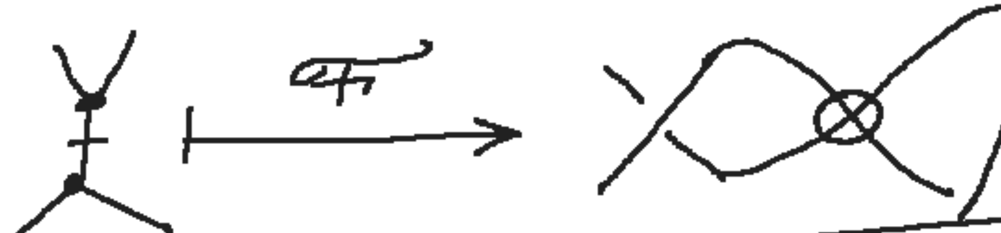


Note: 

$$\left\{\begin{array}{l}\text{Trivalent}\\ \text{Double Virtual}\\ \text{Graphs}\\ \text{with PM}\end{array}\right\} \xrightarrow{F} \left\{\begin{array}{l}\text{Double}\\ \text{Virtual}\\ \text{Knots \& Links}\end{array}\right\}$$

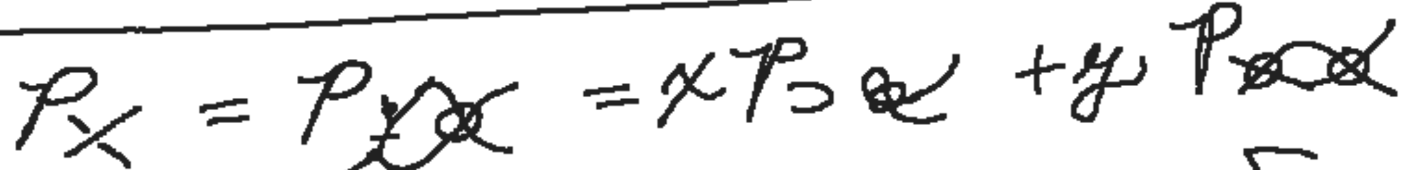


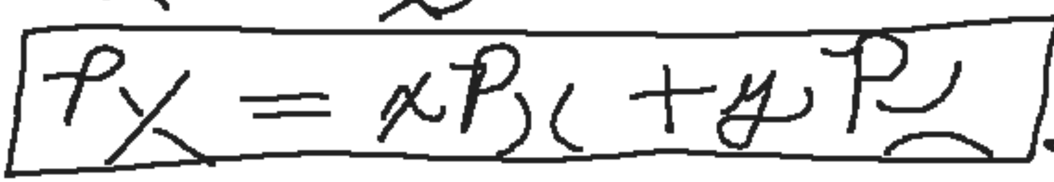
$\langle\langle \longleftrightarrow \rangle\rangle$ can be removed by adding orientations



F sends P to an extended bracket.

Then





$P_{O} = \delta$

context

Virtual Knot Theory

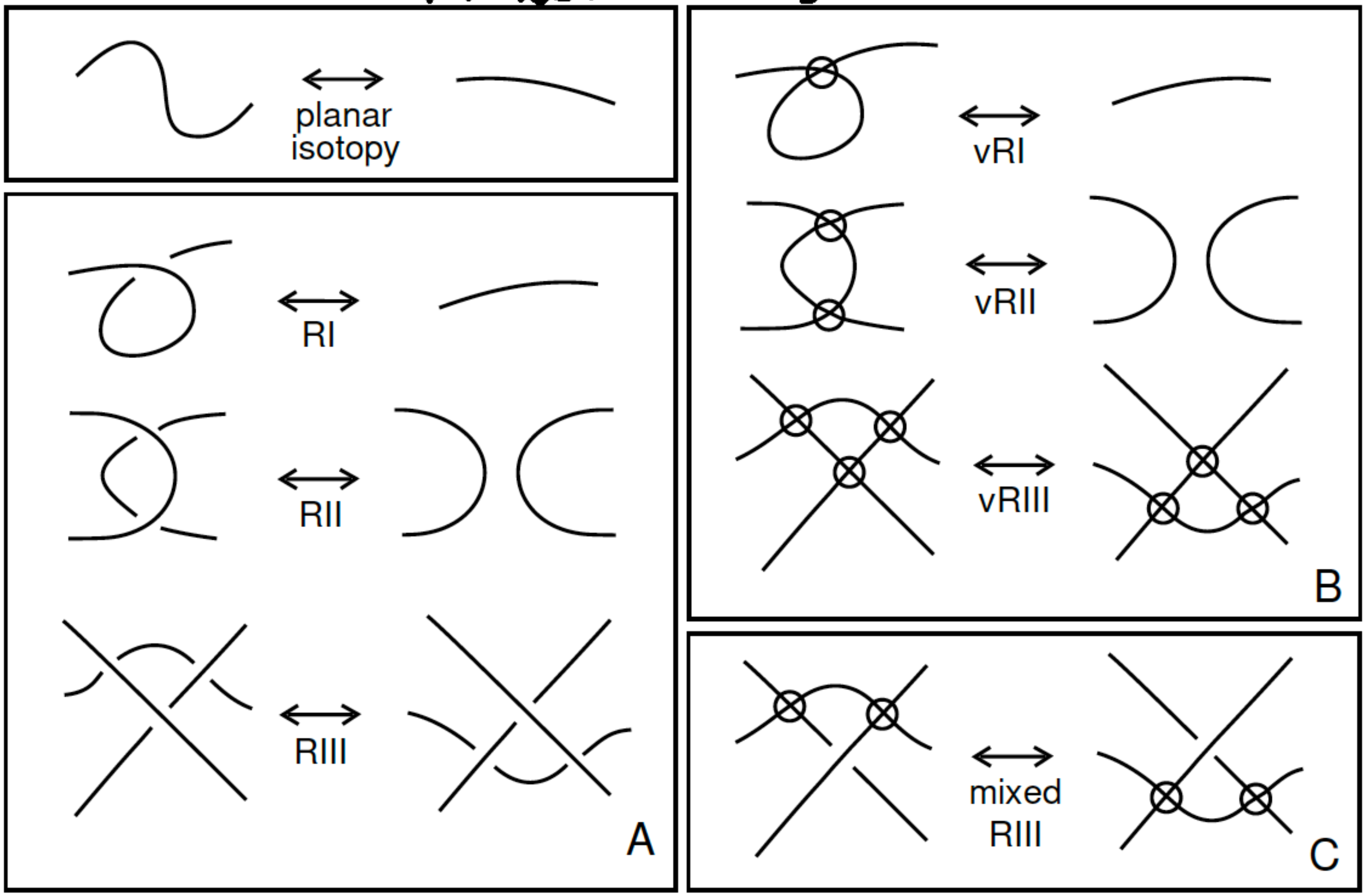


Figure 27: **Moves**

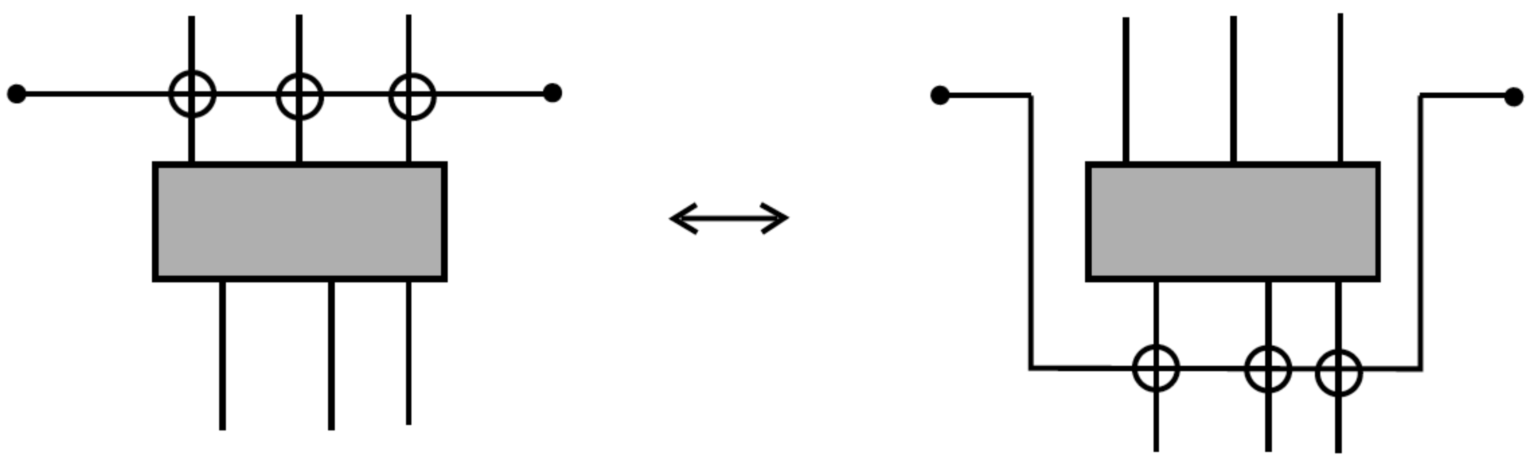


Figure 28: **Detour Move**

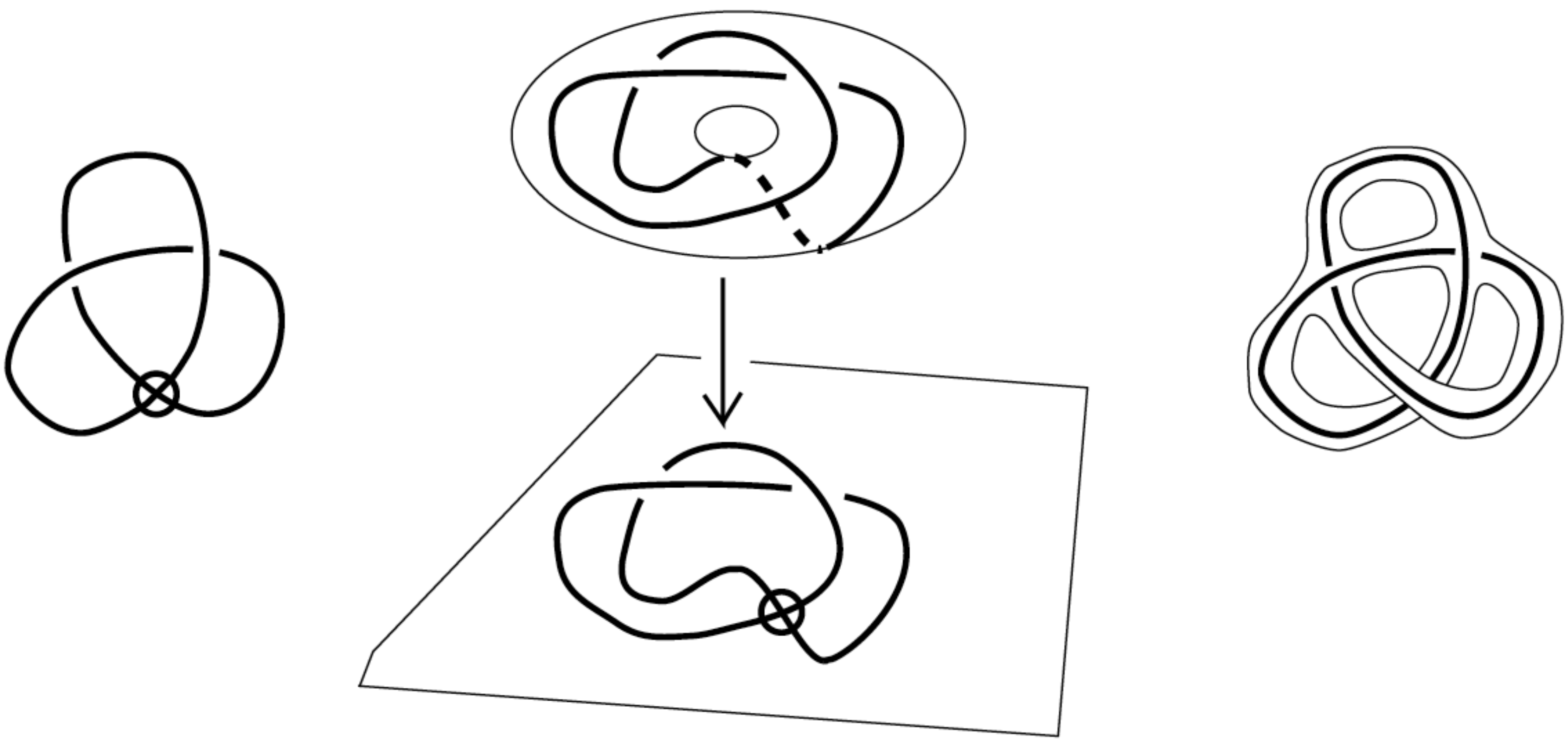


Figure 30: **Surfaces and Virtuals**

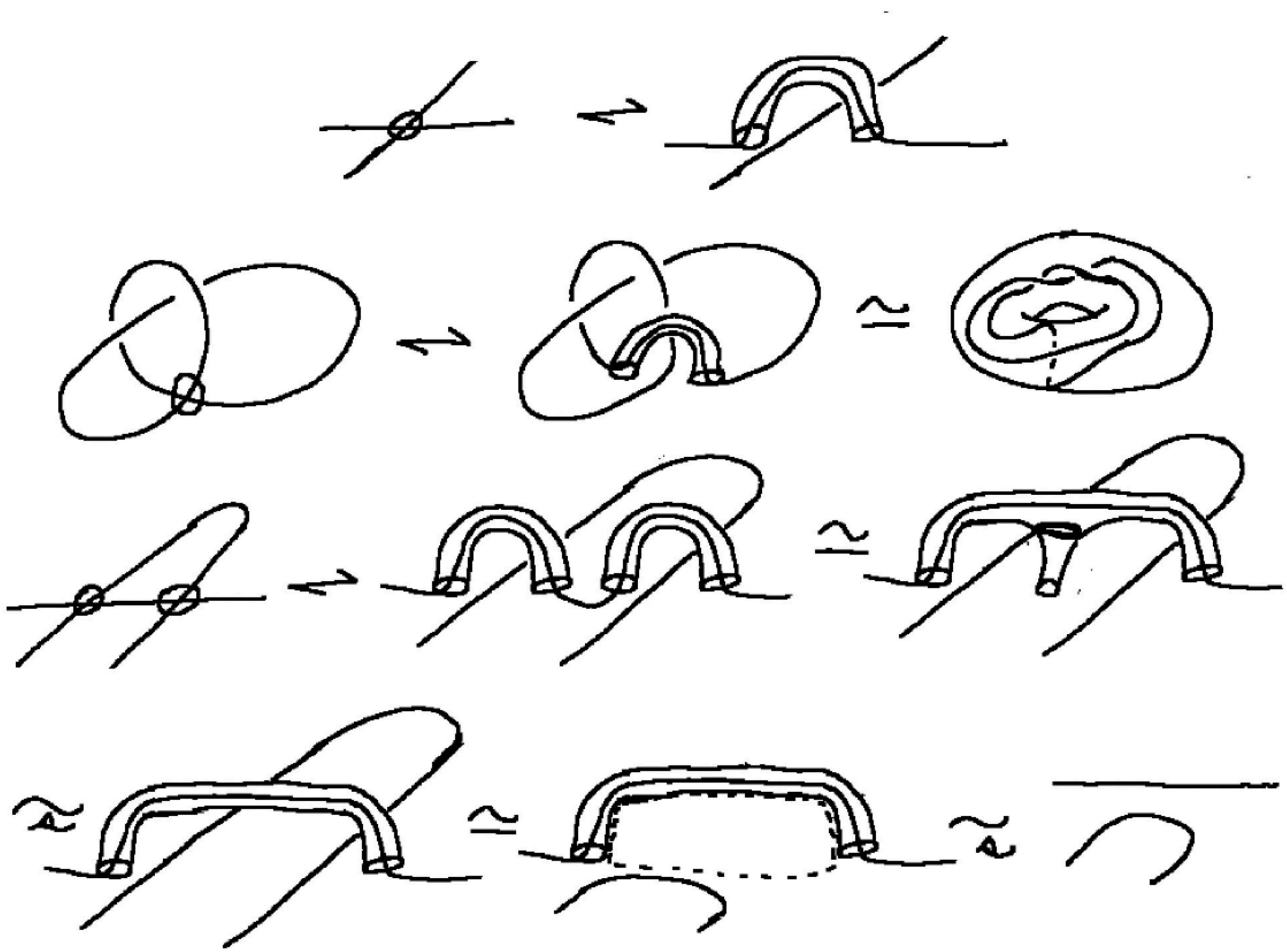
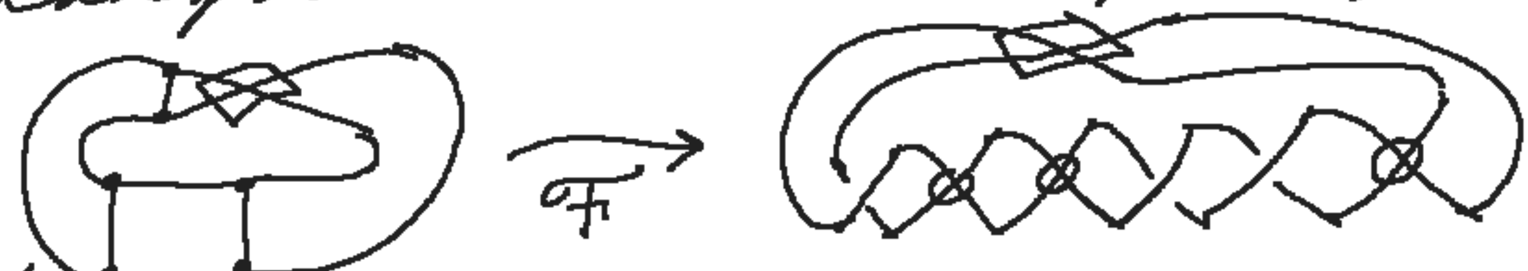
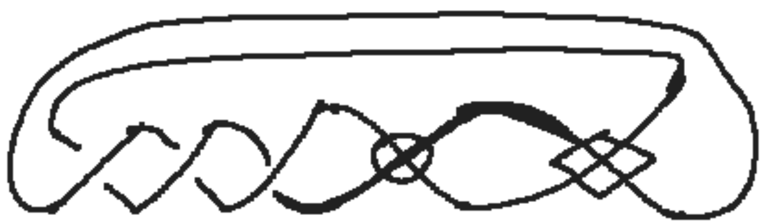


Figure 31: **Replacing Virtual Crossings by Handle Detours**

It is of interest to go back and forth. For example,

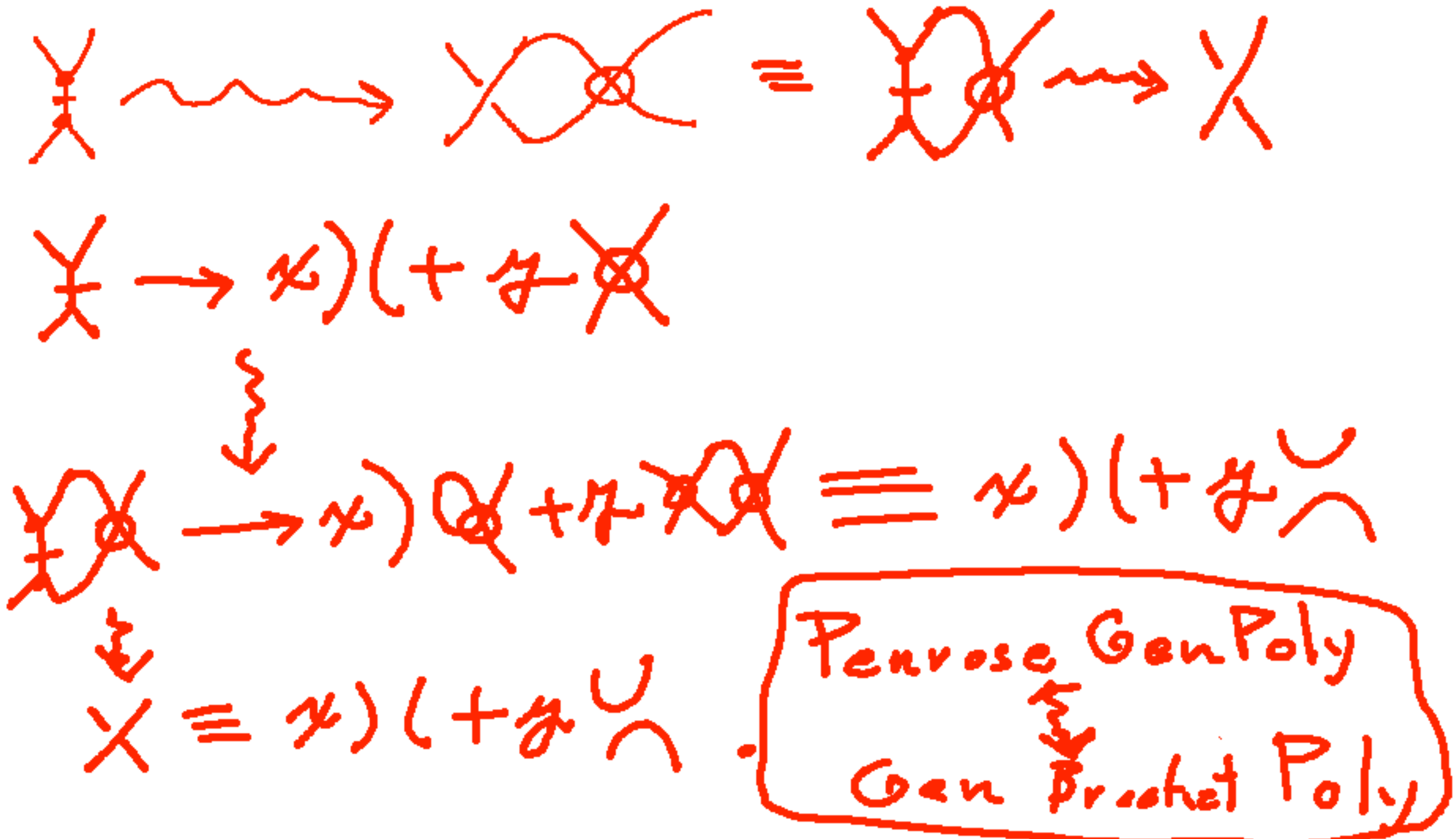


K_{33}

$=$ 

and this is an example of a virtual knot whose topological type is influenced by the doubling.

Transition to Virtual Knot Theory



Thus we will have

<u>multi-virtual knot theory</u>

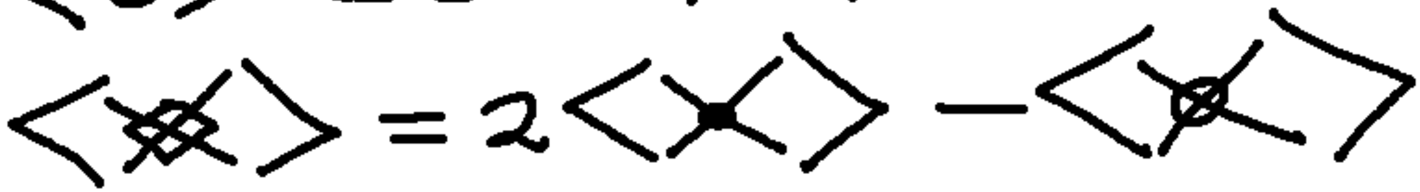
with $\otimes$, $\diamond$ (and $\otimes_\alpha$, $\alpha \in \text{SomeSet}$)

- each virtual crossing detours over all other virtual crossings (and over classical crossings).

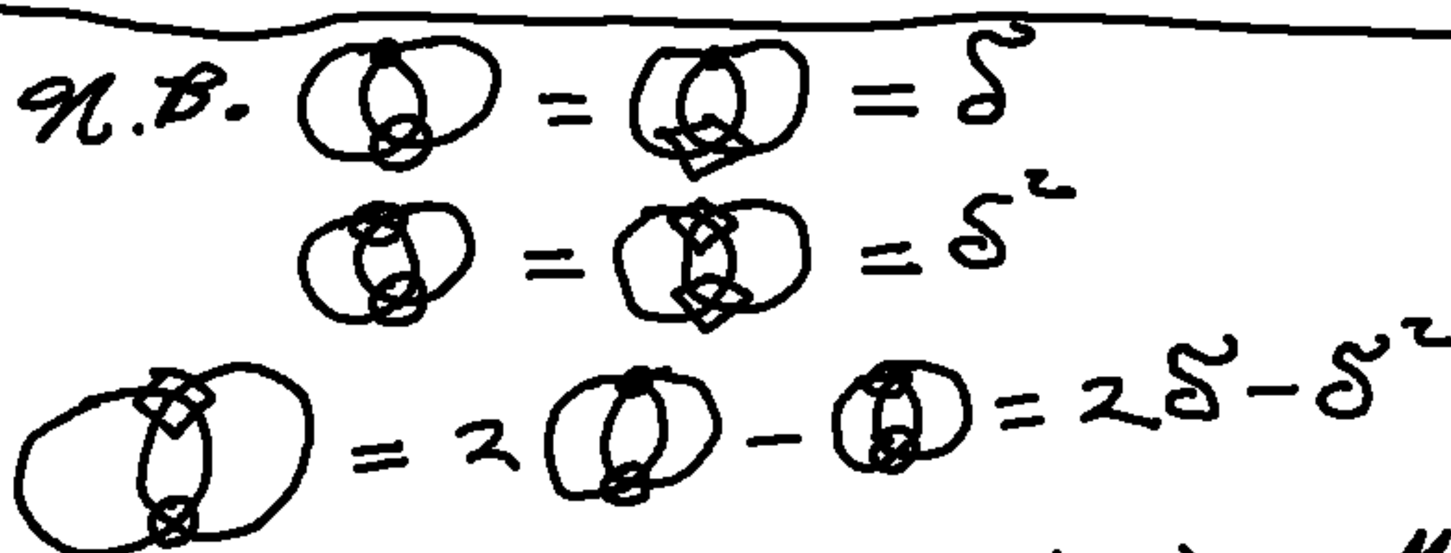
-  this does <u>not</u> reduce.

Generalized MV Bracket

$$\langle O \rangle = \delta = -A^{2} - A^{-2}$$



N.B.



<u>Thm.</u> This gives an MV invariant.

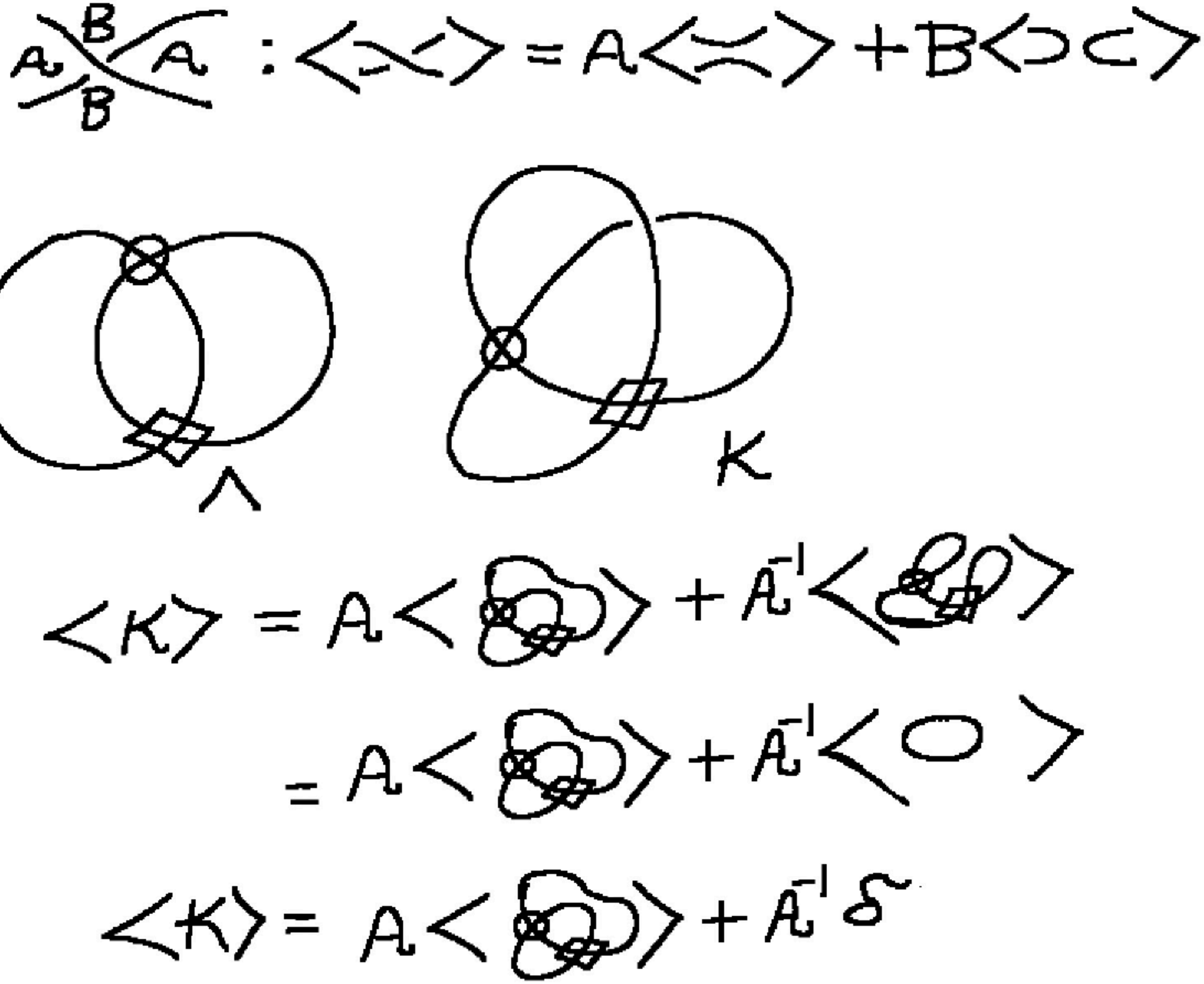


Figure 33: **Double Virtual Link and Double Virtual Knot**

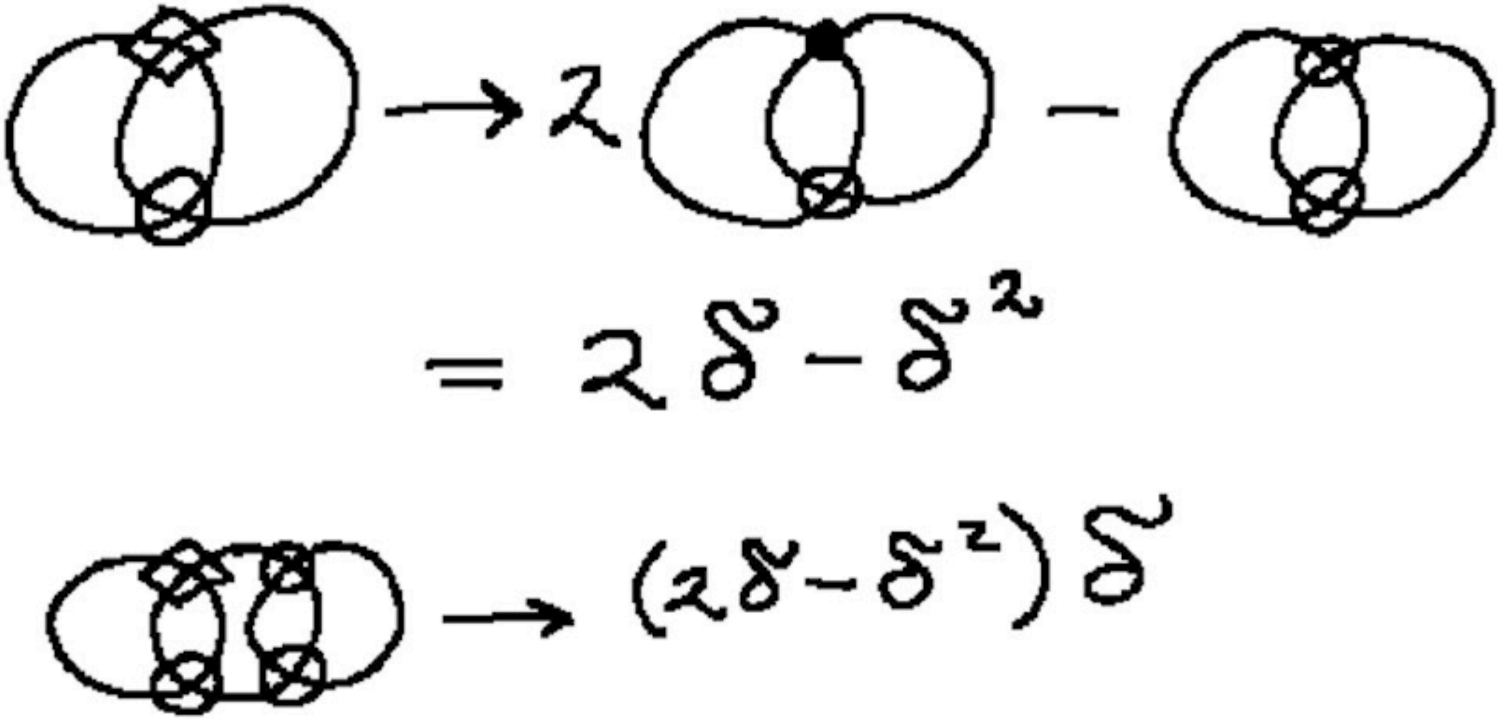


Figure 40: **Loop Evaluations**

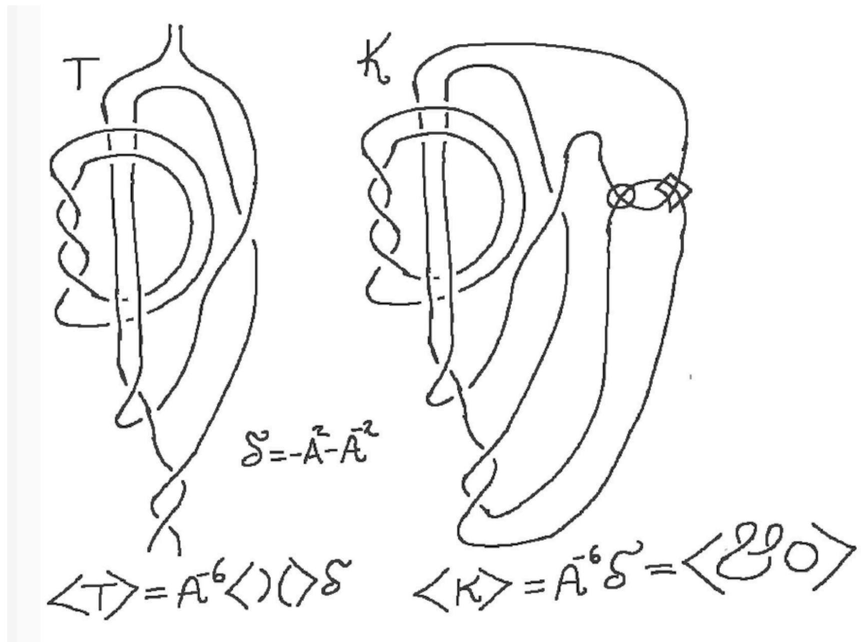
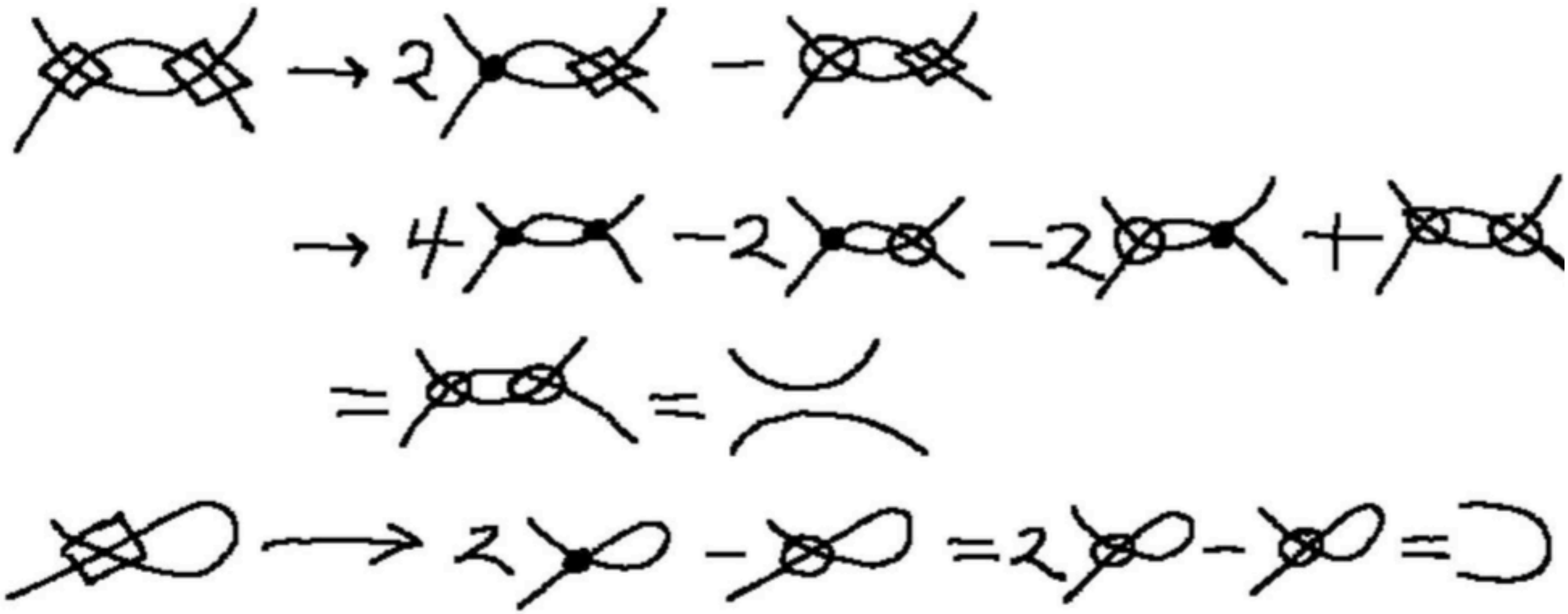
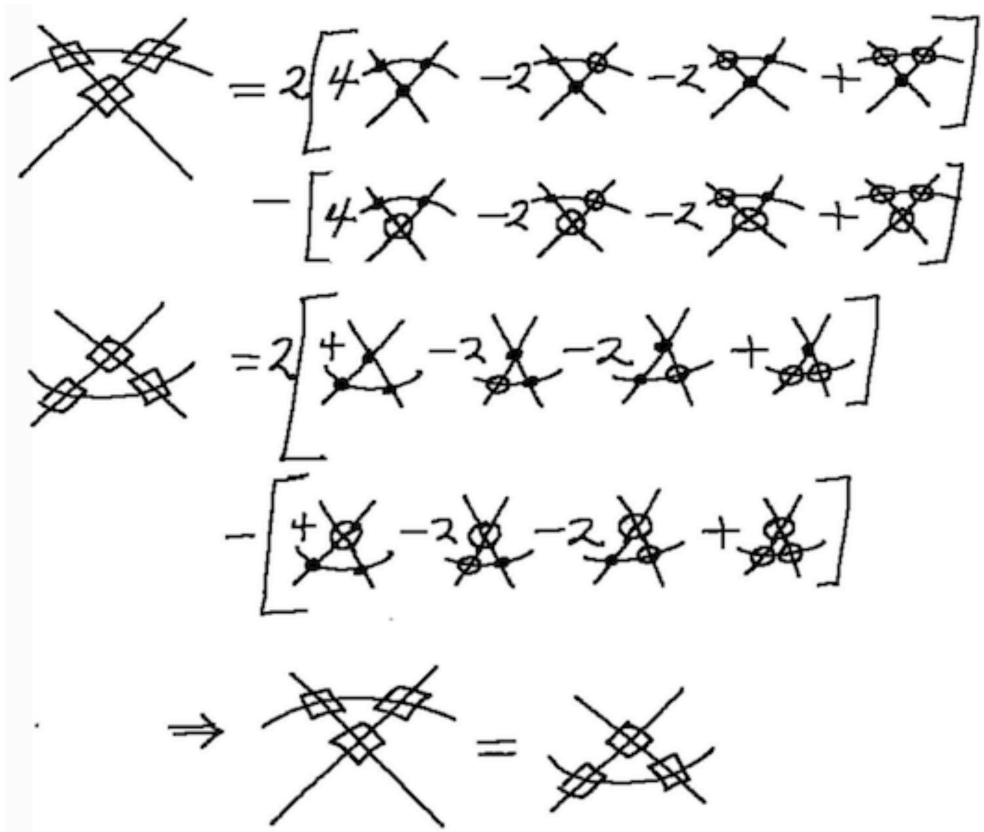


Figure 49: **Example of Non-Trivial Double Virtual Knot whose Virtuality is Invisible to Generalized Bracket**





 $= A$  $+ A^{-1}$ 

$= A$  $+ A^{-1}\delta$

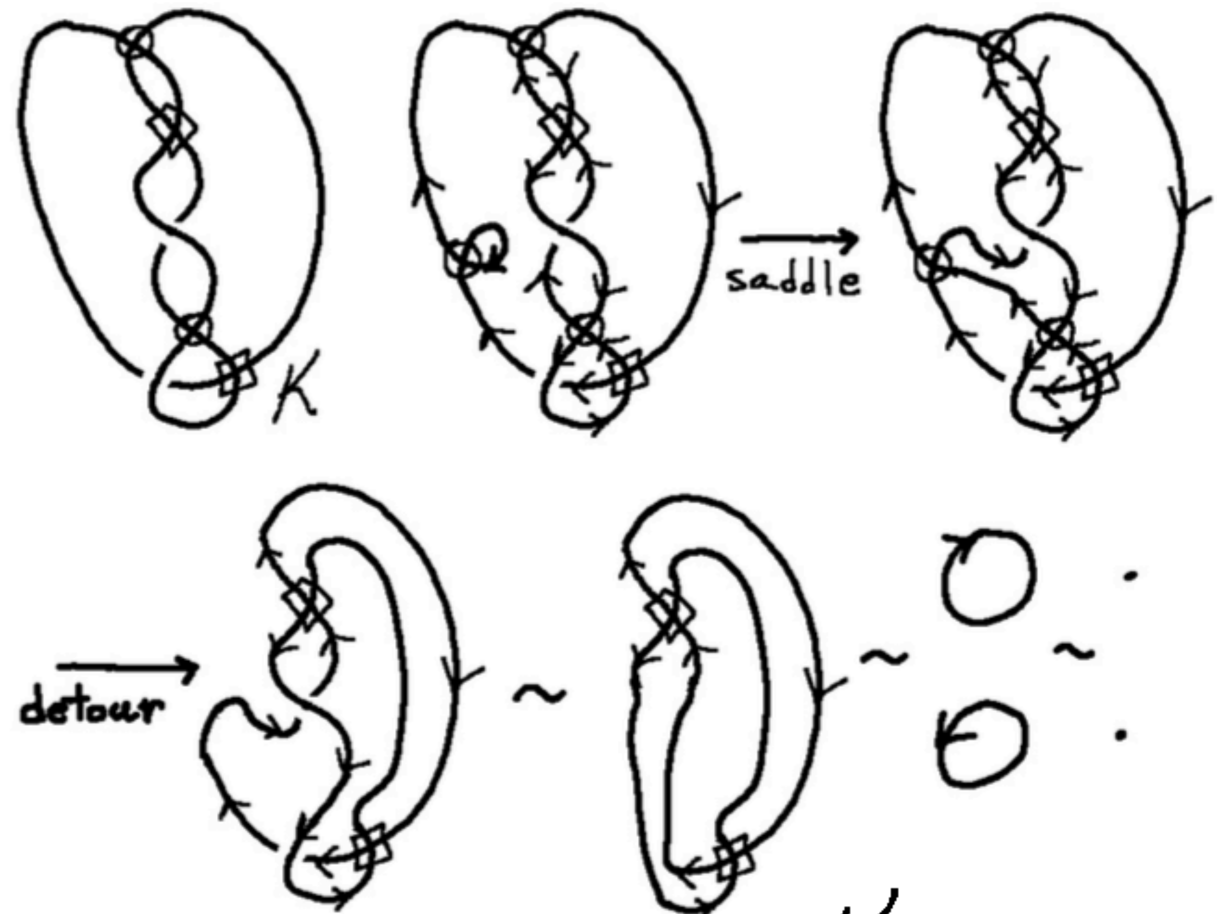
and in evaluating this generalized bracket, we take  $= 2\delta - \delta^2$.

One can leave virtual graphs in an evaluation.

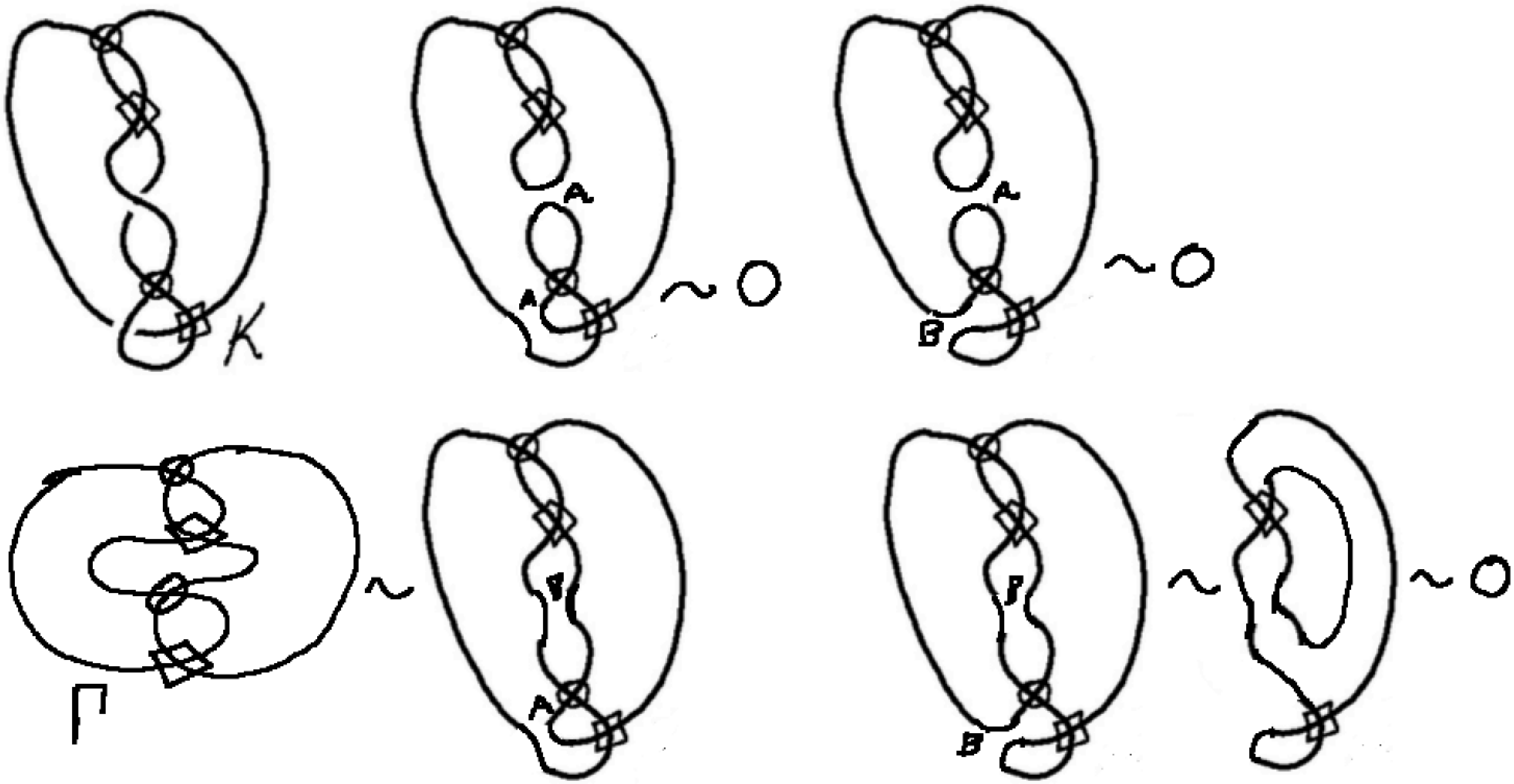
e.g.  is non-trivial but not detected by this method.

Here is an example. K is a slice knot in MV category.

(max, min, saddles + MV isotopy)



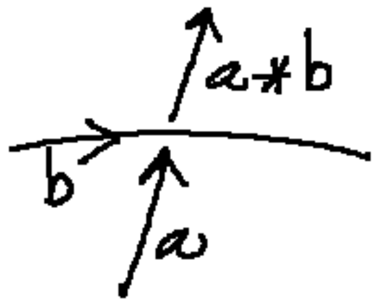
So we want to show that K is a non-trivial MV knot.



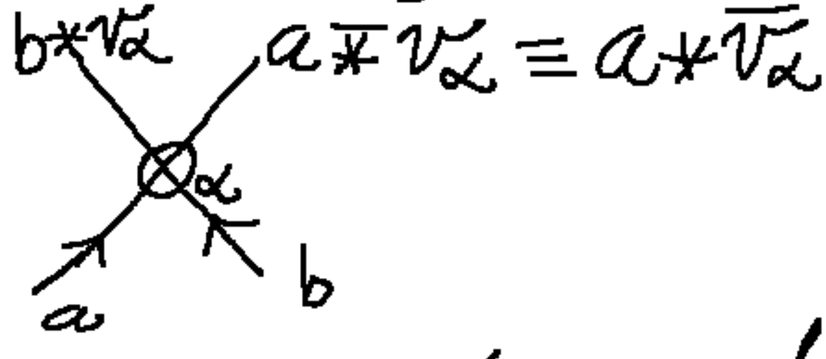
$$\Rightarrow \langle K \rangle = A^2\delta + 2\delta + \Gamma$$

The $\otimes = z \bullet - \otimes$ does not distinguish Γ from OO +
so does not distinguish K from O.

However, the quandle also has an MV generalization.



and



$\{v_\alpha\}$ free gens of quandle automorphisms

and s.t. $(a*v_\alpha)*v_\beta$

$= (a*v_\beta)*v_\alpha$

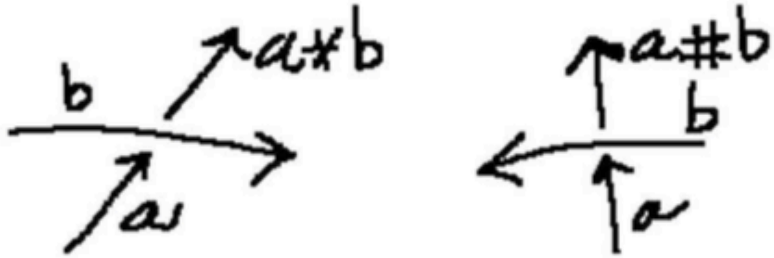
when $\alpha \neq \beta$.

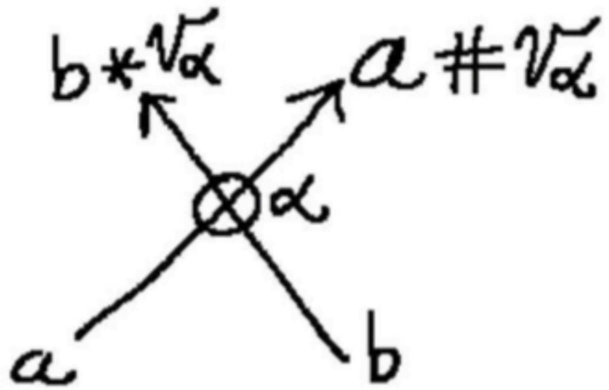
e.g.

$a*v_\alpha = v_\alpha a$

module elt in a Fox Alexander quandle.

Generalized Quandle



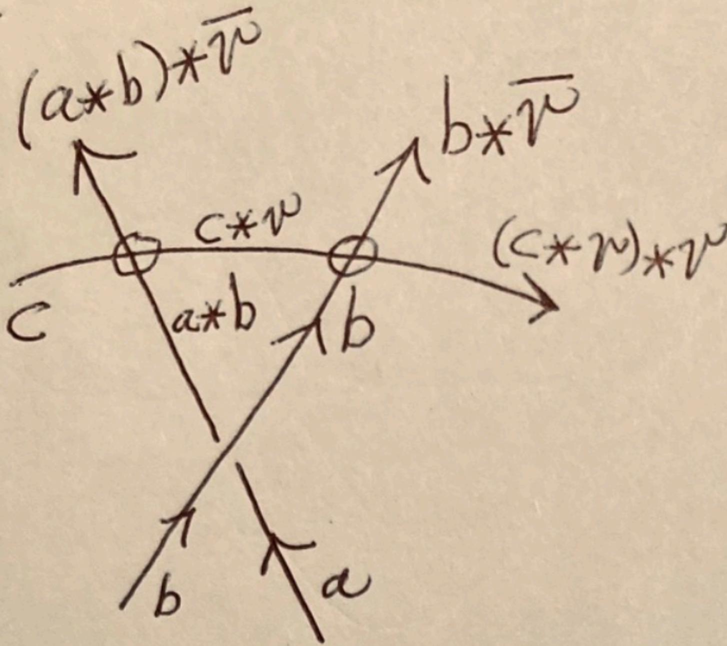


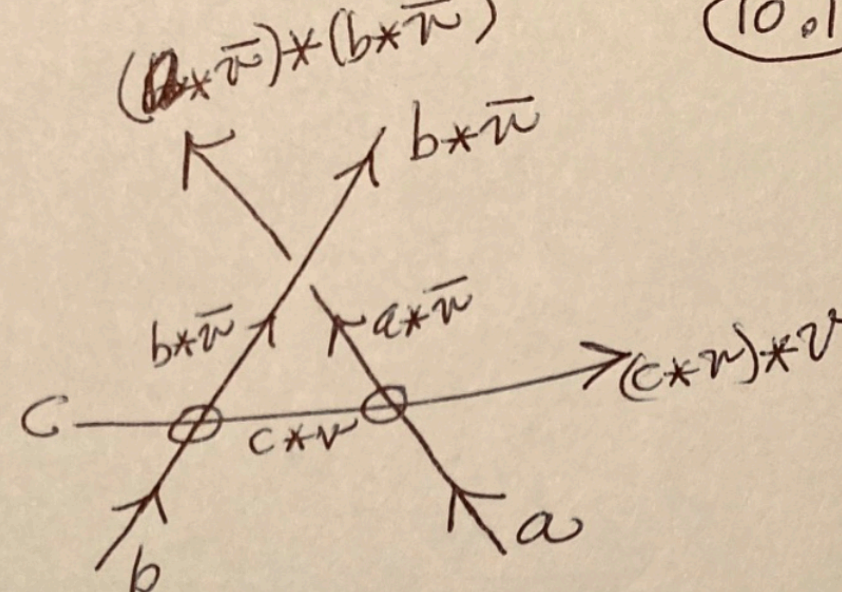
$$(x * V_\alpha) * V_\beta = (x * V_\beta) * V_\alpha$$

$$(x \# V_\alpha) \# V_\beta = (x \# V_\beta) \# V_\alpha$$

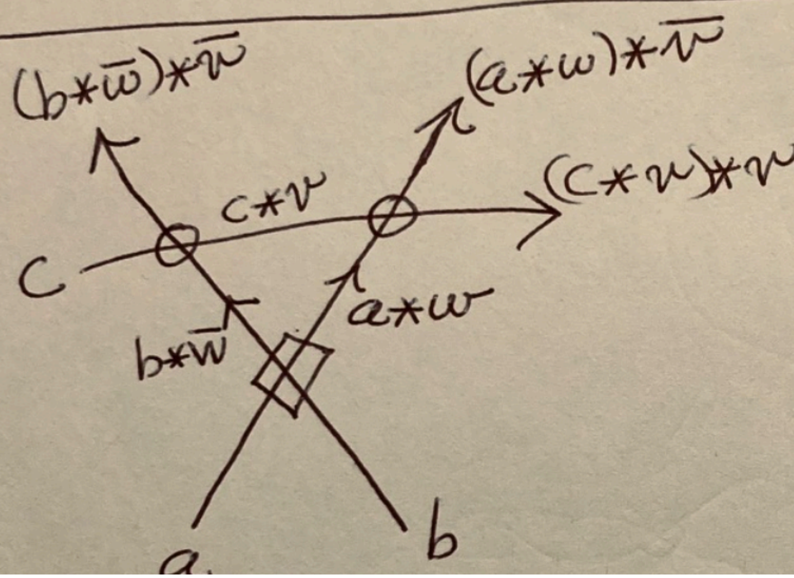
1. $a * a = a$, $a \# a = a$
2. $(a * b) \# b = a$, $(a \# b) * b = a$
3. $(a * b) * c = (a * c) * (b * c)$

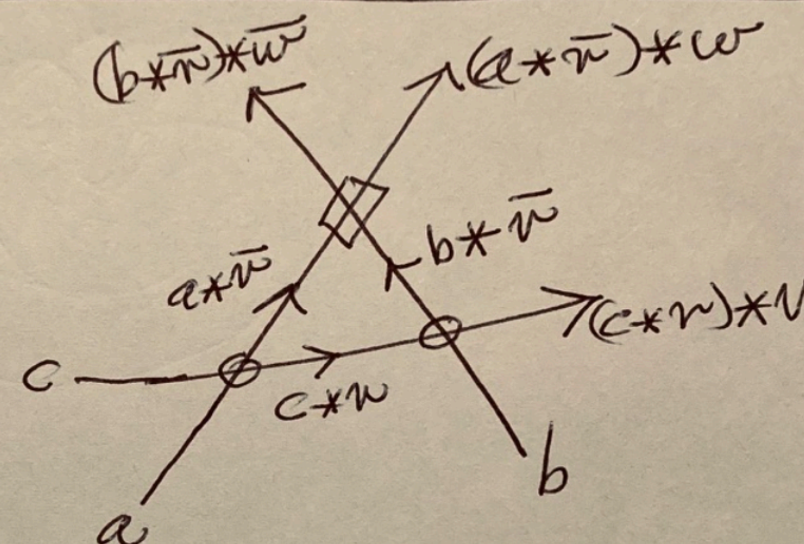
   $(a \# b) \# c = (a \# c) \# (b \# c)$



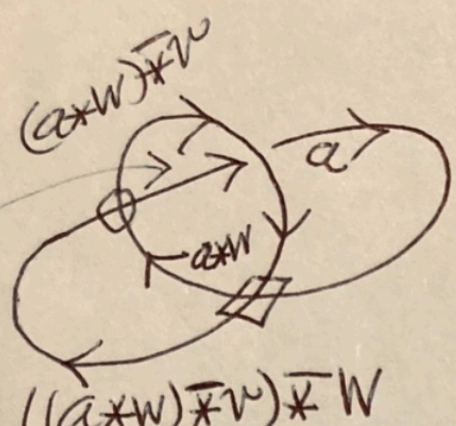


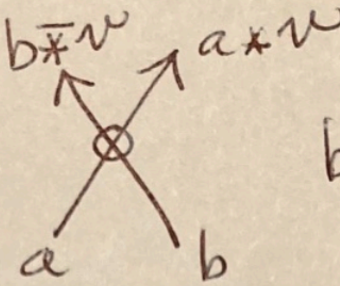
$$(a*b)*\overline{v} = (a*\overline{v})*(b*\overline{v})$$

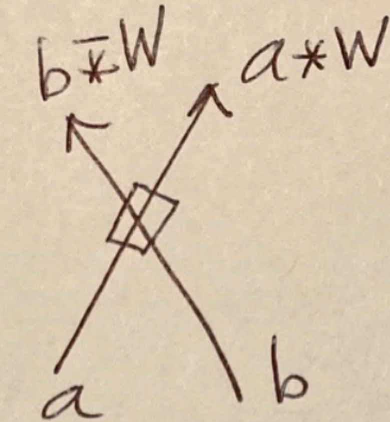




11

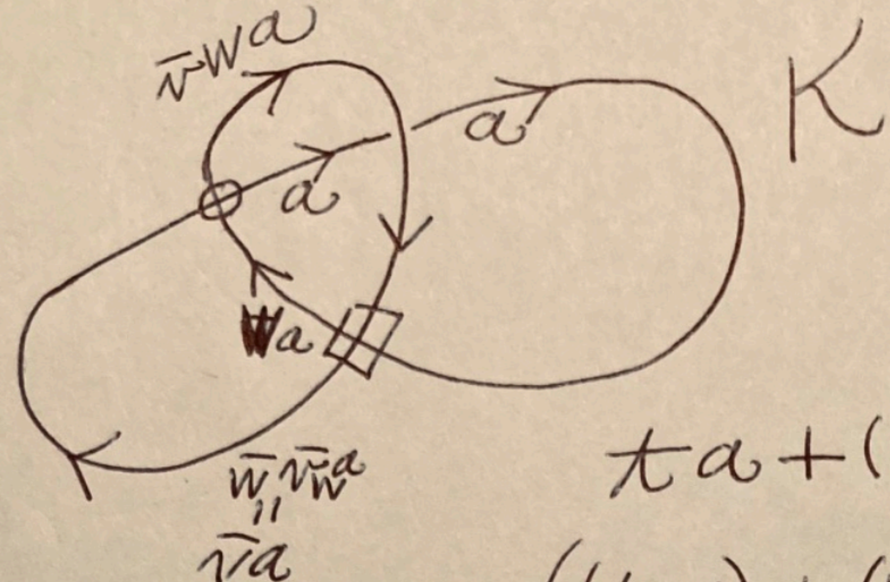






$$(((a*w)\bar{*}v)\bar{*}W)*v$$

$$a=\left[(((a*w)\bar{*}v)\bar{*}W)*v\right]*\left[(a*w)\bar{*}v\right]$$



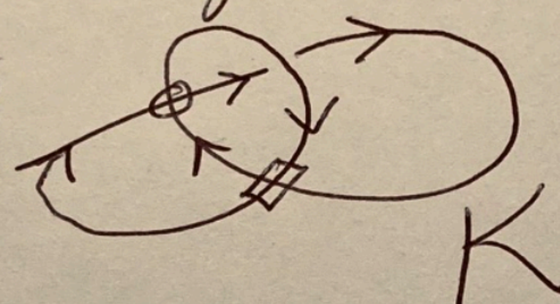
$$ta + (1-t)\bar{v}wa = a$$

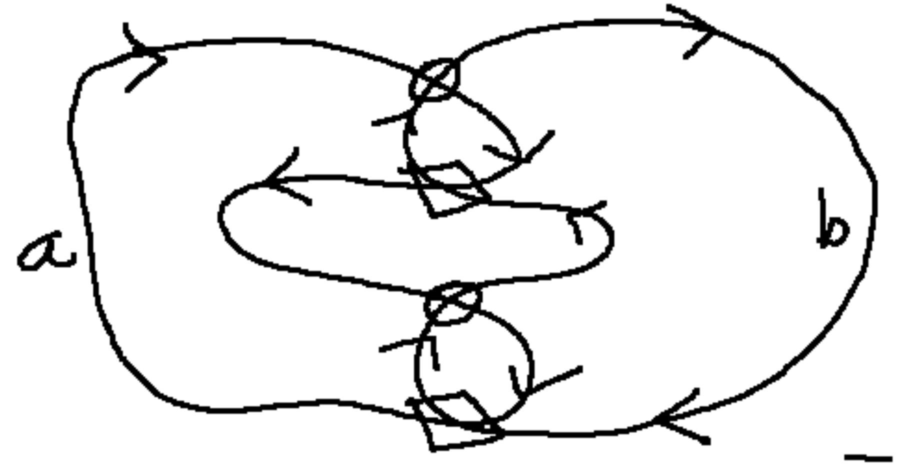
$$((t-1) + (1-t)\bar{v}w)a = 0$$

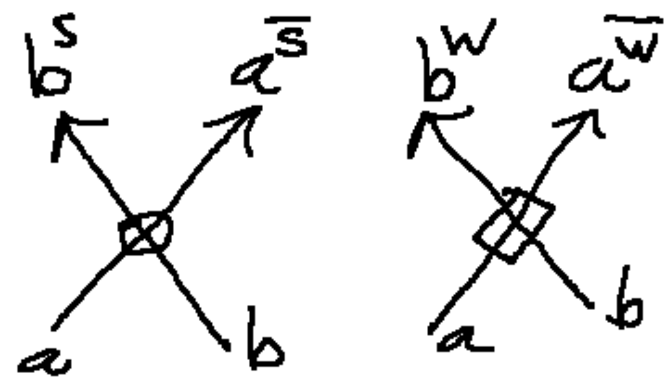
$$\underline{P(t,v,w) = (1-t)(1-\bar{v}w)}$$

Generalized Sawollek Poly

detects



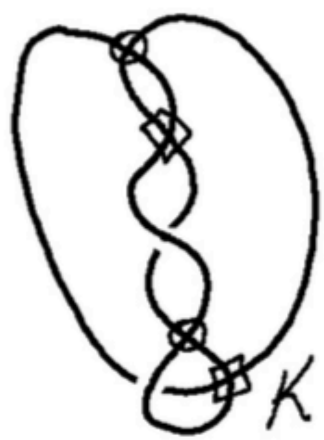




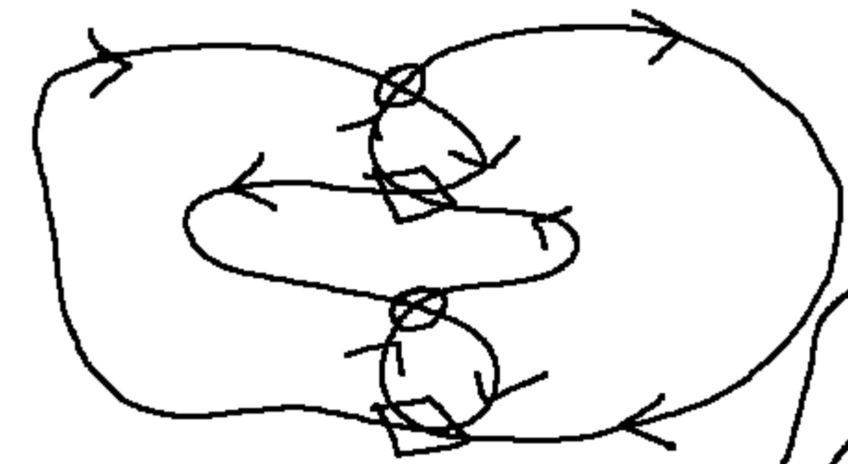
$a = a^{\bar{s} w \bar{s} \bar{w}}$ $\qquad$ $b = b^{\bar{w} s \bar{w} s}$

$\Rightarrow$ Π has non-trivial gen quandle

$\Rightarrow$



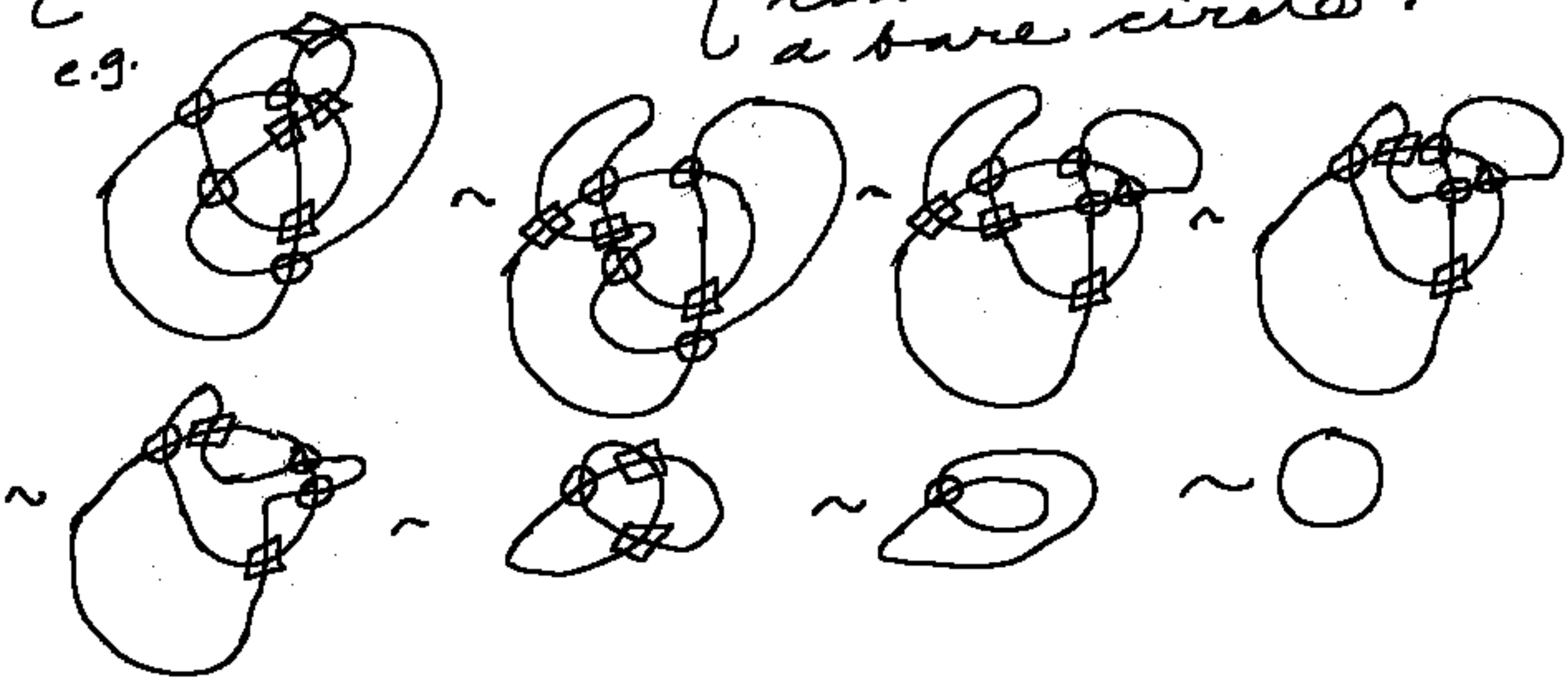
nontrivial
MV slice knot



We have shown that this multiflat is non-trivial.

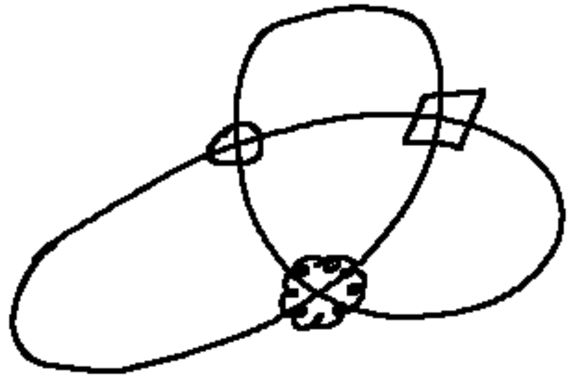
__Conjecture.__ All single component (⌗, ⊗) two virtual curves reduce to a bare circle.

e.g.



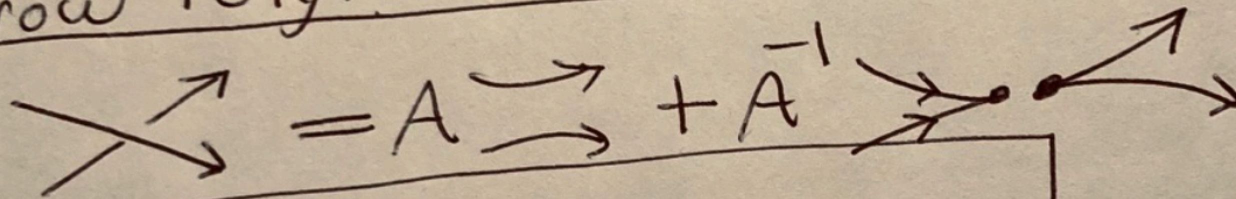
Conjecture. This is not trivial!

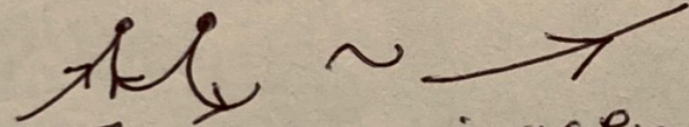
(in 3 MV theory)

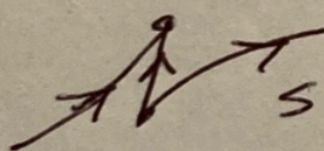


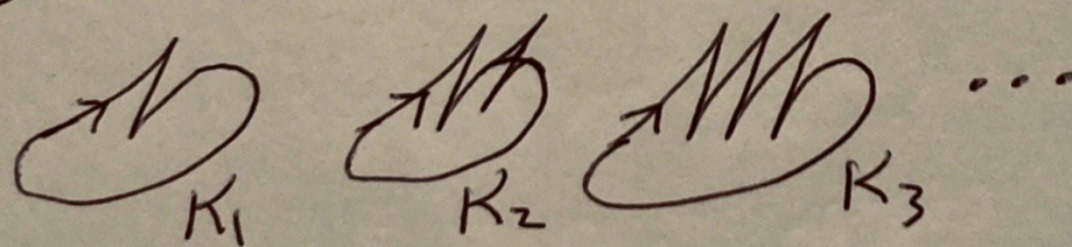
Well of course it is not trivial, but we need a proof. There is a big structure here, largely unexplored.

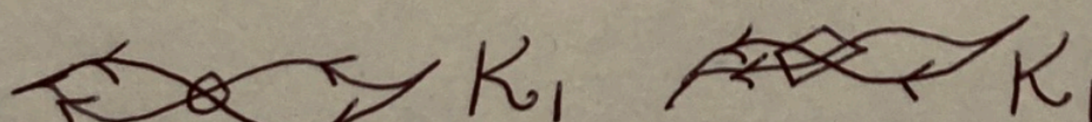
Arrow Polynomial Generalization



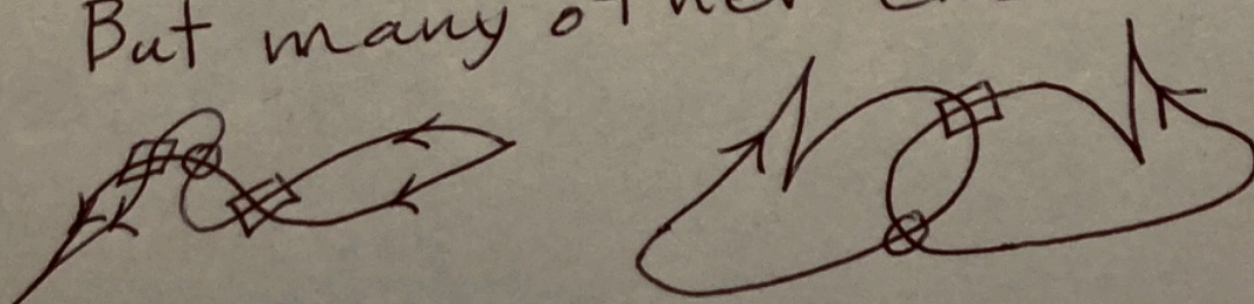
- Need  for invariance.

-  survives.

- 



But many other end states:



Now there is a big structure
of end-states such as

and there need some
graphical classification
to sort out the new
doubled virtual arrow
polynomial.

There are many questions
and this just the beginning
of this development.

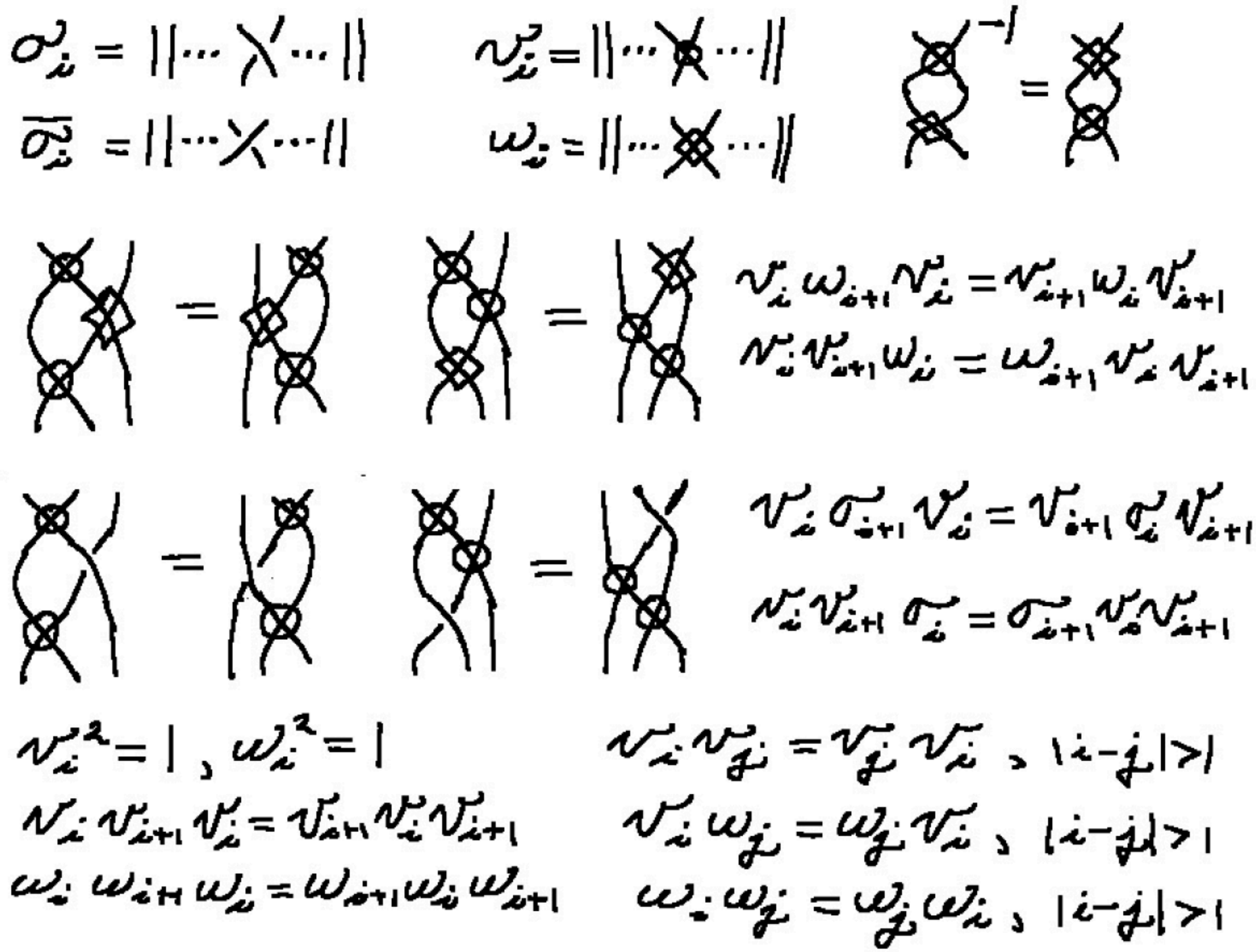


Figure 78: **Multiple Virtual Braid Group**

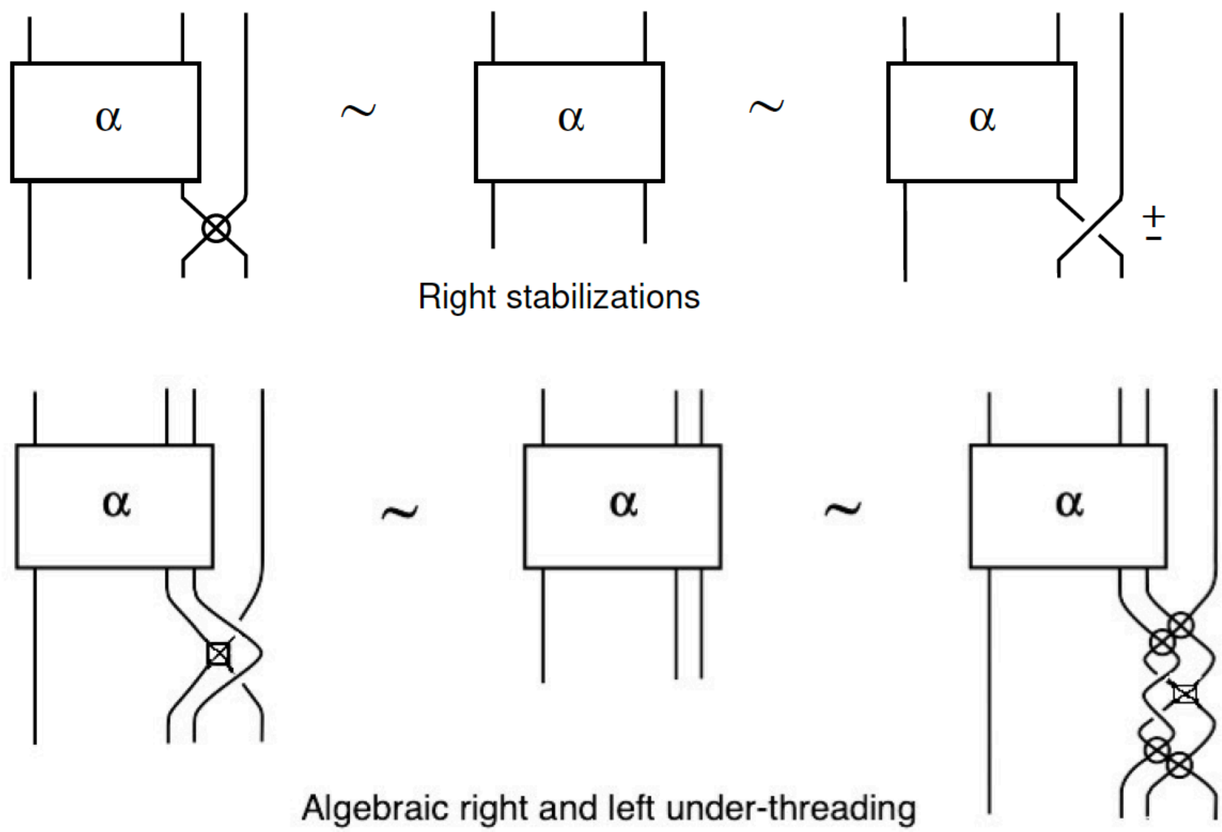


Figure 79: **The Moves (ii), (iii) and (iv) of the Algebraic Markov Theorem.**

Theorem. (Algebraic Markov Theorem for multi-virtuals). Two oriented multi-virtual links are isotopic if and only if any two corresponding virtual braids differ by a finite sequence of braid relations in MVB_∞ and the following moves or their inverses. In the statement below and in Figure 79, v_n stands for any given virtual crossing type.

(i) Virtual and real conjugation: $v_i \alpha v_i \sim \alpha \sim {\sigma_i}^{-1} \alpha \sigma_i$

(ii) Right virtual and real stabilization: $\alpha v_n \sim \alpha \sim \alpha \sigma_n^{\pm 1}$

(iii) Algebraic right under–threading: $\alpha \sim \alpha \sigma_n^{-1} v_{n-1} \sigma_n^{+1}$

(iv) Algebraic left under–threading: $\alpha \sim \alpha v_n v_{n-1} \sigma_{n-1}^{+1} (v_n)' \sigma_{n-1}^{-1} v_{n-1} v_n,$

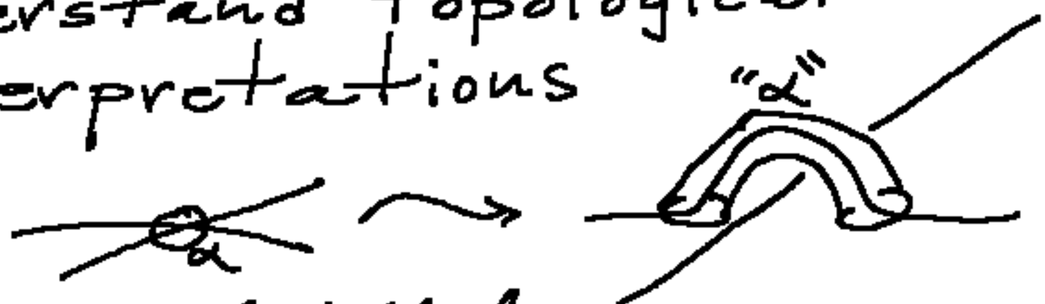
where $\alpha, v_i, \sigma_i \in VB_n$ and $v_n, \sigma_n \in VB_{n+1}$ (see Figure 79) and $(v_n)'$ denotes a possibly different virtual crossing type from v_n . Note that in Figure 79 this possible difference in virtual crossing type is indicated by a box at the crossing rather than a circle.

(This result will be in a paper in preparation by LK and S.Lambropoulou.)

Many Problems

- articulate invariants
- relations with graph theory
- understand topological interpretations

e.g.



labelled handles?

- use for understanding classical knots and knotoids.
- and ...