<u>Problem</u>. An integer $n \geq 1$ is called b-*normal* if all digits $0, 1, \ldots, (b-1)$ appear the same number of times in the expansion of n with respect to base b ($b \geq 2$). We denote by $\mathscr{N}_b$ the set of all b-normal integers. Determine the b's for which the sum

$$S := \sum_{n \in \mathscr{N}_b} \frac{1}{n}$$

is finite.

<u>Solution</u>. Let us denote by $l_b(n)$ the length of the expansion of n with respect to base b, namely the number of digits in the expansion (where, of course, the first digit cannot be 0). Clearly, if n is b-normal, we must have

$$l_b(n) = bk \qquad \text{for some integer } \; k \geq 1.$$

If $\nu_b(bk)$ is the number of b-normal integers n such that $l_b(n) = bk$, it is not difficult to see that

$$\nu_b(bk) = (b-1)\,\frac{(bk-1)!}{(k!)^{b-1}(k-1)!} = \frac{b-1}{b} \cdot \frac{(bk)!}{(k!)^b}. \tag{1}$$

Now,

$$S = \sum_{k=1}^{\infty} \sum_{\substack{n \in \mathscr{N}_b \\ l_b(n)=bk}} \frac{1}{n} \tag{2}$$

where the second sum has $\nu_b(bk)$ terms. Since every number whose b-base expansion has length bk is between b^{bk-1} and b^{bk}, we must have

$$\frac{\nu_b(bk)}{b^{bk}} \leq \sum_{\substack{n \in \mathscr{N}_b \\ l_b(n)=bk}} \frac{1}{n} \leq \frac{\nu_b(bk)}{b^{bk-1}} = \frac{b\nu_b(bk)}{b^{bk}}.$$

Using the above estimate in (2) gives

$$\sum_{k=1}^{\infty} \frac{\nu_b(bk)}{b^{bk}} \leq S \leq b \sum_{k=1}^{\infty} \frac{\nu_b(bk)}{b^{bk}}.$$

Thus, in view of (1), the finiteness of S is equivalent to the finiteness of

$$S' := \sum_{k=1}^{\infty} \frac{(bk)!}{(k!)^b} \cdot \frac{1}{b^{bk}}.$$

By Stirling's formula ($k! \sim k^k e^{-k} \sqrt{2\pi k}$) we have

$$\frac{(bk)!}{(k!)^b} \cdot \frac{1}{b^{bk}} \sim \frac{(bk)^{bk} e^{-bk} \sqrt{2\pi bk}}{k^{bk} e^{-bk} (\sqrt{2\pi k})^b} \cdot \frac{1}{b^{bk}} = \frac{\sqrt{b}}{(2\pi k)^{(b-1)/2}}$$

(where $f(k) \sim g(k)$ means that $f(k)/g(k) \to 1$ as $k \to \infty$). It follows that S' (and, therefore, S) is finite if and only if $b \geq 4$. ■