Problem. (a) Consider the set-valued sequence $\{A_n\}_{n=0}^{\infty}$ defined by the formula

$$A_0 = \{1\}, \qquad A_n = \{\delta(3x+1),\ \delta(3x+3),\ \delta(3x+5)\ :\ x \in A_{n-1}\}, \quad n \geq 1,$$

where $\delta(k)$ denotes the largest odd factor of the integer k (e.g., $\delta(1) = \delta(2) = \delta(4) = 1$, $\delta(3) = \delta(6) = \delta(12) = 3$, $\delta(15) = \delta(60) = 15$, etc). Show that

$$A := \bigcup_{n=0}^{\infty} A_n = \mathbb{N}_{\text{odd}},$$

where $\mathbb{N}_{\text{odd}}$ denotes the set of odd positive integers.
* (b) (**OPEN**) Suppose now that $\{B_n\}_{n=0}^{\infty}$ is the set-valued sequence defined by

$$B_0 = \{1\}, \qquad B_n = \{\delta(3x+1),\ \delta(3x+3)\ :\ x \in B_{n-1}\}, \quad n \geq 1$$

(notice that $B_n \subset A_n$ for every n). Is it still true that

$$B := \bigcup_{n=0}^{\infty} B_n = \mathbb{N}_{\text{odd}}?$$

Solution of Part (a). Clearly, $A \subset \mathbb{N}_{\text{odd}}$. Let $S := \mathbb{N}_{\text{odd}} \setminus A$ and suppose that $S \neq \emptyset$. Then, by the well-ordering principle of the positive integers S must contain a smallest member, say m. Since m must be odd and ≥ 3, we should have one of the following three possibilities:

$$m = 6k+3 \quad \text{or} \quad m = 6k+5 \quad \text{or} \quad m = 6k+7 \qquad \text{for some} \ \ k \geq 0.$$

(i) Suppose $m = 6k+3$ for some $k \geq 0$. Then, $4k+1 < m$ and since $4k+1$ is odd, we must have that $4k+1 \in A_n$ for some n. But, then,

$$3(4k+1)+3 = 12k+6 \qquad \text{and} \qquad \delta(12k+6) = 6k+3 = m,$$

which implies that $m \in A_{n+1} \subset A$, a contradiction.
(ii) Next, suppose $m = 6k+5$ for some $k \geq 0$. Then, $4k+3 < m$ and since $4k+3$ is odd, we must have that $4k+3 \in A_n$ for some n. But, then,

$$3(4k+3)+1 = 12k+10 \qquad \text{and} \qquad \delta(12k+10) = 6k+5 = m,$$

which implies that $m \in A_{n+1} \subset A$, a contradiction.

(iii) Finally, suppose $m = 6k + 7$ for some $k \geq 0$. Then, $4k + 3 < m$ and since $4k + 3$ is odd, we must have that $4k + 3 \in A_n$ for some n. But, then,

$$3(4k + 3) + 5 = 12k + 14 \qquad \text{and} \qquad \delta(12k + 14) = 6k + 7 = m,$$

which implies that $m \in A_{n+1} \subset A$, again a contradiction.
Therefore, in all three possibilities for m we have reached a contradiction. It follows that $S = \emptyset$. ■

(The inspiration for this problem came from the Collatz $3x + 1$ Problem)