A PROBABILITY PROBLEM

A big supermarket chain has the following policy: For every m Euros you spend per buy, you earn one "point" (suppose, e.g., that $m = 3$; in this case, if you spend 8.45 Euros, you get two points, if you spend 9.00 Euros, you get three points, while if you spend 2.85 Euros, you get zero points). After you collect a certain number of points, you can redeem them for a gift certificate.

What is the actual price β of a point?

General Discussion. The amount X spent per buy can be considered a positive random variable with a distribution function $F(x)$ and finite mean $\mu_X = E[X] > 0$. Recall the well-known formula

$$\mu_X = \int_0^\infty P\{X \geq x\}dx = \int_0^\infty P\{X > x\}dx = \int_0^\infty [1 - F(x)]\, dx, \tag{1}$$

which is valid for any nonnegative random variable X.

If $F(m^-) := \lim_{x \to m^-} F(x) = P\{X < m\} = 1$, then it is impossible to earn points and we may say that in this case $\beta = \infty$, which is totally unrealistic and uninteresting. Hence, in order for our problem to be meaningful we should at least have $F(m^-) < 1$. Actually, from now on we will make a slightly stronger assumption, namely

$$F(m) = P\{X \leq m\} < 1. \tag{2}$$

If one visits a store and spends X Euros, the number of points (s)he earns is

$$Y = \lfloor X/m \rfloor, \tag{3}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Thus,

$$\frac{X}{m} - 1 < Y \leq \frac{X}{m}. \tag{4}$$

Since $Y \geq 0$, formula (4) can be slightly improved:

$$\left(\frac{X}{m} - 1\right)^+ = \frac{(X - m)^+}{m} \leq Y \leq \frac{X}{m}, \tag{5}$$

where $(x)^+ = \max\{x, 0\}$. By taking expectations (5) yields

$$\frac{E\left[(X - m)^+\right]}{m} \leq \mu_Y \leq \frac{\mu_X}{m}, \tag{6}$$

where $\mu_Y = E[Y]$. Since

$$E\left[(X-m)^+\right] = \int_0^\infty (x-m)^+ dF(x) = \int_m^\infty (x-m) dF(x) = \int_m^\infty P\{X > x\} dx, \quad (7)$$

we have, in view of (2),

$$E\left[(X-m)^+\right] > 0. \quad (8)$$

Let us, also, notice that it follows easily from (7) that

$$E\left[(X-m)^+\right] \geq (\mu_X - m)^+ . \quad (9)$$

Furthermore, equation (3) implies that

$$\mu_Y = \sum_{k=1}^\infty k\, P\{Y = k\} = \sum_{k=1}^\infty k\, P\{km \leq X < (k+1)m\} = \sum_{k=1}^\infty P\{X \geq km\}, \quad (10)$$

where the last equality follows by summation by parts (notice, also, that $P\{X \geq km\} = 1 - F(km^-)$).

Now, let $X_1, X_2, \ldots, X_n$ be the amounts spent per buy by a random group of consumers. We assume that these amounts are independent copies of X, namely that $\{X_k\}_{1 \leq k \leq n}$ is a collection of independent random variables with common distribution function $F(x)$. For these n buys, the total amount spent is

$$X_1 + X_2 + \cdots + X_n \quad (11)$$

and the total number of points earned is

$$Y_1 + Y_2 + \cdots + Y_n, \qquad \text{where } \; Y_k = \lfloor X_k/m \rfloor, \quad k = 1, 2, \ldots, n. \quad (12)$$

It is, then, reasonable to infer that the actual value of a point is

$$\beta = \lim_n \frac{X_1 + X_2 + \cdots + X_n}{Y_1 + Y_2 + \cdots + Y_n}. \quad (13)$$

Dividing the numerator and the denominator of the above fraction by n and invoking the Law of Large Numbers, we obtain from (13) that

$$\beta = \frac{\mu_X}{\mu_Y}. \quad (14)$$

Remark 1. The limit in (13) exists almost surely. Actually, for the a.s. existence of the limit it is enough to assume that $\{X_k\}_{k=1}^\infty$ is a sequence of *pairwise* independent random variables with common distribution function $F(x)$ (this is Etemadi's version of the Strong Law of Large Numbers—see, e.g., [1]).

Formulas (6) and (9), applied to (14), give us some bounds for β, namely

$$m \leq \beta \leq \frac{\mu_X}{E\left[(X-m)^+\right]}\, m \leq \frac{\mu_X}{(\mu_X - m)^+}\, m = \frac{m}{\left(1-\frac{m}{\mu_X}\right)^+}. \tag{15}$$

Of course, the lower bound of β given in (15), namely $m \leq \beta$, is trivial. An equivalent way to write (15) is

$$1 \leq \lambda \leq \frac{\mu_X}{E\left[(X-m)^+\right]} \leq \frac{1}{\left(1-\frac{m}{\mu_X}\right)^+}, \qquad \text{where we have set } \lambda := \frac{\beta}{m}. \tag{16}$$

Thus, if μ_X is considerably larger that m, then λ is close to 1, i.e. β is close to m.

Remark 2. (a) Suppose we have a whole family (or a sequence) $\{F_K(x)\}$ of distribution functions satisfying our basic assumptions (i) $F_K(0) = 0$, (ii) $F_K(m) < 1$, and (iii) the expectation associated to $F_K(x)$ is finite, i.e. $\int_0^\infty [1 - F_K(x)]\, dx < \infty$. Under this setup, formula (16) implies that

$$\text{if} \quad \lim_{K\to\infty} \mu_X = \infty, \qquad \text{then} \quad \lim_{K\to\infty} \lambda = 1. \tag{17}$$

(b) In the case where our family $\{F_K(x)\}$ is such that $\lim_{K\to\infty} F_K(m^-) = \lim_{K\to\infty} P\{X < m\} = 1$, one might expect that $\lambda \to \infty$ (i.e. $\beta \to \infty$). However, it is easy to find examples where λ stays finite. Actually, we can even have $\lim_{K\to\infty} \mu_X = \infty$, which by (17) will imply that $\lambda \to 1$. For example, for each K choose $F_K(x)$ so that $P\{X < m\} = 1 - (2/K)$ and $P\{X > K^2\} = 1/K$. Then, it is obvious that $\lim_{K\to\infty} P\{X < m\} = 1$, while $\mu_X \geq K$ and hence $\lim_{K\to\infty} \mu_X = \infty$ (thus, (17) implies $\lim_{K\to\infty} \lambda = 1$). In fact, even if, $\lim_{K\to\infty} \mu_X = 0$ (which is, clearly, stronger than $\lim_{K\to\infty} P\{X < m\} = 1$), we have that the limit of λ, if it exists, can take any value ≥ 1 (including ∞). To see an example let us for convenience take $m = 1$:

$$\text{If} \quad F_K(x) = (1 - K^{-1})\mathbf{1}_{[K^{-2},\, 1+K^{-2})}(x) + \mathbf{1}_{[1+K^{-2},\, \infty)}(x), \qquad \text{then} \quad \lim_{K\to\infty} \lambda = 1$$

(where $\mathbf{1}_I(x)$ denotes the indicator function of the interval I), while

$$\text{if} \quad F_K(x) = (1 - K^{-2})\mathbf{1}_{[K^{-1},\, 1+K^{-1})}(x) + \mathbf{1}_{[1+K^{-1},\, \infty)}(x), \qquad \text{then} \quad \lim_{K\to\infty} \lambda = \infty.$$

The Gamma Case. In order to get a more precise estimate for β, we need to have more information regarding the distribution of X. For instance, a plausible assumption is that X follows a Gamma distribution with parameters $a > 0$ and $p > 0$, namely that its probability density function is

$$f(x) = \frac{1}{\Gamma(p)}(ax)^{p-1} a e^{-ax}, \qquad x > 0 \tag{18}$$

(of course, $f(x) = 0$ for $x < 0$), where $\Gamma(\cdot)$ is the Gamma function. Recall that, since $p > 0$, we have

$$\Gamma(p) = \int_0^\infty \xi^{p-1} e^{-\xi} d\xi \tag{19}$$

and that if X has the Gamma density $f(x)$ given by (18), then

$$\mu_X = \frac{p}{a}. \tag{20}$$

Therefore, in view of (14), in order to determine the price β we have to calculate μ_Y. Using (18) in (10) yields

$$\mu_Y = \frac{1}{\Gamma(p)} \sum_{k=1}^\infty k \int_{km}^{(k+1)m} (ax)^{p-1} a e^{-ax} dx = \frac{1}{\Gamma(p)} \sum_{k=1}^\infty k \int_{kam}^{(k+1)am} \xi^{p-1} e^{-\xi} d\xi \tag{21}$$

or

$$\mu_Y = \frac{1}{\Gamma(p)} \sum_{k=1}^\infty \int_{kam}^\infty \xi^{p-1} e^{-\xi} d\xi. \tag{22}$$

By substituting (20) and (22) in (14) we obtain

$$\beta = \frac{p\Gamma(p)}{a \sum_{k=1}^\infty \int_{kam}^\infty \xi^{p-1} e^{-\xi} d\xi} = \frac{\Gamma(p+1)}{a \sum_{k=1}^\infty \int_{kam}^\infty \xi^{p-1} e^{-\xi} d\xi}, \tag{23}$$

which in terms of the ratio λ introduce in (16) becomes

$$\lambda(a,p) := \lambda = \frac{\beta}{m} = \frac{\Gamma(p+1)}{am \sum_{k=1}^\infty \int_{kam}^\infty \xi^{p-1} e^{-\xi} d\xi}. \tag{24}$$

Observe that (17) tells us that

$$\mu_X \to \infty \qquad \text{implies} \qquad \lambda(a,p) \to 1 \tag{25}$$

and, in particular, in view of (20),

$$\lim_{a \to 0^+} \lambda(a,p) = 1 \qquad \text{and} \qquad \lim_{p \to \infty} \lambda(a,p) = 1. \tag{26}$$

It is remarkable that the above limits are not so obvious from (24).

Looking at formula (24) it seems that, unless we have specific numerical values for the parameters m, a, and p, it is not clear how to extract a precise estimate for the ratio $\lambda(a,p)$. For this reason, the qualitative behavior of $\lambda(a,p)$ has some interest.

Conjecture. The function $\lambda(a,p)$ of (24) is (i) strictly increasing in a and (ii) strictly decreasing in p. Furthermore,

$$\lim_{a \to \infty} \lambda(a,p) = \infty \qquad \text{and} \qquad \lim_{p \to 0^+} \lambda(a,p) = \infty. \tag{27}$$

In an attempt to obtain a formula for $\lambda(a,p)$ which is more explicit than (24), let us suppose that X follows an Erlang distribution, namely

$$p = n \in \mathbb{N} := \{1, 2, \ldots\}. \tag{28}$$

Then (24) becomes

$$\lambda(a,n) = \frac{n!}{am\sum_{k=1}^{\infty}\int_{kam}^{\infty}\xi^{n-1}e^{-\xi}d\xi} = \frac{n}{am\sum_{k=1}^{\infty}F_n(kam)}, \tag{29}$$

where we have set

$$F_n(x) := \frac{1}{(n-1)!}\int_x^{\infty}\xi^{n-1}e^{-\xi}d\xi, . \tag{30}$$

We claim that

$$F_n(x) = S_n(x)e^{-x}, \qquad \text{where} \qquad S_n(x) := \sum_{l=0}^{n-1}\frac{x^l}{l!}. \tag{31}$$

For $n = 1$ formula (31) is clearly true. To justify (31) for $n \geq 2$ we first observe that the definition (30) of $F_n(x)$ implies

$$F_n'(x) = -\frac{x^{n-1}}{(n-1)!}\,e^{-x} \qquad \text{and} \qquad F_n(0) = 1. \tag{32}$$

Now, from the definition (31) of $S_n(x)$ we have

$$S_n'(x) = S_{n-1}(x) \qquad \text{and} \qquad S_n(0) = 1. \tag{33}$$

Therefore, $F_n(x)$ and $S_n(x)e^{-x}$ agree at $x = 0$ and by applying (33) and the definition of $S_n(x)$ we get

$$\left[S_n(x)e^{-x}\right]' = S_n'(x)e^{-x} - S_n(x)e^{-x} = S_{n-1}(x)e^{-x} - S_n(x)e^{-x} = -\frac{x^{n-1}}{(n-1)!}\,e^{-x}. \tag{34}$$

By comparing (34) with (32) we get the validity of (31). Using now (31) in (29) yields

$$\lambda(a,n) = \frac{n}{am\sum_{k=1}^{\infty}S_n(kam)e^{-kam}} = \frac{n}{am\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{(kam)^l}{l!}e^{-kam}} \tag{35}$$

or, by interchanging the order of summation in the denominator of the last expression

$$\lambda(a,n) = \frac{n}{am\sum_{l=0}^{n-1}\frac{(am)^l}{l!}\,T_l(am)}, \tag{36}$$

where we have set

$$T_l(x) := \sum_{k=1}^{\infty}k^l e^{-kx}, \qquad\qquad x > 0, \quad l = 0, 1, 2, \ldots \tag{37}$$

(incidentally, $T_l(x)$ of (37) makes sense for any complex number l, actually it is entire in l; also, for any real l, $T_l(x)$ is strictly decreasing in x on $(0, \infty)$, with $T_l(\infty) = 0$; finally, if $l \geq -1$, then $T_l(0^+) = \infty$, while if $l < -1$, then $T_l(0) = \zeta(-l)$). From (37) we have

$$T_0(x) = \sum_{k=1}^{\infty} e^{-kx} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1} \tag{38}$$

and

$$T_{l+1}(x) = -T_l'(x). \tag{39}$$

Therefore,

$$T_l(x) = (-1)^l T_0^{(l)}(x) \tag{40}$$

and equation (36) can be written as

$$\lambda(a, n) = \frac{n}{am \sum_{l=0}^{n-1} \frac{(-am)^l}{l!} T_0^{(l)}(am)}. \tag{41}$$

However, from (41), unless we specify n it is still not clear how λ varies with m, a, and n.

There is something peculiar in formula (41). If we consider the Taylor polynomial

$$Q_n(x; x_0) := \sum_{l=0}^{n-1} \frac{(x - x_0)^l}{l!} T_0^{(l)}(x_0) \tag{42}$$

associated to $T_0(x)$, then the sum appearing in the denominator of (41) is $Q_n(0; am)$, while $T_0(0^{\pm}) = \pm\infty$.

Finally, let us notice that by straightforward induction we can show

$$T_0^{(l)}(x) = (-1)^l \frac{q_l(e^x)}{(e^x - 1)^{l+1}}, \qquad l = 0, 1, \dots, \tag{43}$$

where $q_0(z) \equiv 1$ and

$$q_{l+1}(z) = (l + 1)z q_l(z) - z(z - 1) q_l'(z) \qquad l = 0, 1, \dots. \tag{44}$$

It is easy to see that $q_l(z)$ is a monic polynomial of degree l with $q_l(0) = 0$ and $q_l'(0) = 1$ for all $l \geq 1$. Furthermore, the coefficients of $q_l(z)/z$ are strictly positive for all $l \geq 1$ (also, $q_l(1) = l!$ and $q_l'(1) = (l + 1)!/2$ for all $l \geq 1$). In particular, we have

$$q_1(z) = z, \qquad q_2(z) = z^2 + z, \qquad q_3(z) = z^3 + 4z^2 + z, \qquad q_4(z) = z^4 + 13z^3 + 9z^2 + z. \tag{45}$$

Example 1. Suppose $n = 1$, so that X is exponentially distributed with parameter a. Then (41) yields

$$\lambda(a, 1) = \frac{e^{am} - 1}{am} = 1 + am \sum_{n=0}^{\infty} \frac{(am)^n}{(n + 2)!}. \tag{46}$$

Notice that $\lambda(a,1)$ is an increasing function of a. Furthermore, $\lambda(a,1) \to \infty$ as $a \to \infty$ (equivalently as $\mu_X \to 0$), while $\lambda(a,1) \to 1$ as $a \to 0$ (equivalently as $\mu_X \to \infty$), as expected.

Example 2. Suppose $n = 2$, so that X is Erlang with parameters a and 2. Then (41) yields

$$\lambda(a,2) = \frac{2}{1 + \frac{am}{1-e^{-am}}} \cdot \frac{e^{am}-1}{am} = \frac{2\lambda(a,1)}{1 + \frac{am}{1-e^{-am}}}. \tag{47}$$

Since $\frac{x}{1-e^{-x}}$ is strictly increasing for $x > 0$ (and, hence, > 1), it follows from (47) that $\lambda(a,2) < \lambda(a,1)$ and, furthermore, that $\lambda(a,2)$ is strictly increasing in a and approaches ∞ as $a \to \infty$ (equivalently as $\mu_X \to 0$), while $\lambda(a,2) \to 1$ as $a \to 0$ (equivalently as $\mu_X \to \infty$), as expected.

Example 3. Suppose $n = 3$, so that X is Erlang with parameters a and 3. Then (41) yields

$$\lambda(a,3) = \frac{3}{1 + \frac{am}{1-e^{-am}}\left[1 + \frac{am}{1-e^{-am}} \cdot \frac{1+e^{-am}}{2}\right]} \cdot \frac{e^{am}-1}{am} = \frac{3\lambda(a,1)}{1 + \frac{am}{1-e^{-am}}\left[1 + \frac{am}{1-e^{-am}} \cdot \frac{1+e^{-am}}{2}\right]}. \tag{48}$$

Since $\frac{x}{1-e^{-x}} \cdot \frac{1+e^{-x}}{2}$ is strictly increasing for $x > 0$ (and, hence, > 1), it follows from (47) and (48) that $\lambda(a,3) < \lambda(a,2)$ and, furthermore, that $\lambda(a,3)$ is strictly increasing in a and approaches ∞ as $a \to \infty$ (equivalently as $\mu_X \to 0$), while $\lambda(a,3) \to 1$ as $a \to 0$ (equivalently as $\mu_X \to \infty$), as expected.

References

[1] R. Durrett, *Probability: Theory and Examples*, Third Edition, Duxbury Advanced Series, Brooks/Cole—Thomson Learning. Belmont, CA, USA, 2005.