A Question

1 Introductory material

Let $\Omega = \mathbb{N} = \{1, 2, \ldots\}$ (the set of natural numbers). For each $n \geq 1$ and $j \in \{1, 2, 3, \ldots, 2^n\}$ we consider the subsets of $\mathbb{N}$

$$E_n(j) := \{j + k\, 2^n \,:\, k = 1, 2, \ldots\}. \tag{1.1}$$

In other words, for each n the set $E_n(j)$ is the equivalence class $[j] \pmod{2^n}$ (i.e. $x, y \in E_n(j)$ if and only if $x \equiv y \pmod{2^n}$). In particular, for each n the sets $E_n(j)$, $j = 1, 2, \ldots, 2^n$, form a partition of $\mathbb{N}$.
Next, for each $n \geq 1$ we consider the algebra $\mathcal{F}_n$ generated by the sets $E_n(j)$, $j = 1, 2, \ldots, 2^n$. Clearly, $\mathcal{F}_n$ contains 2^{2^n} sets, hence, being finite, it is also a σ-algebra and we can write

$$\mathcal{F}_n := \sigma(E_n(j),\ j = 1, 2, \ldots, 2^n). \tag{1.2}$$

Furthermore,

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \qquad \text{for all } \ n \geq 1. \tag{1.3}$$

We continue by considering the probability spaces

$$(\Omega, \mathcal{F}_n, P_n), \qquad n = 1, 2, \ldots, \tag{1.4}$$

where, for each $n \geq 1$ the probability measure P_n is defined by the formula

$$P_n\{E_n(j)\} := 2^{-n}, \qquad j = 1, 2, \ldots, 2^n. \tag{1.5}$$

It is easy to see that P_n is the restriction of P_{n+1} on $\mathcal{F}_n$, symbolically

$$P_{n+1}|_{\mathcal{F}_n} = P_n, \qquad \text{for all } \ n \geq 1. \tag{1.6}$$

Let us now set

$$\mathcal{A} := \bigcup_{n=1}^{\infty} \mathcal{F}_n. \tag{1.7}$$

Clearly, $\mathcal{A}$ is a (countably infinite) algebra of subsets of $\mathbb{N}$, while the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ is $2^{\mathbb{N}}$, namely the power set of $\mathbb{N}$ (i.e. the class of all subsets of $\mathbb{N}$). We define a set function $P : \mathcal{A} \to [0, 1]$ as follows. Let $A \in \mathcal{A}$. Then, by (1.7) we have that $A \in \mathcal{F}_n$ for some n. We set

$$P(A) := P_n(A) \tag{1.8}$$

and it is clear by (1.6) that this definition is independent of n, hence P is well defined. Furthermore, it is clear that P is finitely additive on $\mathcal{A}$.

However, P cannot be extended to a probability measure $\bar{P}$ on $\sigma(\mathcal{A})$, since, if such an extension existed, then from the equation

$$\{j\} = \bigcap_{n=j}^{\infty} E_n(j) \tag{1.9}$$

we would have that $\bar{P}(\{j\}) = \lim_n \bar{P}\{E_n(j)\} = \lim_n P_n\{E_n(j)\} = \lim_n 2^{-n} = 0$ for all singletons $\{j\}$, and this (together with the countable additivity of $\bar{P}$) is in contradiction with the fact that $\bar{P}(\mathbb{N}) = P(\mathbb{N}) = 1$.

It follows that P must violate the hypothesis of the well-known Caratheodory Extension Theorem (also known as Caratheodory-Hahn Extension Theorem or Caratheodory-Hahn-Kolmogorov Extension Theorem). Hence, there are disjoint sets $A_1, A_2, \ldots$ in $\mathcal{A}$ such that

$$A := \bigcup_{m=1}^{\infty} A_m \in \mathcal{A} \tag{1.10}$$

and

$$\sum_{m=1}^{\infty} P(A_m) < P(A). \tag{1.11}$$

Question. Can you construct a specific collection of disjoint sets $A_1, A_2, \ldots$ in $\mathcal{A}$ satisfying (1.10) and (1.11)?

A complete answer to the above question has been given by **Chris Bowen**.