Problem. Let $m \geq 2$ be an integer. Consider the polynomial

$$P(z) = (m-1)z^{m+1} - (m+1)z^m + (m+1)z - (m-1).$$

Show that all roots of $P(z)$ lie on the unit circle.

Solution. Notice that $P(z)$ can be written in the form

$$P(z) = m[(z^m + 1)(z-1) - (1/m)(z^m - 1)(z+1)].$$

Hence, the following proposition answers our problem immediately:

Proposition. Let $m \geq 2$ be an integer and $\lambda \in (-\infty, 1/m]$. Then, all zeros of the polynomial

$$p(z) = p(z;\lambda) = (z^m + 1)(z-1) - \lambda(z^m - 1)(z+1) \tag{1}$$

lie on the unit circle $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$.

Proof of the Proposition. We start with two observations.

Observation 1: The polynomial p of (1) can be also written as

$$p(z) = (1-\lambda)z^{m+1} - (1+\lambda)z^m + (1+\lambda)z - (1-\lambda). \tag{2}$$

From (2) it follows that, if $p(r) = 0$, for some $r \in \mathbb{C}$, then

$$p(\overline{r}) = p(1/r) = p(1/\overline{r}) = 0.$$

Hence, the zeros of p occur in quadruplets, unless they lie on Γ.

Observation 2: For $\lambda = 0$, the polynomial $p(z;\lambda)$ of (1) becomes

$$p(z;0) = (z^m + 1)(z-1),$$

which, obviously, has $m+1$ simple zeros on Γ.

Suppose now that λ starts moving away from 0 in a continuous manner. Then, the zeros of p start moving continuously. Due to the Observations 1 and 2, initially the zeros stay on Γ. As λ is moving, a zero r of p, in order to escape Γ it must first become a multiple zero of p on Γ. Thus, to establish the proposition, it suffices to show that p does not have multiple zeros on Γ, for $\lambda \in (-\infty, 1/m)$. We will prove this by contradiction.

Let $\lambda \in (-\infty, 1/m)$ and assume that r is a multiple zero of p, with $|r| = 1$. Then

$$p(r) = (1-\lambda)r^{m+1} - (1+\lambda)r^m + (1+\lambda)r - (1-\lambda) = 0 \tag{3}$$

and

$$p'(r) = (m+1)(1-\lambda)r^m - m(1+\lambda)r^{m-1} + (1+\lambda) = 0. \tag{4}$$

We will exploit the fact that most coefficients of p are 0. Multiplying (3) by $(m+1)$, (4) by r and subtracting the two resulting equations yields

$$r^m = mr - \frac{(m+1)(1-\lambda)}{1+\lambda} \tag{5}$$

(alternatively, (5) comes from (4) by observing that $p'(1/r) = 0$, since $|r| = 1$ and hence $1/r = \overline{r}$). Next, we replace r^m in equation (3) by its value given in (5). The result is

$$m(1-\lambda^2)r^2 - 2(m - 2\lambda + m\lambda^2)r + m(1-\lambda^2) = 0. \tag{6}$$

The discriminant of the quadratic equation (6) is

$$\Delta = -16\lambda(m-\lambda)(1-m\lambda).$$

If $\lambda < 0$, then $\Delta > 0$ and one can easily see that r of (6) cannot lie on Γ. Thus, we only need to examine the case $\lambda \in (0, 1/m)$.

By squaring both of its sides, (5) implies

$$(1+\lambda)^2 r^{2m} = m^2(1+\lambda)^2 r^2 - 2m(m+1)(1-\lambda^2)r + (m+1)^2(1-\lambda)^2. \tag{7}$$

On the other hand, (6) gives

$$r^2 = \frac{2(m - 2\lambda + m\lambda^2)}{m(1-\lambda^2)} r - 1.$$

Substituting r^2, as given by the above formula, in (7) gives (after some straightforward manipulations)

$$(1+\lambda)^2(1-\lambda)r^{2m} = 2m(1+\lambda)(2m\lambda - \lambda^2 - 1)r + (1-\lambda)(1-\lambda-2m\lambda)(1-\lambda+2m). \tag{8}$$

We continue by rewriting (5) as

$$r = (1/m)r^m + \frac{(m+1)(1-\lambda)}{m(1+\lambda)}$$

and substituting this expression for r in (8). This leads us to the equation

$$(1-\lambda^2)r^{2m} - 2(2m\lambda - \lambda^2 - 1)r^m + (1-\lambda^2) = 0. \tag{9}$$

From equations (6) and (9) it is not hard to get a contradiction. For example, (6) and (9) imply respectively

$$r - 1 = \frac{2\lambda(m\lambda - 1) \pm 2iE}{m(1-\lambda^2)} \tag{10}$$

and

$$r^m - 1 = \frac{2(m\lambda - 1) \pm 2iE}{1 - \lambda^2}, \tag{11}$$

where

$$E := \sqrt{\lambda(m - \lambda)(1 - m\lambda)} > 0,$$

since $\lambda \in (0, 1/m)$. Then (10) and (11) imply

$$\left|\frac{r^m - 1}{r - 1}\right| = m \left|\frac{(m\lambda - 1) \pm iE}{\lambda(m\lambda - 1) \pm iE}\right| = m\sqrt{\frac{(m\lambda - 1)^2 + E^2}{\lambda^2(m\lambda - 1)^2 + E^2}} > m, \tag{12}$$

where the inequality follows from the fact that, since $0 < \lambda < 1/m < 1$, the numerator in the square root in (12) is larger than the denominator. However, the assumption $|r| = 1$ implies (thanks to the triangle inequality for the absolute value)

$$\left|\frac{r^m - 1}{r - 1}\right| = \left|r^{m-1} + r^{m-2} + \cdots + r + 1\right| \leq m,$$

which contradicts (12). ■

Remark. The polynomial $P(z) = p(z; 1/m)$ has a triple root at $z = 1$. A consequence of the above proof is that, if $r \neq 1$ is a root of $P(z)$, then it is a simple root (and, of course, lies on the unit circle).

Comment. The polynomial $P(z)$ arises in the study of the so-called *transmission eigenvalues in the spherically symmetric case*. The property that the roots of $P(z)$ lie on the unit circle is equivalent to the fact that, if the index of refraction equals an integer $m \geq 2$, then all transmission eigenvalues are real.