Interpolating the Derivatives of the Gamma Function

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Abstract

We consider a function $G(\lambda, z)$, entire in λ, which interpolates the derivatives of the Gamma function in the sense that $G(m, z) = \Gamma^{(m)}(z)$ for any integer $m \geq 0$ and we calculate the asymptotics of $G(\lambda, z)$ as $\lambda \to +\infty$.

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1 Introduction—the function $G(\lambda, z)$

Starting with the familiar integral representation of the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \Re(z) > 0, \tag{1}$$

and then taking derivatives, one obtains

$$\Gamma^{(m)}(z) = \int_0^\infty e^{-t} (\ln t)^m t^{z-1} dt, \qquad m = 0, 1, 2, \dots, \quad \Re(z) > 0. \tag{2}$$

The goal of this article is to construct and study an analytic function G of two variables λ and z which, for $\lambda = m$ becomes the m-th derivative $\Gamma^{(m)}(z)$ of the Gamma function.

To construct such a function we simply set

$$G(\lambda, z) := \int_{C_r} e^{-\zeta} (\ln \zeta)^\lambda \, \zeta^{z-1} d\zeta, \qquad \Re(z) > 0, \tag{3}$$

where:
(i) The contour C_r consists of the intervals $[0, 1-r]$ and $[1+r, \infty)$ of the positive real semiaxis ($0 < r < 1$) connected with the semicircle $\{\zeta \in \mathbb{C} : \zeta = 1 + re^{i\theta},\ 0 < \theta < \pi\}$, lying in the upper half-plane. The orientation of C_r is $0 \to \infty$.
(ii) $w = \ln \zeta$ denotes the branch of the logarithm which maps the open upper half-plane $\Im(\zeta) > 0$ conformally onto the open strip $0 < \Im(w) < \pi$. In particular, $\ln 1 = 0$ and $\ln(-1) = \pi i$.
(iii) $(\ln \zeta)^\lambda = e^{\lambda \ln \ln \zeta}$ and $\zeta^{z-1} = e^{(z-1)\ln \zeta}$. In particular $1^\lambda = 1^{z-1} = 1$.
It is clear that the value of the integral in (3) is independent of $r \in (0,1)$ and hence it defines a unique function $G(\lambda, z)$, which is entire in λ. Moreover, it is not hard to see that, as a function of λ, $G(\lambda, z)$ is of order 1 and of maximal (i.e. infinite) type for any z with $\Re(z) > 0$.
If $\Re(\lambda) > -1$ (and $\Re(z) > 0$), we can let $r \to 0^+$ in (3) and arrive at the formula

$$G(\lambda, z) = e^{\pi i \lambda} \int_0^1 e^{-t} (-\ln t)^\lambda\, t^{z-1} dt + \int_1^\infty e^{-t} (\ln t)^\lambda\, t^{z-1} dt, \tag{4}$$

since for the negative values of $\ln t$ we have

$$(\ln t)^\lambda = e^{\lambda \ln \ln t} = e^{\lambda[\ln(-\ln t) + \pi i]} = e^{\pi i \lambda} (-\ln t)^\lambda. \tag{5}$$

Formula (4) can be also written as

$$G(\lambda, z) = \int_0^\infty e^{-t} (\ln t)^\lambda\, t^{z-1} dt, \qquad \Re(\lambda) > -1, \quad \Re(z) > 0, \tag{6}$$

from which we can see immediately that $G(m, z) = \Gamma^{(m)}(z)$ for $m = 0, 1, \dots$ and $\Re(z) > 0$.

2 Some basic properties of $G(\lambda, z)$

Before we start our analysis let us introduce the notation

$$\mathbb{C}_\xi := \{z \in \mathbb{C} : \Re(z) > \xi\}, \tag{7}$$

where ξ is a real number.

As we have, essentially, already observed, it follows from (3) that $G(\lambda, z)$ is analytic in $\mathbb{C} \times \mathbb{C}_0$.
We continue with some basic properties of $G(\lambda, z)$.

Property 1. For any $\lambda \in \mathbb{C}_{-1}$ we have

$$L(\lambda) := \lim_{\substack{z \to 0 \\ z \in \mathbb{C}_0}} z^{\lambda+1} G(\lambda, z) = e^{\pi i \lambda} \Gamma(\lambda + 1), \tag{8}$$

where $z^{\lambda+1} = e^{(\lambda+1)\ln z}$, $\ln z$ being continuous in $\mathbb{C}_0$ (in particular $1^{\lambda+1} = 1$).

Proof. In view of (6) we have

$$G(\lambda, z) = \int_0^\varepsilon e^{-t}(\ln t)^\lambda\, t^{z-1} dt + \int_\varepsilon^\infty e^{-t}(\ln t)^\lambda\, t^{z-1} dt, \quad \Re(\lambda) > -1,\ \Re(z) > 0, \tag{9}$$

for any fixed $\varepsilon > 0$. Now, the integral

$$\int_\varepsilon^\infty e^{-t}(\ln t)^\lambda\, t^{z-1} dt \tag{10}$$

is entire in z and, in particular, finite at $z = 0$; thus it does not contribute to the value of the limit $L(\lambda)$. Consequently, for the limit in (8) we have

$$L(\lambda) = \lim_{\substack{z \to 0 \\ z \in \mathbb{C}_0}} z^{\lambda+1} \int_0^\varepsilon e^{-t}(\ln t)^\lambda\, t^{z-1} dt, \tag{11}$$

independently of ε, as long as $\varepsilon > 0$. Since for $t \in [0, \varepsilon]$ the quantity e^{-t} can be made as close to 1 as we wish by choosing ε sufficiently small, it follows that

$$L(\lambda) = \lim_{\substack{z \to 0 \\ z \in \mathbb{C}_0}} z^{\lambda+1} \int_0^\varepsilon (\ln t)^\lambda\, t^{z-1} dt. \tag{12}$$

But the limit in (12) is, also, independent of ε, as long as $\varepsilon > 0$. Therefore, we can take $\varepsilon = 1$ in (12) and get

$$\begin{aligned} L(\lambda) &= \lim_{\substack{z \to 0 \\ z \in \mathbb{C}_0}} z^{\lambda+1} \int_0^1 (\ln t)^\lambda\, t^{z-1} dt \\ &= e^{\pi i \lambda} \lim_{\substack{z \to 0 \\ z \in \mathbb{C}_0}} z^{\lambda+1} \int_0^1 (-\ln t)^\lambda\, t^{z-1} dt \\ &= e^{\pi i \lambda} \lim_{\substack{z \to 0 \\ z \in \mathbb{C}_0}} z^{\lambda+1} \int_0^\infty e^{-z\tau} \tau^\lambda d\tau \end{aligned} \tag{13}$$

(the last integral is obtained by substituting $\tau = -\ln t$ in the previous integral). Finally, since for real and positive z, the substitution $t = z\tau$ yields

$$z^{\lambda+1} \int_0^\infty e^{-z\tau} \tau^\lambda d\tau = \Gamma(\lambda + 1), \qquad \lambda \in \mathbb{C}_{-1}, \tag{14}$$

it follows by analytic continuation that the equality (14) holds for any $z \in \mathbb{C}_0$. ■

Property 2. If ∂_z denotes the derivative operator with respect to z, then

$$\partial_z G(\lambda, z) = G(\lambda + 1, z), \qquad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}_0. \tag{15}$$

Property 2 follows by differentiating (3) with respect to z.

Property 3. The function $G(\lambda, z)$ satisfies the functional equation

$$G(\lambda, z+1) = zG(\lambda, z) + \lambda G(\lambda-1, z), \qquad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}_0. \tag{16}$$

Proof. For $\Re(\lambda) > 1$ and $\Re(z) > 0$ integration by parts yields

$$\begin{aligned} z\int_0^\infty e^{-t}(\ln t)^\lambda\, t^{z-1}dt &= e^{-t}(\ln t)^\lambda\, t^z\Big|_{t=0}^\infty + \int_0^\infty \left[e^{-t}(\ln t)^\lambda\, t^z - \lambda e^{-t}(\ln t)^{\lambda-1}t^{z-1}\right] dt \\ &= \int_0^\infty e^{-t}(\ln t)^\lambda\, t^z dt - \lambda \int_0^\infty e^{-t}(\ln t)^{\lambda-1}t^{z-1}dt \\ &= G(\lambda, z+1) - \lambda G(\lambda-1, z). \end{aligned}$$

Thus, $G(\lambda, z)$ satisfies (16) for $\Re(\lambda) > 1$ and, consequently, for all $\lambda \in \mathbb{C}$ by analytic continuation. ∎

We can now combine Property 2 with Property 3 and get

$$G(\lambda, z+1) = z\partial_z G(\lambda-1, z) + \lambda G(\lambda-1, z), \qquad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}_0. \tag{17}$$

Multiplying (17) by $z^{\lambda-1}$ yields

$$z^{\lambda-1}G(\lambda, z+1) = \partial_z\left[z^\lambda G(\lambda-1, z)\right], \qquad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}_0, \tag{18}$$

so that, by integrating (18) from z_0 to z we obtain

$$z^\lambda G(\lambda-1, z) - z_0^\lambda G(\lambda-1, z_0) = \int_{z_0}^z \zeta^{\lambda-1}G(\lambda, \zeta+1)\, d\zeta, \qquad \lambda \in \mathbb{C}, \tag{19}$$

where the integral is taken over a rectifiable arc joining z_0 and z, and lying entirely in the open right half-plane $\mathbb{C}_0$.
Incidentally, if we restrict $\lambda \in \mathbb{C}_0$, then we can let $z_0 \to 0$ in (19) and invoke Property 1 in order to get

$$z^\lambda G(\lambda-1, z) + e^{\pi i\lambda}\Gamma(\lambda) = \int_0^z \zeta^{\lambda-1}G(\lambda, \zeta+1)\, d\zeta, \qquad \lambda, z \in \mathbb{C}_0. \tag{20}$$

Noticing that, for any $\lambda \in \mathbb{C}$, the integral in the right-hand side of (19) defines a (multivalued) analytic function of z in the half-plane $\mathbb{C}_{-1}$ we can conclude that $z^\lambda G(\lambda-1, z)$ and, consequently, $G(\lambda-1, z)$ have a meromorphic extension for $z \in \mathbb{C}_{-1}$, with a branch point at $z = 0$. It is convenient (and harmless) to denote by $G(\lambda-1, z)$ too the meromorphic extension of $G(\lambda-1, z)$.
It, then, follows that for a fixed $z_0 \in \mathbb{C}_0$ formula (19) remains valid for $z \in \mathbb{C}_{-1}$ (as long as the integral is taken over an arc joining z_0 and z, and lying entirely in $\mathbb{C}_{-1}$). Therefore, by the same argument $G(\lambda-1, z)$ has a (multivalued) meromorphic extension for $z \in \mathbb{C}_{-2}$. As the dummy variable ζ of the integral in (19) takes the value -1, the function $G(\lambda, \zeta+1)$ appearing in the integrand is evaluated at its branch point 0. Hence $z = -1$ is a branch point of the integral and, hence, of $G(\lambda-1, z)$. Keeping arguing in the same manner, we

can conclude that $G(\lambda-1,z)$ has a (multivalued) meromorphic extension in $\mathbb{C}_{-n}$ for any $n=1,2,\ldots$, with branch points at $z=0,1,\ldots,n-1$. And since n is arbitrary, we can finally conclude that $G(\lambda-1,z)$ has a (multivalued) meromorphic extension in $\mathbb{C}$ with branch points at the nonpositive integers. For the integral values of λ, namely for $\lambda=m$, $m=1,2,\ldots$, we know that $G(\lambda-1,z)=\Gamma^{(m-1)}(z)$ has poles of order m at $z=-n$, where $n=0,1,2,\ldots$. From the above discussion it also follows that the limit in (8) of Proposition 1 remains valid as long as z approaches zero in a continuous fashion, without ever crossing the negative real semiaxis.
Let us summarize the above observations in the following property.

Property 4. For any complex λ, the function $G(\lambda,z)$, viewed as a function of z, has a (multivalued) meromorphic extension in $\mathbb{C}$ with branch points at $z=0,-1,-2,\ldots$. Furthermore, Property 1 can be strengthen as

$$\lim_{\substack{z\to 0\\ z\in\gamma}} z^{\lambda+1}G(\lambda,z)=e^{\pi i\lambda}\Gamma(\lambda+1), \tag{21}$$

where γ is a continuous curve which does not cross the negative real semiaxis, while $z^{\lambda+1}$ is defined so that it is continuous in z and $1^{\lambda+1}=1$.

Finally, from (19) it follows easily the following property:

Property 5. For any $z\neq 0,-1,-2,\ldots$, the function $G(\lambda,z)$ is entire in λ.

3 Asymptotics of $G(\lambda,z)$ as $\lambda\to+\infty$

For typographical convenience we write (4) as

$$G(\lambda,z)=G_0(\lambda,z)+G_1(\lambda,z), \tag{22}$$

where

$$G_0(\lambda,z):=e^{\pi i\lambda}\int_0^1 e^{-t}(-\ln t)^{\lambda}\,t^{z-1}dt, \qquad \Re(\lambda)>-1, \quad \Re(z)>0, \tag{23}$$

and

$$G_1(\lambda,z):=\int_1^{\infty} e^{-(t-\lambda\ln\ln t)}t^{z-1}dt, \qquad \Re(\lambda)>-1, \quad \Re(z)>0 \tag{24}$$

(in fact, $G_1(\lambda,z)$ is entire in z).

3.1 The expansion of $G_0(\lambda,z)$

By expanding e^{-t} in (23) as a power series we obtain

$$G_0(\lambda,z)=e^{\pi i\lambda}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1(-\ln t)^{\lambda}\,t^{n+z-1}dt, \qquad \Re(\lambda)>-1, \quad \Re(z)>0. \tag{25}$$

Now, as in (13)-(14),

$$\int_0^1 (-\ln t)^\lambda \, t^{n+z-1} dt = \int_0^\infty e^{-(n+z)\tau} \tau^\lambda d\tau = \frac{\Gamma(\lambda+1)}{(n+z)^{\lambda+1}}. \tag{26}$$

Thus, (25) becomes

$$G_0(\lambda, z) = e^{\pi i \lambda} \Gamma(\lambda+1) \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)^{\lambda+1}}, \qquad \Re(\lambda) > -1, \quad \Re(z) > 0, \tag{27}$$

and the series converges very rapidly. Since for $\Re(z) > 0$

$$\frac{1}{(n+1+z)^{\lambda+1}} \ll \frac{1}{(n+z)^{\lambda+1}} \qquad \text{as } \ \lambda \to +\infty,$$

for any $n = 0, 1, 2, \ldots$, the expansion in (27) describes completely the behavior of $G_0(\lambda, z)$ as $\lambda \to +\infty$. In particular (27) implies that for any fixed $z \in \mathbb{C}_0$ we have

$$G_0(\lambda, z) = e^{\pi i \lambda} \frac{\Gamma(\lambda+1)}{z^{\lambda+1}} \left[1 + O\left(\left(\frac{z}{z+1} \right)^{\lambda+1} \right) \right], \qquad \lambda \to +\infty. \tag{28}$$

3.2 Asymptotics of $G_1(\lambda, z)$ as $\lambda \to +\infty$

Let us, first, introduce some notation. We set

$$\psi(t) := t \ln t, \qquad t \in [1, \infty) \tag{29}$$

and

$$\omega(\lambda) := \psi^{-1}(\lambda), \qquad \lambda \in [0, \infty). \tag{30}$$

The function $\omega(\lambda)$ is well defined, smooth, and strictly increasing on $[0, \infty)$, with $\omega(0) = 1$, since $\psi(t)$ is smooth and strictly increasing on $[1, \infty)$, with $\psi(1) = 0$. Furthermore, it is easy to see that

$$\omega(\lambda) = \frac{\lambda}{\ln \lambda} \left[1 + O\left(\frac{\ln \ln \lambda}{\ln \lambda} \right) \right], \qquad \lambda \to +\infty. \tag{31}$$

Lemma 1. For any fixed $z \in \mathbb{C}_0$ and for any fixed α such that

$$0 < \alpha < \frac{1}{2}, \tag{32}$$

the function $G_1(\lambda, z)$ of (24) satisfies

$$G_1(\lambda, z) = \frac{\sqrt{\pi}\, e^{\lambda \ln \ln \omega(\lambda)} e^{-\omega(\lambda)}}{\sqrt{A}} \, \omega(\lambda)^z \left[1 + O\left(\frac{1}{\lambda^\alpha} \right) \right], \qquad \lambda \to +\infty, \tag{33}$$

where $\omega(\lambda)$ is given by (29)-(30) and

$$A := \lambda \frac{1 + \ln \omega(\lambda)}{2 \ln^2 \omega(\lambda)} \tag{34}$$

(thus, from (31) we have that $A \sim \lambda/(2 \ln \lambda)$ as $\lambda \to +\infty$).

Proof. The integral of (24) can be viewed as a "Laplace integral" with large parameter λ [1]. Thus, we will try to apply the so-called Laplace method for integrals.

We begin by looking for the value of t which minimizes the function

$$h(t) := t - \lambda \ln \ln t, \qquad t \in [1, \infty), \tag{35}$$

appearing in the exponent of the integrand in (24). Since

$$h'(t) = 1 - \frac{\lambda}{t \ln t} \tag{36}$$

we have that $h(t)$ has a unique minimum attained when

$$t \ln t = \lambda$$

namely when (recall (29)-(30))

$$t = \omega(\lambda) \tag{37}$$

and, since $\omega(\lambda) > 1$ for $\lambda > 0$, this minimum is attained in the interior of the interval $[1, \infty)$, for every $\lambda > 0$.

From the theory of Laplace integrals we know that the main contribution to the value of the integral in (24), as λ gets large, comes from the values of t around $\omega(\lambda)$. In order to avoid the dependence in λ we make the substitution $t = \omega(\lambda)x$. Then, formula (24) becomes

$$G_1(\lambda, z) = \omega(\lambda)^z \int_{1/\omega(\lambda)}^{\infty} e^{-p(x)} x^{z-1} dx, \tag{38}$$

where for typographical convenience we have set

$$p(x) := h\big(\omega(\lambda)x\big) = \omega(\lambda)x - \lambda \ln(\ln \omega(\lambda) + \ln x), \qquad x \in [1/\omega(\lambda), \infty)\,. \tag{39}$$

The minimum of $p(x)$ on $[1/\omega(\lambda), \infty)$ is attained at $x = 1$. Since $p'(1) = 0$ the Taylor expansion of $p(x)$ (with remainder) about $x = 1$ up to the cubic term is

$$p(x) = p(1) + \frac{p''(1)}{2}(x-1)^2 + \frac{p'''(c)}{6}(x-1)^3, \tag{40}$$

where c is some number between 1 and x.

From (39) we have

$$p(1) = \omega(\lambda) - \lambda \ln \ln \omega(\lambda), \qquad\qquad p''(1) = \lambda \frac{1 + \ln \omega(\lambda)}{\ln^2 \omega(\lambda)}, \tag{41}$$

and

$$p'''(c) = -\lambda \left\{ \frac{2[\ln \omega(\lambda) + \ln c]^2 + 3[\ln \omega(\lambda) + \ln c] + 2}{c^3[\ln \omega(\lambda) + \ln c]^3} \right\}. \tag{42}$$

In particular, formula (42), combined with (31), imply that

$$p'''(c) = O\left(\frac{\lambda}{\ln \lambda}\right) \qquad \text{as} \quad \lambda \to +\infty, \tag{43}$$

provided $\delta \leq x \leq 1/\delta$ for some fixed $\delta \in (0,1)$. Moreover, by (40) and (41) we can see that the quadratic Taylor polynomial of $p(x)$ about $x = 1$ is

$$P(x;\lambda) := \omega(\lambda) - \lambda \ln \ln \omega(\lambda) + \lambda \frac{1 + \ln \omega(\lambda)}{2 \ln^2 \omega(\lambda)}(x-1)^2 \tag{44}$$

and it follows easily from (40), (41), (42), (44), and (32) that for all λ sufficiently large we must have

$$|p(x) - P(x;\lambda)| \leq \frac{1}{\lambda^{3\alpha-1}}, \qquad x \in I_\lambda, \tag{45}$$

where I_λ is the interval

$$I_\lambda := \left[1 - \frac{1}{\lambda^\alpha}, 1 + \frac{1}{\lambda^\alpha}\right]. \tag{46}$$

From now on let us assume, without loss of generality, that $\alpha > 1/3$, so that $1/\lambda^{3\alpha-1} \to 0$ as $\lambda \to +\infty$. Then, from (45) it follows that

$$e^{-p(x)} = e^{-P(x;\lambda)} + R(x;\lambda), \qquad x \in I_\lambda, \tag{47}$$

where

$$|R(x;\lambda)| \leq \frac{2}{\lambda^{3\alpha-1}}\, e^{-P(x;\lambda)}, \qquad x \in I_\lambda, \tag{48}$$

for all sufficiently large λ.
Multiplying by x^{z-1} and then taking integrals in (47) yields

$$\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-p(x)} x^{z-1} dx = \int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-P(x;\lambda)} x^{z-1} dx + \int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} R(x;\lambda) x^{z-1} dx. \tag{49}$$

Having formula (49), in order to complete the proof of the lemma, we need to verify the following three claims.

Claim 1. For the first integral in the right-hand side of (49) we have

$$\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-P(x;\lambda)} x^{z-1} dx = \frac{\sqrt{\pi}\, e^{-\omega(\lambda) + \lambda \ln \ln \omega(\lambda)}}{\sqrt{A}} \left[1 + O\left(\frac{1}{\lambda^\alpha}\right)\right], \qquad \lambda \to +\infty. \tag{50}$$

Claim 2. For the second integral in the right-hand side of (49) we have

$$\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} R(x;\lambda)x^{z-1}dx = \frac{e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)}}{\sqrt{A}}\, O\left(\frac{1}{\lambda^{3\alpha-1}}\right), \qquad \lambda\to+\infty. \quad (51)$$

Claim 3. For the integral in the left-hand side of (49) we have

$$\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-p(x)}x^{z-1}dx = \int_{1/\omega(\lambda)}^{\infty} e^{-p(x)}x^{z-1}dx + \frac{e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)}}{\sqrt{A}}\, o\left(\frac{1}{\lambda^{\alpha}}\right) \quad (52)$$

as $\lambda\to+\infty$.

Proof of Claim 1. In view of (44) we have

$$\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-P(x;\lambda)}x^{z-1}dx = e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)}\int_{-\lambda^{-\alpha}}^{\lambda^{-\alpha}} e^{-\lambda\frac{1+\ln\omega(\lambda)}{2\ln^2\omega(\lambda)}\xi^2}(1+\xi)^{z-1}d\xi. \quad (53)$$

Now, for $|\xi|\le\lambda^{-\alpha}$ we have (since z is fixed)

$$(1+\xi)^{z-1} = 1+O\left(\frac{1}{\lambda^{\alpha}}\right), \qquad \lambda\to+\infty. \quad (54)$$

Thus, (53) implies

$$\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-P(x;\lambda)}x^{z-1}dx = e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)}\left[1+O\left(\frac{1}{\lambda^{\alpha}}\right)\right]\int_{-\lambda^{-\alpha}}^{\lambda^{-\alpha}} e^{-A\xi^2}d\xi, \quad (55)$$

where A is given by (34).
Finally, since from (32), (31), and (34) we have that $\sqrt{A}\,\lambda^{-\alpha}\gg\lambda^{(1-2\alpha)/4}\to+\infty$, while

$$\int_0^{\sqrt{A}\,\lambda^{-\alpha}} e^{-u^2}du = \frac{\sqrt{\pi}}{2}+O\left(\frac{e^{-A\,\lambda^{-2\alpha}}}{\sqrt{A}\,\lambda^{-\alpha}}\right) \quad (56)$$

as $\lambda\to+\infty$, formula (55) implies (50).

Proof of Claim 2. Using (48) we get

$$\left|\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} R(x;\lambda)x^{z-1}dx\right| \le \frac{2}{\lambda^{3\alpha-1}}\int_{1-\lambda^{-\alpha}}^{1+\lambda^{-\alpha}} e^{-P(x;\lambda)}x^{\Re(z)-1}dx, \quad (57)$$

thus (51) follows by using (50) in (57).

Proof of Claim 3. We assume, without loss of generality, that $\lambda>1$. Then

$$\left|\int_{1+\lambda^{-\alpha}}^{\infty} e^{-p(x)}x^{z-1}dx\right| \le \int_{1+\lambda^{-\alpha}}^{2} e^{-p(x)}x^{\Re(z)-1}dx + \int_2^{\infty} e^{-p(x)}x^{\Re(z)-1}dx \quad (58)$$

From (39) we have that $p''(x) > 0$ for all $x \in [1/\omega(\lambda), \infty)$, while $p'(1) = 0$ by (29)-(30). Thus, $p(x)$ is convex on $[1/\omega(\lambda), \infty)$ and increasing on $[1, \infty)$. Consequently, for the first integral in the right-hand side of (58) we have

$$\int_{1+\lambda^{-\alpha}}^{2} e^{-p(x)} x^{\Re(z)-1} dx \leq 2^{\Re(z)} \int_{1+\lambda^{-\alpha}}^{2} e^{-p(x)} dx$$
$$\leq 2^{\Re(z)} e^{-p(1+\lambda^{-\alpha})} \int_{1+\lambda^{-\alpha}}^{2} e^{-p'(1+\lambda^{-\alpha})(x-1-\lambda^{-\alpha})} dx \tag{59}$$

(since $p(x) \geq p(1+\lambda^{-\alpha}) + p'(1+\lambda^{-\alpha})(x-1-\lambda^{-\alpha})$ on $[1/\omega(\lambda), \infty)$ by convexity). Formula (59) implies

$$\int_{1+\lambda^{-\alpha}}^{2} e^{-p(x)} x^{\Re(z)-1} dx \leq 2^{\Re(z)} \frac{e^{-p(1+\lambda^{-\alpha})}}{p'(1+\lambda^{-\alpha})}. \tag{60}$$

By (45) we have

$$p(1+\lambda^{-\alpha}) = \omega(\lambda) - \lambda \ln\ln\omega(\lambda) + \frac{1+\ln\omega(\lambda)}{2\ln^2\omega(\lambda)} \lambda^{1-2\alpha} + O\left(\frac{1}{\lambda^{3\alpha-1}}\right), \tag{61}$$

thus

$$e^{-p(1+\lambda^{-\alpha})} = e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)} \, e^{-\frac{1+\ln\omega(\lambda)}{2\ln^2\omega(\lambda)}\lambda^{1-2\alpha}} \left[1 + O\left(\frac{1}{\lambda^{3\alpha-1}}\right)\right]. \tag{62}$$

Furthermore, from the Taylor expansion with remainder of $p'(x)$ about $x = 1$, namely

$$p'(x) = p'(1) + p''(1)(x-1) + \frac{p'''(c)}{2}(x-1)^2, \qquad x \in I_\lambda,$$

together with (41), (43), and the fact that $p'(1) = 0$, we obtain

$$p'(1+\lambda^{-\alpha}) = \frac{\lambda^{1-\alpha}}{\ln\omega(\lambda)} \left[1 + O\left(\lambda^{1-2\alpha}\right)\right]. \tag{63}$$

Hence, by using (62) and (63) in (60) (and recalling (31) and the fact that $1 - 2\alpha > 0$) it follows that

$$\int_{1+\lambda^{-\alpha}}^{2} e^{-p(x)} x^{\Re(z)-1} dx = \frac{e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)}}{\sqrt{A}} \, o\left(\frac{1}{\lambda^{\alpha}}\right), \qquad \lambda \to +\infty. \tag{64}$$

As for the second integral in the right-hand side of (58), it is easy to see that it is much smaller than the bound given by (64) for the first integral in the right-hand side of (58). We can see that, e.g., by writing

$$\int_{2}^{\infty} e^{-p(x)} x^{\Re(z)-1} dx = \int_{2}^{\infty} e^{-[p(x)-p(x-3/2)]} e^{-p(x-3/2)} x^{\Re(z)-1} dx \tag{65}$$

and observing that the function $e^{-p(x-3/2)}x^{\Re(z)-1}$ is bounded on $[2,\infty)$ uniformly in λ (for, say, $\lambda > 2$).
Finally, in the same manner we can show that

$$\int_{1/\omega(\lambda)}^{1-\lambda^{-\alpha}} e^{-p(x)}x^{z-1}dx = \frac{e^{-\omega(\lambda)+\lambda\ln\ln\omega(\lambda)}}{\sqrt{A}}\, o\left(\frac{1}{\lambda^{\alpha}}\right), \qquad \lambda \to +\infty. \tag{66}$$

Therefore, (52) follows immediately by using (64) and (66) in (58). ∎

If we use the estimate (31) in (33) we get the following corollary

Corollary 1. For the function $G_1(\lambda, z)$ of (24) we have

$$G_1(\lambda, z) \sim \sqrt{\frac{\pi}{2}}\,\big[\ln\omega(\lambda)\big]^{\lambda}e^{-\omega(\lambda)}\left(\frac{\lambda}{\ln\lambda}\right)^{z-(1/2)} \qquad \text{as } \lambda \to +\infty, \tag{67}$$

where $\omega(\lambda)$ is given by (29)-(30).

Remark 1. For the function $G_1(\lambda, z)$ of (24) we have

$$G_1(m, z) = \Gamma^{(m)}(z, 1), \qquad m = 0, 1, \ldots, \tag{68}$$

where

$$\Gamma(z, 1) := \int_1^{\infty} e^{-t}t^{z-1}dt, \qquad z \in \mathbb{C}, \tag{69}$$

is the so-called upper incomplete Gamma function. Since $\Gamma(z, 1)$ is entire in z, the radius of convergence of its Taylor series about any z_0 is infinite, and this agrees with the asymptotics of $G_1(\lambda, z)$ for large λ, as given by (67).
As for the function $G_0(\lambda, z)$ of (23) we have

$$G_0(m, z) = \gamma^{(m)}(z, 1), \qquad m = 0, 1, \ldots, \tag{70}$$

where

$$\gamma(z, 1) := \int_0^{1} e^{-t}t^{z-1}dt, \qquad \Re(z) > 0, \tag{71}$$

is the so-called lower incomplete Gamma function. Since $\gamma(z, 1)$ has a simple pole at $z = 0$, the radius of convergence of its Taylor series about any z_0 with $\Re(z) > 0$ is $|z_0|$, and this agrees with the asymptotics of $G_0(\lambda, z)$ for large λ, as given by (28). Thus $G_0(\lambda, z)$ grows much faster than $G_1(\lambda, z)$ as $\lambda \to \infty$. This is, also, reflected in the corollary that follows.

Corollary 2. For the function $G(\lambda, z)$ of (3) we have

$$G(\lambda, z) = e^{\pi i\lambda}\Gamma(\lambda+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+z)^{\lambda+1}} + \sqrt{\frac{\pi}{2}}\,\big[\ln\omega(\lambda)\big]^{\lambda}e^{-\omega(\lambda)}\left(\frac{\lambda}{\ln\lambda}\right)^{z-(1/2)}[1+o(1)], \qquad \lambda \to +\infty, \tag{72}$$

where z is fixed with $\Re(z) > 0$ and $\omega(\lambda)$ is given by (29)-(30).

Corollary 2 follows immediately from (27) and Corollary 1. Since $G(m, z) = \Gamma^{(m)}(z)$ we obtain immediately from (72) the behavior of $\Gamma^{(m)}(z)$ as $m \to \infty$.

Corollary 3. For $\Re(z) > 0$ we have

$$\Gamma^{(m)}(z) = (-1)^m m! \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+z)^{m+1}} + \sqrt{\frac{\pi}{2}}\,\left[\ln \omega(m)\right]^m e^{-\omega(m)} \left(\frac{m}{\ln m}\right)^{z-(1/2)} [1 + o(1)], \qquad \lambda \to +\infty, \tag{73}$$

where $\omega(\cdot)$ is given by (29)-(30). In particular,

$$\Gamma^{(m)}(z) \sim \frac{(-1)^m m!}{z^{m+1}}, \qquad \lambda \to +\infty. \tag{74}$$

3.3 Examples

1. If we set $z = 1$ in (73), we obtain

$$\Gamma^{(m)}(1) = (-1)^m m! \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)^{m+1}} + \sqrt{\frac{\pi}{2}}\,\left[\ln \omega(m)\right]^m e^{-\omega(m)} \sqrt{\frac{m}{\ln m}}\,[1 + o(1)], \qquad \lambda \to +\infty, \tag{75}$$

in particular, $\Gamma^{(m)}(1) \sim (-1)^m m!$. Also, in view of (68), formula (67) gives

$$\Gamma^{(m)}(1,1) \sim \sqrt{\frac{\pi}{2}}\,\left[\ln \omega(m)\right]^m e^{-\omega(m)} \sqrt{\frac{m}{\ln m}} \qquad \text{as } \ m \to \infty, \tag{76}$$

where $\omega(\cdot)$ is given by (29)-(30).

2. Suppose we want the asymptotics of $G(\lambda, z)$ as $\lambda \to \infty$, in the case where z is a given complex number with $\Re(z) < 0$. Then, we can employ formula (19) or (20). For instance, for $z = -1/2$ formula (20) gives

$$G\left(\lambda - 1, -\frac{1}{2}\right) = -2^\lambda \Gamma(\lambda) - 2^\lambda e^{-\pi i \lambda} \int_{-1/2}^{0} x^{\lambda-1} G(\lambda, x+1)\, dx,$$

or

$$G\left(\lambda - 1, -\frac{1}{2}\right) = -2^\lambda \Gamma(\lambda) - e^{-\pi i \lambda} \int_0^1 (\xi - 1)^{\lambda-1} G\left(\lambda, \frac{\xi+1}{2}\right) d\xi. \tag{77}$$

We can now use (72) in (77) and get (as $\lambda \to +\infty$)

$$G\left(\lambda - 1, -\frac{1}{2}\right) = -\,2^{\lambda}\Gamma(\lambda) + e^{\pi i \lambda} 2^{\lambda+1}\Gamma(\lambda+1) J(\lambda) + \left[\sqrt{\frac{\pi}{2}} + o(1)\right]\left[\ln\omega(\lambda)\right]^{\lambda} e^{-\omega(\lambda)} \int_0^1 (\xi - 1)^{\lambda-1}\left(\frac{\lambda}{\ln\lambda}\right)^{\xi/2} d\xi \tag{78}$$

where we have set

$$J(\lambda) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 \frac{(1-\xi)^{\lambda-1}}{(2n+1+\xi)^{\lambda+1}}\, d\xi. \tag{79}$$

By invoking the formula

$$-\frac{1}{(a+b)\lambda}\,\frac{d}{d\xi}\left[\frac{(a-\xi)^{\lambda}}{(b+\xi)^{\lambda}}\right] = \frac{(a-\xi)^{\lambda-1}}{(b+\xi)^{\lambda+1}}, \tag{80}$$

(79) becomes

$$J(\lambda) = \frac{1}{2\lambda}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\,\frac{1}{(2n+1)^{\lambda}}. \tag{81}$$

Also, it is not hard to show that

$$\int_0^1 (\xi - 1)^{\lambda-1}\left(\frac{\lambda}{\ln\lambda}\right)^{\xi/2} d\xi \sim \frac{1}{\lambda}, \qquad \lambda \to +\infty. \tag{82}$$

Using (81) and (82) in (78) yields

$$G\left(\lambda - 1, -\frac{1}{2}\right) = \left(e^{\pi i \lambda} - 1\right) 2^{\lambda}\Gamma(\lambda) + e^{\pi i \lambda} 2^{\lambda}\Gamma(\lambda) \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}\,\frac{1}{(2n+1)^{\lambda}} + \frac{\left[\ln\omega(\lambda)\right]^{\lambda}}{\lambda}\, e^{-\omega(\lambda)}\left[\sqrt{\frac{\pi}{2}} + o(1)\right], \qquad \lambda \to +\infty. \tag{83}$$

By analytic continuation we have that $G(m, z) = \Gamma^{(m)}(z)$ for all $z \neq 0, -1, \ldots$. Hence, (83) implies

$$\Gamma^{(m)}\left(-\frac{1}{2}\right) = 2^{m+1} m! \left\{ \left[(-1)^{m+1} - 1\right] - (-1)^m \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}\,\frac{1}{(2n+1)^{m+1}} \right\} + \frac{\left[\ln\omega(m+1)\right]^{m+1}}{m+1}\, e^{-\omega(m+1)}\left[\sqrt{\frac{\pi}{2}} + o(1)\right], \qquad m \to \infty, \tag{84}$$

in particular,

$$\Gamma^{(2k)}\left(-\frac{1}{2}\right) \sim -2^{2k+2}(2k)! \qquad \text{and} \qquad \Gamma^{(2k+1)}\left(-\frac{1}{2}\right) \sim -\frac{2^{2k+1}(2k+1)!}{3^{2k+2}}$$

as $k \to \infty$. The above asymptotic formulas are in agreement with the fact that for $z \in (-1, 0)$ the "dominant" part of $\Gamma(z)$ is

$$D_\Gamma(z) := \frac{1}{z} - \frac{1}{z+1}$$

for which we have $D_\Gamma^{(2k)}(-1/2) = -2^{2k+2}(2k)!$, while $D_\Gamma^{(2k+1)}(-1/2) = 0$. Let us also mention that from the functional equation

$$\Gamma^{(m)}(z+1) = z\Gamma^{(m)}(1) + m\Gamma^{(m-1)}(z), \qquad z \in \mathbb{C}, \quad m = 0, 1, \ldots, \tag{85}$$

and the limits

$$\Gamma^{(2k)}(0^-) = \Gamma^{(2k)}(-1^+) = -\infty, \qquad \Gamma^{(2k+1)}(0^-) = -\Gamma^{(2k)}(-1^+) = -\infty, \tag{86}$$

it follows that $\Gamma^{(2k)}(x) < 0$ for $x \in (-1, 0)$ and hence that $\Gamma^{(2k+1)}(x)$ is decreasing on $(-1, 0)$. Consequently, for each $k = 0, 1, \ldots$ the function $\Gamma^{(2k+1)}(z)$ has a unique zero, say η_k, in $(-1, 0)$. A natural question here is whether η_k, $k \geq 0$, is a monotone sequence and, furthermore, whether $\eta_k \to -1/2$ as $k \to \infty$.

3. Let us, finally, consider the case where z is real and positive. Then formula (2) implies that $\Gamma^{(2k)}(z) > 0$ for all $k \geq 0$. Consequently, all odd derivatives $\Gamma^{(2k+1)}(z)$, $k \geq 0$, are increasing in $(0, \infty)$, with $\Gamma^{(2k+1)}(0^+) = -\infty$ and $\Gamma^{(2k+1)}(+\infty) = +\infty$. Hence, for each $k \geq 0$ there is a unique $\zeta_k \in (0, \infty)$ such that

$$\Gamma^{(2k+1)}(\zeta_k) = 0. \tag{87}$$

Suppose that the sequence ζ_k, $k \geq 0$, has a bounded subsequence, namely there is an $M > 0$ such that $\zeta_k < M$ for infinitely many values of k. Then, we should have $\Gamma^{(2k+1)}(M) > 0$ for infinitely many values of k. But this is impossible, since formula (73), or just (74), implies that $\Gamma^{(2k+1)}(M) < 0$ for all sufficiently large k. Therefore $\zeta_k \to \infty$. A natural question here is whether ζ_k is increasing.

References

[1] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Springer-Verlag, New York, 1999.