Similarity Solutions for a Multidimensional Replicator Dynamics Equation.

Vassilis G. Papanicolaou *

Department of Mathematics
National Technical University of Athens
Zografou Campus
157 80, Athens, GREECE.
e-mail: *papanico @ math.ntua.gr*

George Smyrlis

Department of Mathematics
Technological Educational Institute of Athens
Ag. Spyridonos Str.
Egaleo 12210, Athens, GREECE.
e-mail: *gsmyrlis@syros.aegean.gr*

Abstract

We construct an one-parameter family of self-similar solutions for a nonlinear degenerate multidimensional parabolic equation containing a nonlocal term. All these solutions are strictly positive and their integral over the whole space is 1. The equation serves as a replicator dynamics model where the set of strategies is a continuum.

Key words: replicator dynamics problem, self–similar solutions, nonlinear degenerate parabolic PDE with a nonlocal term, Laplacian, Dominated Convergence Theorem.

---

1 Introduction

The *replicator dynamics models* are popular models in evolutionary game theory. They have significant applications in economics, population biology, as well as in other areas of science.

* Corresponding author.
Email address: `papanico @ math.ntua.gr` (Vassilis G. Papanicolaou).

Replicator dynamics have been studied extensively in the finite dimensional case:

Let $A = (a_{ij})$ be an $m \times m$ negative matrix. The typical replicator dynamics equation is

$$u_i'(t) = \left[\sum_{j=1}^{m} a_{ij} u_j(t) - \sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} u_k(t) u_j(t) \right] u_i(t), \qquad t > 0, \quad i = 1, ..., m, (1)$$

which symbolically can be also written in the form

$$u_t = [Au - (u, Au)]\, u$$

(where $(Au)u$ is the vector whose i-th component is the product of the i-th components of (Au) and u). The matrix A is called the *payoff matrix* while $S = \{1, ..., m\}$ is the *strategy space* and the vector

$$u = (u_1(t), ..., u_m(t))^{\top},$$

is a probability distribution on S, hence we must have

$$u_j(t) \geq 0, \quad \text{for } j = 1, ..., m, \qquad \text{and} \qquad \sum_{j=1}^{m} u_j(t) = 1. \tag{2}$$

It is easy to see that if the conditions (2) are satisfied for $t = 0$, then they are satisfied for all $t \geq 0$ (under the flow (1)).

The term in the square brackets in the right hand side of equation (1) is a measure of the success of strategy i and it is assumed to be the difference of the payoff of the players playing strategy i from the average payoff of the population. It is then assumed that the logarithmic derivative of $u_i(t)$, where u_i is the percentage of the population playing i, is equal to this success measure, i.e. that agents update their strategies proportionally to the success of the strategy i. This model was introduced in [7] and [8] (see also *Wikipedia* or [3] where a stochastic version of the model is discussed).

Infinite dimensional versions of this evolutionary strategy models have been proposed, e.g., in [1] and [5] (see also the companion paper [6]) in connection to certain economic examples. However, the abstract form of the proposed equations does not allow one to obtain insight on the form of solutions. In order to make some progress in this direction, in the recent work [4] the authors restricted their attention to the case where the strategy space S is the set $\mathbb{R}$

(i.e. the real line) and the payoff operator A is the differential operator d^2/dx^2. Then (1) becomes the evolution law

$$u_t = [u_{xx} - (u, u_{xx})]\, u, \tag{3}$$

where $(\cdot, \cdot)$ denotes the usual inner product in the Hilbert space $L_2(\mathbb{R})$ of the squared-integrable real-valued functions defined on $\mathbb{R}$. The initial condition, $u(x, 0)$ is taken to be the density of a probability measure on $\mathbb{R}$.

Equation (3) is a nonlinear degenerate parabolic PDE with a nonlocal term. In [4] the authors constucted an one-parameter family of self-similar solutions for (3), namely solutions u of the form

$$u(t, x) = \frac{1}{t^\alpha} g\left(\frac{x}{t^\beta}\right).$$

All these similarity solutions are probability densities on $\mathbb{R}$, for every $t > 0$.

It is worth saying that there are situations where strategies correspond to geographical points and hence it is natural to model the set of strategies by a continuum. Also, the infinite-dimensional models lead to interesting mathematics (nonlinear, non local, degenerate parabolic PDE's with rich structure).

In the present work we study the d-dimensional case (with $d \geq 2$) where the strategy space is $S = \mathbb{R}^d$, while $A = \Delta$, namely the Laplacian acting on $\mathbb{R}^d$. In this case the corresponding replicator dynamics problem takes the form

$$u_t = [\Delta u - (u, \Delta u)]\, u, \qquad t > 0,\ x \in \mathbb{R}^d, \tag{4}$$

with

$$\int_{\mathbb{R}^d} u(0, x) dx = 1 \qquad \text{and} \qquad u(0, x) \geq 0, \quad \text{for } x \in \mathbb{R}^d. \tag{5}$$

Here $(\cdot, \cdot)$ denotes the usual inner product on the Hilbert space $L_2(\mathbb{R}^d)$, i.e.

$$(f, g) = \int_{\mathbb{R}^d} f(x) g(x) dx.$$

The main result of the article is the construction of an one-parameter family of self-similar solutions for (4)–(5), namely solutions u of the form

$$u(t, x) = \frac{1}{t^\alpha} g_d\left(\frac{r}{t^\beta}\right), \qquad \text{where} \quad r = |x| = \sqrt{x_1^2 + \cdots + x_d^2}. \tag{6}$$

All the solutions we obtained are probability densities on $\mathbb{R}^d$, for all $t>0$. It is rather unusual for a parabolic problem to have an infinitude of such solutions, since they all approach $\delta(x)$, as $t \to 0^+$.

2 The Equation for g_d

Let $u(t,x)$ be a solution of (4). By applying integration by parts (i.e. the Divergence Theorem for the vector field $u\nabla u$—also known as Green's 1st identity) one obtains

$$(u, \Delta u) = \int_{\mathbb{R}^d} u \Delta u = -\int_{\mathbb{R}^d} \nabla u \cdot \nabla u = -\int_{\mathbb{R}^d} |\nabla u|^2,$$

provided that

$$\lim_{R\to\infty} \int_{S^{d-1}(R)} u \frac{\partial u}{\partial n} = 0, \tag{7}$$

where $S^{d-1}(R)$ is the sphere of radius R in $\mathbb{R}^d$, centered at the origin, and n its outward unit normal vector.

Thus, under (7), equation (4) can be written in the equivalent form

$$u_t = \left(\Delta u + \int_{\mathbb{R}^d} |\nabla u|^2 \right) u, \qquad t>0,\ x \in \mathbb{R}^d. \tag{8}$$

Let us introduce the variable

$$s = t^{-\beta} r \tag{9}$$

(notice that $0 < s < \infty$). Then u of (6) can be also written as

$$u(t,x) = t^{-\alpha} g(s). \tag{10}$$

It follows that

$$u_t = -\alpha t^{-(\alpha+1)} g_d(s) - t^{-\alpha} g_d'(s)\, \beta t^{-(\beta+1)} r = -t^{-(\alpha+1)} \left[\alpha g_d(s) + \beta s g_d'(s) \right]. \tag{11}$$

Also, since u of (6) is radial in x, we have

$$\Delta u = u_{rr} + \frac{d-1}{r} u_r,$$

thus

$$\Delta u = t^{-\alpha-2\beta} g_d''(s) + \frac{d-1}{r} t^{-\alpha-\beta} g_d'(s) = t^{-\alpha-2\beta}\left[g_d''(s) + \frac{d-1}{s} g_d'(s) \right]. \quad (12)$$

Next let us focus on the nonlocal term of (8), namely the term $\int_{\mathbb{R}^d} |\nabla u|^2$. An easy calculation gives that for radial functions we have

$$|\nabla u|^2 = u_r^2.$$

Thus,

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \int_{\mathbb{R}^d} u_r^2 dx = \int_{S^{d-1}} \int_0^\infty u_r^2 r^{d-1} dr \, d\sigma,$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and σ the measure on S^{d-1} induced by the d-dimensional Lebesgue measure. It follows that

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \sigma_d \int_0^\infty u_r^2 r^{d-1} dr, \quad (13)$$

where σ_d is the total measure of S^{d-1}, namely

$$\sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

($\Gamma(\cdot)$ denotes, of course, the Gamma function). Now

$$u_r = t^{-\alpha-\beta} g_d'(s),$$

thus (13) yields

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = t^{-2\alpha-(2-d)\beta} K_d[g], \quad (14)$$

where we have set

$$K_d[g_d] = \sigma_d \int_0^\infty g_d'(s)^2 s^{d-1} ds. \tag{15}$$

Substituting (10), (11), (12), and (14) in (8) gives

$$-\alpha g_d(s) - \beta s g_d'(s) = t^{1-\alpha-2\beta} \left[g_d''(s) + \frac{d-1}{s} g_d'(s) \right] g_d(s) + t^{1-2\alpha-(2-d)\beta} K_d[g_d] g_d(s).$$

The only way that the above is a meaningful equation is that it does not contain t, thus

$$1 - \alpha - 2\beta = 0 \qquad \text{and} \qquad 1 - 2\alpha - (2-d)\beta = 0,$$

which gives

$$\alpha = \frac{d}{d+2}, \qquad \beta = \frac{1}{d+2}. \tag{16}$$

Finally we notice that, under (16), (6) implies

$$\int_{\mathbb{R}^d} u(t,x) dx = \sigma_d \int_0^\infty \frac{1}{t^\alpha} g_d\left(\frac{r}{t^\beta}\right) r^{d-1} dr = \sigma_d \int_0^\infty g_d(s) s^{d-1} ds,$$

independently of t. Thus, if we set

$$\sigma_d \int_0^\infty g_d(s) s^{d-1} ds = 1, \tag{17}$$

then

$$\int_{\mathbb{R}^d} u(t,x) dx = 1, \qquad \text{for all } t \geq 0.$$

The following lemma summarizes what we have done so far.

Lemma 1. Let $d \geq 2$. Then

$$u(t,x) = \frac{1}{t^\alpha} g_d\left(\frac{r}{t^\beta}\right)$$

satisfies (8) and (5) if and only if

$$\alpha = \frac{d}{d+2}, \qquad \beta = \frac{1}{d+2},$$

$$\sigma_d \int_0^\infty g_d(s)\, s^{d-1} ds = 1, \tag{18}$$

and

$$g_d(s)\, g_d''(s) + \frac{d-1}{s} g_d(s)\, g_d'(s) + \frac{s}{d+2} g_d'(s) + \frac{d}{d+2} g_d(s) + K_d[g_d] g_d(s) = 0, \tag{19}$$

where

$$K_d[g_d] = \sigma_d \int_0^\infty g_d'(s)^2 s^{d-1} ds. \tag{20}$$

We, therefore, need to show that there exist function(s) $g_d(s)$ satisfying (19) together with (20), and (18). In order to do that we must first consider an auxiliary problem.

3 The Auxiliary Problem

Consider the problem

$$q(s)\, q''(s) + \frac{d-1}{s} q(s) q'(s) + \frac{s}{d+2} q'(s) + \mu q(s) = 0, \qquad s > 0, \tag{21}$$

$$q(0) = A > 0, \qquad q'(0) = 0, \tag{22}$$

where $d \geq 2$ is a natural number and μ is a real parameter satisfying

$$\mu > \frac{d}{d+2}. \tag{23}$$

We note that the above initial conditions (22) are interpreted in the sense of limits as $s \to 0^+$.

Equation (21) can be written in the form

$$q''(s) + \frac{d-1}{s} q'(s) + \frac{s q'(s)}{(d+2) q(s)} + \mu = 0, \tag{24}$$

as long as $q(s) \neq 0$. By Proposition A1 of the Appendix we have that there is an $\varepsilon > 0$ such that (21)–(22) has a unique solution $q(s)$, for $s \in [0, \varepsilon]$.

Lemma 2. The solution $q(s)$ of (21)–(22) exists for all $s \geq 0$ and it is a strictly positive function which is decreasing on $(0, \infty)$. Moreover,

$$\int_0^\infty q'(s)^2 ds < \infty, \quad \lim_{s \to \infty} q'(s) = 0$$

and

$$\int_0^\infty q(s) ds < \infty.$$

Proof. Let $[0, b)$ $(0 < b \leq \infty)$ be the maximal existence interval of q. We shall show that $b = \infty$.

Suppose on the contrary that $0 < b < \infty$.

Then, by a well-known theorem in the theory of ordinary differential equations (see, e.g., [2, Th.4.1, Chapter 1]), we have three alternative cases:

(i) $\lim_{s \to b^-} q(s) = \infty$;

(ii) $\lim_{s \to b^-} q(s) = 0$;

(iii) $\lim_{s \to b^-} |q'(s)| = \infty$.

In order to drop (i)-(iii) we shall need two claims:

Claim 1: If q is positive on some interval $(0, s_1)$, $0 < s_1 < b$, then q' remains negative on $(0, s_1)$.

If this is not the case, set $s_1' = \inf\{s \in [0, s_1) : q'(s) \geq 0\}$. Note that $s_1' > 0$. Indeed, by view of (49) (see the Appendix) we have that $q''(0) = -\mu/d < 0$, so we may find an $r \in (0, b)$ such that $q'(s) < q'(0) = 0$, for $s \in (0, r)$.

Now we have $q'(s) < 0$, for $s \in (0, s_1')$ and $q'(s_1') = 0$. This implies that $q''(s_1') \geq 0$. However, by (24)

$$q''(s_1') = -\mu < 0,$$

a contradiction.

Claim 2: q is positive and decreasing on $[0,b)$, whereas q' is negative on $(0,b)$.

By Claim 1, it suffices to show that q is positive on $[0,b)$. If this is not true, let s_1 be the first positive zero of q. Multiplying both sides of (24) with s^{d-1} and integrating from 0 to $s\in(0,s_1)$ we get

$$\int_0^s \xi^{d-1}q''(\xi)d\xi+(d-1)\int_0^s \xi^{d-2}q'(\xi)d\xi+\frac{1}{d+2}\int_0^s \xi^{d}[\ln q(\xi)]'d\xi+\frac{\mu}{d}s^d=0.$$

Integration by parts in the above equation combined with the fact that $q'(0)=0$ gives

$$s^{d-1}q'(s)+\frac{s^d}{d+2}\ln q(s)-\frac{d}{d+2}\int_0^s \xi^{d-1}\ln q(\xi)d\xi+\frac{\mu}{d}s^d=0. \tag{25}$$

Now observe that since $q'<0$ (Claim 1) and $q>0$, on $(0,s_1)$, the function $f=-\frac{1}{d+2}\ln q$ has positive derivative on $(0,s_1)$ whereas it tends to ∞ as $s\to s_1^-$. Hence, Proposition A2 of the Appendix applied to f gives

$$\lim_{s\to s_1^-} s^{d-1}q'(s)=\infty,$$

which is impossible, since, as we have seen, q' stays negative in $(0,s_1)$.

To proceed, observe that Claim 2 immediately drops case (i), since $0<q(s)\leq q(0)=A,$ for each $s\in[0,b)$.

Moreover, since (25) holds for all $s\in(0,b)$, the Proposition A2 of the Appendix yields that case (ii) cannot be valid either.

Finally, suppose that case (iii) holds. The fact that $q'<0$ (Claim 2) forces

$$\lim_{s\to b^-} q'(s)=-\infty$$

and hence

$$\underline{\lim}_{s\to b^-} q''(s)=-\infty,$$

which contradicts (24) (recall that $q(s)>0$ by Claim 2) and also drops case (iii).

Consequently, solution $q(s)$ of (21)–(22) exists for all $s \geq 0$ and it is a strictly positive and decreasing function on $(0, \infty)$

We continue the proof of Lemma 2 by noticing that $\lim_{s\to\infty} q(s) = l \in (0, A]$. Then

$$\int_0^\infty q'(s)ds = \lim_{s\to\infty} q(s) - q(0) = l - A, \tag{26}$$

hence $q' \in L_1(0, \infty)$ (since q' is negative).

Suppose

$$\underline{\lim}_{s\to\infty} q'(s) < 0. \tag{27}$$

Then, in view of (26), (27) there is a sequence $s_n \to \infty$ such that q' attains a local minimum at s_n and

$$\lim_n q'(s_n) = -\delta, \qquad \text{for some } \delta > 0. \tag{28}$$

But since $q'(s_n)$ is a local minimum we must have $q''(s_n) = 0$, hence (24) gives

$$|q'(s_n)| = \frac{\mu(d+2)s_n q(s_n)}{(d-1)(d+2)q(s_n) + s_n^2} \leq \frac{A\mu(d+2)}{s_n}$$

(recall that q is strictly positive and decreasing on $[0, \infty)$ with $q(0) = A$).

Thus $\lim_n q'(s_n) = 0$, contradicting (28) and hence (27). We have, thus, established that

$$\lim_{s\to\infty} q'(s) = 0. \tag{29}$$

This, together with the fact that q' is integrable, implies $q' \in L_2((0, \infty))$, i.e.

$$\int_0^\infty q'(s)^2 ds < \infty.$$

Finally, integrating both sides of (21) from 1 to s ($s > 1$) and using integration by parts we obtain

$$\left(\mu - \frac{1}{d+2}\right) \int_1^s q(\xi)d\xi = -q(s)q'(s) + q(1)q'(1) + \int_1^s (q'(\xi))^2 d\xi - \frac{d-1}{2}\frac{q(s)^2}{s} +$$

$$+ \frac{d-1}{2}q(1)^2 - \frac{d-1}{2}\int_1^s \frac{q(\xi)^2}{\xi^2}d\xi - \frac{sq(s)}{d+2} + \frac{q(1)}{d+2}\,. \quad (30)$$

Recalling again that $q'(s) < 0$ and $0 < q(s) \leq A$, we get from (30) that, for each $s > 1$,

$$\left(\mu - \frac{1}{d+2}\right) \int_1^s q(\xi)d\xi \leq -Aq'(s) + \int_1^s (q'(\xi))^2 d\xi + \frac{d-1}{2}A^2 + \frac{A}{d+2}. \quad (31)$$

Since $q' \in L^2(0,\infty)$, $\lim_{s\to\infty} q'(s) = 0$, and $\mu > d/(d+2) > 1/(d+2)$, the last inequality ensures that q is integrable over $[1,\infty)$ and thus, over $[0,\infty)$.

The proof of the lemma is now complete. □

Lemma 3. Let $q\,(s)$ be the solution of the problem (21)–(22). Then the following hold:

(i) $-\int_0^\infty s^d q'(s)ds = d\int_0^\infty s^{d-1}q(s)ds < \infty$;

(ii) $\lim_{s\to\infty} s^d q(s) = 0$;

(iii) $\int_0^\infty s^d q'(s)^2 ds < \infty$.

Proof. First we notice that, since by Lemma 2 q' is negative and $\lim_{s\to\infty} q'(s) = 0$, (iii) follows immediately from (i).

We will use an inductive argument to show that for each $n \in \{1, 2, \ldots, d\}$ we have

(i′) $-\int_0^\infty s^n q'(s)ds = n\int_0^\infty s^{n-1}q(s)ds < \infty$

and

(ii′) $\lim_{s\to\infty} s^n q(s) = 0.$

To begin, observe that (i$'$) and (ii$'$) are valid for $n = 1$. Indeed, by Lemma 2 we have

$$\int_0^\infty q(s)ds < \infty,$$

whereas we know that q is positive and decreasing. By a standard result of calculus we infer that $\lim_{s\to\infty} sq(s) = 0$. Also,

$$-\int_0^\infty sq'(s)ds = -\lim_{s\to\infty} sq(s) + \int_0^\infty q(s)ds = \int_0^\infty q(s)ds < \infty.$$

Next, fix $n \in \{2, \dots, d\}$ and suppose that (i$'$) and (ii$'$) hold for $k \in \{1, 2, \dots, n-1\}$. We will show that (i$'$) and (ii$'$) also hold for $k = n$.

Multiplying both sides of (21) with s^{n-1} and integrating from 0 to s we get

$$\int_0^s \xi^{n-1}q(\xi)q''(\xi)d\xi + (d-1)\int_0^s \xi^{n-2}q(\xi)q'(\xi)d\xi + \frac{1}{d+2}\int_0^s \xi^n q'(\xi)d\xi + \mu\int_0^s \xi^{n-1}q(\xi)d\xi = 0.$$

Integration by parts applied in the above equality gives rise to

$$s^{n-1}q(s)q'(s) - \int_0^s \xi^{n-1}q'(\xi)^2 d\xi + (d-n)\int_0^s \xi^{n-2}q(\xi)q'(\xi)d\xi + \\ + \frac{q(s)}{d+2}s^n + \left(\mu - \frac{n}{d+2}\right)\int_0^s \xi^{n-1}q(\xi)d\xi = 0. \tag{32}$$

Formula (32) implies that

$$\left(\mu - \frac{n}{d+2}\right)\int_0^s \xi^{n-1}q(\xi)d\xi \le -s^{n-1}q(s)q'(s) + \int_0^s \xi^{n-1}q'(\xi)^2 d\xi - (d-n)\int_0^s \xi^{n-2}q(\xi)q'(\xi)d\xi.$$

Exploiting our hypothesis that (i$'$) , (ii$'$) hold for $k = n-1$ and taking into account that q' is negative with $\lim_{s\to\infty} q'(s) = 0$, we deduce that

$$\lim_{s\to\infty} s^{n-1}q(s) = 0, \quad \int_0^\infty \xi^{n-1}q'(\xi)^2 d\xi < \infty.$$

Meanwhile, if $n \geq 3$, our hypothesis enables us to apply (i$'$) for $k = n-2$ and get

$$\int_0^\infty \xi^{n-2} |q'(\xi)| d\xi < \infty,$$

thus,

$$(d-n) \int_0^\infty \xi^{n-2} q(\xi) |q'(\xi)| d\xi \leq A(d-n) \int_0^\infty \xi^{n-2} |q'(\xi)| d\xi < \infty.$$

Note that if $n = 2$, the above inequality still holds. Indeed,

$$\int_0^\infty q(\xi)|q'(\xi)| d\xi = -\lim_{s\to\infty} \frac{q(s)^2}{2} + \frac{A^2}{2} = \frac{A^2}{2} < A^2 = A \int_0^\infty |q'(\xi)| d\xi.$$

The above arguments combined with the facts that $\lim_{s\to\infty} q'(s) = 0$ and $\mu > d/(d+2) \geq n/(d+2)$ yield

$$\int_0^\infty s^{n-1} q(s) ds < \infty.$$

Now (32) implies that $\lim_{s\to\infty} s^n q(s) = L \in \mathbb{R}$. But, if $L \neq 0$, then $s^{n-1}q(s)$ is asymptotic to L/s, as $s \to \infty$, contradicting the fact that $s^{n-1}q(s)$ is integrable. Therefore $L = 0$, and

$$-\int_0^\infty s^n q'(s) ds = -\lim_{s\to\infty} s^n q(s) + n \int_0^\infty s^{n-1} q(s) ds = n \int_0^\infty s^{n-1} q(s) ds < \infty.$$

We have, therefore, established that (i$'$) and (ii$'$) hold for $k = n$, where $n \in \{2, \ldots, d\}$. This finishes the proof of the lemma. $\square$

Corollary 1. Let $q(s)$ be the solution of the problem (21)–(22). Then

$$\int_0^\infty s^{d-1} q'(s)^2 ds = \left(\mu - \frac{d}{d+2}\right) \int_0^\infty s^{d-1} q(s) ds.$$

Proof. For $n = d$, (32) becomes

$$s^{d-1}q(s)q'(s) - \int_0^s \xi^{d-1}q'(\xi)^2 d\xi + \frac{q(s)}{d+2}s^d + \left(\mu - \frac{d}{d+2}\right)\int_0^s \xi^{d-1}q(\xi)d\xi = 0.$$

Taking the limit, as $s \to \infty$, and employing Lemma 3, yields the desired formula. □

4 The Construction of the Self Similar Solutions

We start with two lemmas.

Lemma 4. Let $q(s)$ be the solution of the problem (21)–(22). Then

$$\|q'\|_\infty \leq \mu\sqrt{(d+2)A}, \tag{33}$$

where $\|q'\|_\infty$ denotes the supnorm of q' on $[0, \infty)$ and

$$\int_0^\infty s^{d-1}q(s)ds \geq \frac{A^{1+d/2}}{d(d+1)(\mu\sqrt{d+2}\ )^d}\ . \tag{34}$$

Proof. Since $q'(s) < 0$ in $(0, \infty)$, with $q'(0) = \lim_{s\to\infty} q'(s) = 0$, it follows that q' attains its absolute minimum at some s_m in $(0, \infty)$, and hence

$$\|q'\|_\infty = -q'(s_m) = |q'(s_m)|.$$

Also, $q''(s_m) = 0$, thus (24) implies

$$q'(s_m) = -\frac{\mu(d+2)s_m q(s_m)}{(d-1)(d+2)q(s_m) + s_m^2},$$

therefore

$$\|q'\|_\infty \leq \frac{\mu(d+2)A}{s_m} \tag{35}$$

(since q is positive, decreasing in $(0, \infty)$, and $q(0) = A$). On the other hand, since $q > 0$ and $q' < 0$, (24) implies

$$q''(s) \geq -\mu, \qquad \text{for all } s \geq 0$$

and consequently

$$q'(s) \geq -\mu s, \qquad \text{for all } s \geq 0$$

(recall that $q'(0) = 0$), in particular

$$\|q'\|_\infty = -q'(s_m) \leq \mu s_m. \tag{36}$$

By combining (35) and (36) we obtain

$$\|q'\|_\infty \leq \min\left\{\frac{\mu(d+2)A}{s_m}, \;\; \mu s_m\right\}.$$

But, no matter what s_m is, the minimum of $(d+2)\mu A/s_m$ and μs_m (since the first is decreasing in s_m, while the second is increasing) is always at most M, where

$$M = \frac{(d+2)\mu A}{s} = \mu s.$$

Thus, $s = \sqrt{(d+2)A}$ and $M = \mu\sqrt{(d+2)A}$. This establishes (33).

Next, we notice that, since

$$q(s) \geq q(0) - \|q'\|_\infty s, \qquad \text{for all } s \geq 0,$$

(33) implies that

$$q(s) \geq A - \mu\sqrt{(d+2)A}s, \qquad \text{for all } s \geq 0,$$

in particular for

$$0 \leq s \leq \frac{\sqrt{A}}{\mu\sqrt{d+2}}.$$

Thus (since $q > 0$),

$$\int\limits_0^\infty s^{d-1} q(s)ds \geq \int\limits_0^{\sqrt{A}/(\mu\sqrt{d+2})} s^{d-1} q(s)ds \geq \int\limits_0^{\sqrt{A}/(\mu\sqrt{d+2})} s^{d-1}\left(A - \mu\sqrt{(d+2)A}s\right) ds$$

$$= \frac{A^{1+d/2}}{d(d+1)(\mu\sqrt{d+2}\,)^d}\ ,$$

which is (34). $\square$

Lemma 5. If $q(s)$ is the solution of (21)–(22), then

$$q(s) \leq \frac{1}{s^{\mu(d+2)}} \exp[\,-(d+2)\mu/d\,] A^d \exp[\,d(d+2)A\,] \exp\left[\frac{\mu(d+2)^{3/2}\sqrt{A}}{s}\right], \tag{37}$$

for all $s \geq 1$.

Proof. Let us set

$$F(s) = -\frac{d}{d+2}\int\limits_0^s \xi^{d-1} \ln q(\xi)\, d\xi. \tag{38}$$

Then (25) can be written as

$$sF'(s) - dF(s) = ds^{d-1}q'(s) + \mu s^d, \tag{39}$$

which implies

$$\left[\frac{F(s)}{s^d}\right]' = \frac{\mu}{s} + \frac{d}{s^2}q'(s).$$

We, now, pick an $s \geq 1$ and integrate both sides of the above equation from 1 to s. This results to

$$\frac{F(s)}{s^d} = F(1) + \mu \ln s + d\int\limits_1^s \frac{q'(\xi)}{\xi^2} d\xi. \tag{40}$$

Since $q' < 0$ on $(0, \infty)$,

$$0 \geq d\int\limits_1^s \frac{q'(\xi)}{\xi^2} d\xi \geq d\int\limits_1^s q'(\xi)\, d\xi \geq d\int\limits_0^\infty q'(s) ds = -dq(0) = -dA,$$

hence formula (40) implies

$$\frac{F(s)}{s^d} \geq \mu \ln s + F(1) - dA.$$

Invoking (38) and (39) gives

$$\frac{F(s)}{s^d} = \frac{F'(s)}{ds^{d-1}} - \frac{q'(s)}{s} - \frac{\mu}{d} = -\frac{\ln[q(s)]}{d+2} - \frac{q'(s)}{s} - \frac{\mu}{d} \ ,$$

and the previous inequality becomes

$$-\frac{\ln[q(s)]}{d+2} - \frac{q'(s)}{s} - \frac{\mu}{d} \geq \mu \ln s + F(1) - dA,$$

which, combined with (33) implies

$$-\frac{\ln[q(s)]}{d+2} \geq \mu \ln s + F(1) + \frac{\mu}{d} - \frac{\mu\sqrt{(d+2)A}}{s} - dA.$$

It follows that

$$\ln[q(s)] \leq -\mu(d+2)\ln s - (d+2)F(1) - \frac{(d+2)\mu}{d} + \frac{\mu(d+2)^{3/2}\sqrt{A}}{s} + d(d+2)A$$

or

$$q(s) \leq \frac{1}{s^{\mu(d+2)}} \exp[-(d+2)F(1)] \exp[\ -(d+2)\mu/d\ ] \exp[\ d(d+2)A\ ] \exp[\ \mu\sqrt{(d+2)A}(d+2)/s$$

Since q is decreasing in $(0,\infty)$, we have (see (38))

$$-\frac{d+2}{d}F(1) = \int_0^1 \ln[q(s)]ds \leq \ln[q(0)] = \ln A$$

or

$$\exp[-(d+2)F(1)] \leq A^d.$$

Consequently,

$$q(s) \leq \frac{1}{s^{\mu(d+2)}} \exp[\ -(d+2)\mu/d\ ] A^d \exp[\ d(d+2)A\ ] \exp\left[\ \frac{\mu(d+2)^{3/2}\sqrt{A}}{s}\ \right].$$

□

Corollary 2. If $q\,(s)$ satisfies (21)–(22), then

$$\lim_{A\to 0^+}\int_0^\infty s^{d-1}q'(s)^2 ds = 0 \tag{41}$$

and

$$\lim_{A\to \infty}\int_0^\infty s^{d-1}q'(s)^2 ds = \infty. \tag{42}$$

Proof. By Lemma 5 we have that for each $s \geq 1$,

$$\int_1^s \xi^{d-1}q(\xi)d\xi \leq A^d \exp[d(d+2)A]\exp[\,\mu(d+2)^{3/2}\sqrt{A}\,]\int_1^s \frac{d\xi}{\xi^{\mu(d+2)-d+1}}$$

and hence (as $s\to\infty$)

$$\int_1^\infty s^{d-1}q(s)ds \leq A^d \exp[d(d+2)A]\exp[\,\mu(d+2)^{3/2}\sqrt{A}\,]\int_1^\infty \frac{ds}{s^{\mu(d+2)-d+1}}\ .$$

Since

$$\int_1^\infty \frac{ds}{s^{\mu(d+2)-d+1}} < \infty$$

(recall that $\mu > d/(d+2)$), we get that

$$\lim_{A\to 0^+}\int_1^\infty s^{d-1}q(s)ds = 0.$$

Also,

$$\int_0^1 s^{d-1}q(s)ds \leq q(0) = A,$$

thus

$$\lim_{A\to 0^+}\int_0^\infty s^{d-1}q(s)ds = 0. \tag{43}$$

Now (34) and (43) in conjunction with Corollary 1 give (41) and (42). □

Corollary 3. Let $q\,(s)$ satisfy (21)–(22), in particular $q(0) = A > 0$. Then, as a function of A, the quantity

$$I(A) = \int_0^\infty s^{d-1} q(s) ds \tag{44}$$

is continuous in $(0, \infty)$.

Proof. Let $q\,(s) = q\,(s; A)$ be the (unique) solution of the problem (21)–(22). By using an argument similar to the one in the proof of Proposition A1 of the Appendix one can verify that $q\,(s; A)$ is continuous in A, for $A > 0$. Thus $\overline{q}(s; A) = s^{d-1} q(s; A)$ too is continuous in A, for $A > 0$. For fixed A_1, A_2, with $0 < A_1 < A_2 < \infty$, the estimate (37), the monotonicity of q and the fact $\mu > d/(d+2)$ imply that the family $\{\overline{q}\,(\cdot\,; A) : A \in [A_1, A_2]\}$ is dominated by the integrable function $h(s),\ s \geq 0$, where

$$h(s) = \begin{cases} A_2, & 0 \leq s \leq 1, \\ \\ \dfrac{A_2^d}{s^{(d+2)\mu - d + 1}} \exp[\; d(d+2)A_2 \;] \exp[\; \mu \sqrt{(d+2)A_2} (d+2) \;], & s > 1. \end{cases}$$

Hence, the continuity of $I(A)$ follows by invoking the Dominated Convergence Theorem. □

We are now ready for our main theorem.

Theorem 1. For each number $\kappa \in (0, \infty)$ there is a self-similar solution of (4) and (5), namely a function $g_d(s)$ satisfying (19) together with (15), (17), such that

$$K[g_d] = \kappa.$$

Proof. Let us consider again the (unique) solution $q\,(s) = q\,(s; A)$ of the problem (21)–(22) with $\mu = \kappa + d/(d+2)$, that is

$$q\,(s)\, q''\,(s) + \frac{d-1}{s} q(s) q'(s) + \frac{s}{d+2} q'\,(s) + \frac{d}{d+2} q(s) + \kappa q\,(s) = 0, \quad s \geq 0,$$

$$q(0) = A > 0, \qquad q'(0) = 0,$$

and set

$$Q(A) = \sigma_d \int_0^\infty s^{d-1} q'(s;A)^2 ds. \tag{45}$$

Then by Corollary 1,

$$Q(A) = \sigma_d \kappa \int_0^\infty s^{d-1} q(s;A) ds,$$

hence Corollary 3 tells us that $Q(A)$ is continuous on $(0,\infty)$. Furthermore, (41) and (42) of Corollary 2 read

$$\lim_{A\to 0^+} Q(A) = 0 \qquad \text{and} \qquad \lim_{A\to\infty} Q(A) = \infty.$$

Thus $Q(A)$ takes every value between 0 and ∞. In particular there is an $A = A_\kappa$ such that

$$Q(A_\kappa) = \kappa.$$

Set

$$g_d(s) = q(s;A_\kappa).$$

Then

$$K_d[g_d] = \sigma_d \int_0^\infty s^{d-1} (g_d'(s))^2 ds = Q(A_\kappa) = \kappa,$$

thus $g_d(s)$ satisfies (20), (19). Furthermore, by Corollary 1,

$$\sigma_d \int_0^\infty s^{d-1} g_d(s)\, ds = \sigma_d \int_0^\infty s^{d-1} q(s;A_\kappa)\, ds = \frac{\sigma_d}{\kappa} \int_0^\infty q'(s;A_\kappa)^2 ds = \frac{1}{\kappa} Q(A_\kappa) = 1,$$

hence $g_d(s)$ also satisfies (18). □

Remarks. (a) As we have already pointed out, it is rather surprising that there is an infinitude of self-similar solutions.

(b) Estimate (37) and the fact that $q'(s) \to 0$, as $s \to \infty$, imply that our similarity solutions satisfy (7). Hence they also satisfy (4).

Acknowledgments. (i) The work was partially supported by a Π.Ε.Β.Ε. Grant of the National Technical University of Athens. (ii) This work was motivated by ideas and suggestions offered to us by Prof. N. Alikakos.

APPENDIX

Proposition A1. There is an $\varepsilon > 0$ such that (21)–(22) has a unique solution $q(s)$, for $s \in [0, \varepsilon]$.

Proof. By multiplying both sides of (24) by s^{d-1}, then integrating from 0 to $s > 0$, and using the initial condition $q'(0) = 0$ one gets the equivalent problem

$$s^{d-1}q'(s) + \frac{1}{d+2}\int_0^s \frac{\tau^d q'(\tau)}{q(\tau)}d\tau + \frac{\mu}{d}s^d = 0, \qquad q(0) = A.$$

Setting $p = q'$ we obtain the system

$$p(s) = -\frac{1}{(d+2)s^{d-1}}\int_0^s \frac{\tau^d p(\tau)}{q(\tau)}d\tau - \frac{\mu}{d}s\ , \tag{46}$$

$$q(s) = \int_0^s p(\tau)d\tau + A, \tag{47}$$

which, of course, is also equivalent to (21)–(22).

We intend to prove that for some $\varepsilon > 0$, the above system possesses a unique solution $(p(s), q(s))$, for $s \in [0, \varepsilon]$.

Fix an $\varepsilon > 0$ and consider the linear space $C[0, \varepsilon]$ of all real-valued continuous functions defined on the interval $[0, \varepsilon]$. Of course, $C[0, \varepsilon]$ is a Banach space under the supnorm $||\cdot||_\infty$. Furthermore, $C[0, \varepsilon] \times C[0, \varepsilon]$ is also a Banach space with norm

$$||(u, v)|| = ||u||_\infty + ||v||_\infty. \tag{48}$$

Now let us set

$$Y_\varepsilon = \{(q, p) \in C[0, \varepsilon] \times C[0, \varepsilon] : ||q - A||_\infty \leq A/2\ ,\ ||p||_\infty \leq 1\ \}.$$

Obviously, Y_ε is a closed subset of $C[0, \varepsilon] \times C[0, \varepsilon]$ and thus, it is a complete metric space with respect to the metric induced by the norm (48).

To proceed, define the mapping $\Phi_\varepsilon : Y_\varepsilon \to C[0,\varepsilon] \times C[0,\varepsilon]$ by

$$\Phi_\varepsilon[q,p](s) = \left(\int_0^s p(\tau)d\tau + A, \quad -\frac{1}{(d+2)s^{d-1}} \int_0^s \frac{\tau^d p(\tau)}{q(\tau)} d\tau - \frac{\mu}{d} s, \right).$$

It is straightforward to check that by taking ε sufficiently small, the mapping Φ_ε can be made a contraction from Y_ε into itself. Therefore Φ_ε admits a unique fixed point which is the unique solution to the system (46)–(47) on the interval $[0,\varepsilon]$. □

Remark: Differentiating both sides of (46) with respect to $s > 0$ and then taking the limit as $s \to 0^+$ we get (recall that $p = q'$)

$$\lim_{s\to 0^+} q''(s) = \lim_{s\to 0^+} p'(s) = -\mu/d\ .$$

Hence

$$q''(0) = -\mu/d\ . \tag{49}$$

Proposition A2. Let f be a differentiable function defined on the interval $(0,b)$, for some $b > 0$, with

$$f'(x) > 0,\ x \in (0,b), \qquad \text{and} \qquad \lim_{x\to b^-} f(x) = \infty.$$

Then for each $d \geq 1$,

$$\lim_{x\to b^-} \left[x^d f(x) \ - \ d \int_0^x \xi^{d-1} f(\xi) d\xi \right] = \infty.$$

Proof. Let $a \in (0,b)$. Set

$$F(x) = x^d f(x) \ - \ d \int_0^x \xi^{d-1} f(\xi) d\xi - a^d[f(x) - f(a)], \quad x \in (0,b).$$

Then $F'(x) = (x^d - a^d) f'(x)$, $x \in (0,b)$, and since $f' > 0$ on $(0,b)$, the function F attains its (global) minimum on $(0,b)$ at $x = a$. Thus for each $x \in (0,b)$,

$$F(x) \geq F(a) = a^d f(a) - d \int_0^a \xi^{d-1} f(\xi) d\xi = \int_0^a \xi^d f'(\xi) d\xi \geq 0$$

(where the last equality follows by integration by parts).

Therefore

$$x^d f(x) \ - \ d\int_0^x \xi^{d-1} f(\xi) d\xi \geq a^d[f(x) - f(a)], \quad x \in (0, b),$$

and the right-hand side of the above inequality tends to ∞, as $x \to b^-$. □

References

[1] Bomze, I., Dynamical aspects of evolutionary stability, Monaish. Mathematik **110** (1990), 189–206.

[2] Coddington, E. A. and Levinson, N. : *Theory of Ordinary Differential Equations*, Robert E. Krieger Publishing Company, Malabar, Florida, 1987.

[3] Imhof, L. A., The long-run behavior of the stochastic replicator dynamics, Ann. Appl. Probab. **15**, no. 1B (2005), 1019–1045.

[4] Kravvaritis, D., Papanicolaou, V. G., and Yannacopoulos, A., Similarity Solutions for a Replicator Dynamics Equation, Indiana Univ. Math. Journal **57**, no. 4 (2008).

[5] Oechssler, J. and Riedel, F., Evolutionary dynamics on infinite strategy spaces, Economic Theory **17** (2001), 141–162.

[6] Oechssler, J. and Riedel, F., On the dynamic foundation of evolutionary stability in continuous models, Journal of Economic Theory **107** (2002), 223–252.

[7] Smith, J. Maynard: *Evolution and the Theory of Games*, Cambridge University Press, Cambridge, UK, 1982.

[8] Taylor, P. D. and Jonker, L. B., Evolutionary Stable Strategies and Game Dynamics, Mathematical Biosciences **40** (1978), 145–156.