Erratum: Trace Formulas and the Behaviour of Large Eigenvalues [SIAM J. Math. Anal. 20 (1995), pp. 218–237]

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In [10] the value of the regularized trace of the operator $L - L_0$ was mistakenly taken to be

$$\lim_{t \downarrow 0} \frac{1}{t} \operatorname{tr}\left(e^{-tL} - e^{-tL_0}\right) = \lim_{t \downarrow 0} \frac{1}{t} \sum_{n=1}^{\infty} \left(e^{-t\mu_n} - e^{-t\nu_n}\right).$$

The error in the above expression is a minus sign omission. The expression for the regularized trace of $L - L_0$ should have been

$$-\lim_{t \downarrow 0} \frac{1}{t} \operatorname{tr}\left(e^{-tL} - e^{-tL_0}\right) = -\lim_{t \downarrow 0} \frac{1}{t} \sum_{n=1}^{\infty} \left(e^{-t\mu_n} - e^{-t\nu_n}\right). \tag{1}$$

Here L and L_0 are self-adjoint differential operators with eigenvalues μ_n and ν_n, $n \geq 1$, respectively (counting multiplicities), such that $\lim_n \mu_n = \lim_n \nu_n = +\infty$. We can think of L_0 as the "unperturbed" operator. Notice that if $\sum_{n=1}^{\infty} |\mu_n - \nu_n| < \infty$, then the right-hand side of (1) simplifies as $\sum_{n=1}^{\infty} (\mu_n - \nu_n)$, which is a neat expression of the regularized trace of $L - L_0$. Now by calculating the short-time asymptotics of the heat kernel of L and using it in the left-hand side of (1), one can obtain another expression for the regularized trace of $L - L_0$. And by equating the two expressions a trace formula is obtained. This approach of deriving trace formulas was suggested in [4] and was followed in [10]. One of its advantages is that it can be also applied to multi-dimensional operators.

The aforementioned sign omission induced some sign errors in the trace formulas presented in [10]. Below we give a list of sign corrections.

1. In the case where $L_0 = -d^2/dx^2$ acting on $L^2(0,b)$, with Dirichlet boundary conditions at $x = 0$ and $x = b$, we have $\nu_n = \pi^2 n^2/b^2$, $n \geq 1$. If $L = L_0 + q$, where $q \in C^2[0,b]$ with

$$\int_0^b q(x)dx = 0, \tag{2}$$

then we have the trace formula

$$\sum_{n=1}^{\infty}\left(\mu_n - \frac{\pi^2 n^2}{b^2}\right) = -\frac{q(0)+q(b)}{4}. \tag{3}$$

In [10] formula (3) appears twice on page 218 and, also, in formula (1.9) on page 220. In all these three cases the minus sign is missing.
Let us mention that formula (3) was first derived in [6] (in the formula presented in [6] the minus sign was also missing!). Various derivations of (3) have been presented in [3],[8], [5], [1], and [9] (in chronological order). The derivation given in [9] also covers the case of a nonreal function $q(x)$.

2. In the formula on page 222 of [10], just above formula (1.15), the minus sign is missing.

3. Let Q_0 be an $r \times r$ real symmetric matrix with constant elements. Consider the operators $L_0 = -d^2/dx^2 + Q_0$ acting on $\bigoplus_{j=1}^r L^2(0,b)$, with Dirichlet boundary conditions at $x = 0$ and $x = b$, and $L = L_0 + Q$, where Q is the $r \times r$ matrix $[q_{jk}(x)]_{j,k=1}^r$, with $q_{jk} \in C^2[0,b]$, $1 \leq j,k \leq r$, and

$$\int_0^b q_{jk}(x)dx = 0, \qquad 1 \leq j,k \leq r. \tag{4}$$

Then, we have the trace formula

$$\sum_{n=1}^{\infty}(\mu_n - \nu_n) = -\frac{\mathrm{tr}Q(0) + \mathrm{tr}Q(b)}{4}, \tag{5}$$

where μ_n and ν_n, $n \geq 1$, are the eigenvalues of L and L_0 respectively, counting multiplicities. Formula (5) is the corrected version of formula $(2.8')$ on page 225 of [10] (in $(2.8')$ the minus sign is missing).

Let us mention that the first complete proofs of formula (5) were given in [7] and [2] (in chronological order).

4. In the case where $L_0 = (-d^2/dx^2)^m$, $m = 2, 3, \ldots$, acting on $L^2(0, b)$, with Dirichlet boundary conditions at $x = 0$ and $x = b$, we have $\nu_n = \pi^{2m} n^{2m}/b^{2m}$, $n \geq 1$. If $L = L_0 + q$, where $q \in C^2[0, b]$ satisfies (2), then we have the trace formula

$$\sum_{n=1}^{\infty} (\mu_n - \nu_n) = -\frac{q(0) + q(b)}{4}. \tag{6}$$

Formula (6) is the corrected version of formula (4.15) on page 233 of [10] (in (4.15) the minus sign is missing).

5. Let $L_0 = (-\Delta)^m$, $m = 4, 5, \ldots$, acting on $L^2(D)$, $D = (0, b_1) \times (0, b_2)$, with Dirichlet boundary conditions on ∂D, and $L = L_0 + q$, where $q \in C^3[\bar{D}]$ with

$$\int_0^{b_1} q(x_1, x_2) dx_1 \equiv \int_0^{b_2} q(x_1, x_2) dx_2 \equiv 0. \tag{7}$$

Then, we have the trace formula

$$\sum_{n=1}^{\infty} (\mu_n - \nu_n) = -\frac{q(0,0) + q(b_1, 0) + q(0, b_2) + q(b_1, b_2)}{16}. \tag{8}$$

Formula (8) is the corrected version of formula (5.4) on page 236 of [10] (in (5.4) the minus sign is missing).

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