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Trace formulas and the behaviour of large eigenvalues. (English summary)
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This very interesting paper contains several results on the so-called regularized sums of eigenvalues of ordinary and partial differential equations. For differential equations, the classical notion of a trace has no sense because the eigenvalues grow unboundedly. But, if the eigenvalues obey some sharp asymptotics, it is possible to obtain convergent sums by subtracting from every eigenvalue several terms of the asympotics. The convergent series obtained in this way are called regularized sums of eigenvalues or regularized trace formulas. The problem consists in expressing the regularized sum as a functional of the coefficients of the corresponding differential equation.

In the present paper, five trace formulas are proved which we will describe briefly below.

1. Preliminaries: Let $\mu_1 < \mu_2 < \cdots < \mu_n < \cdots$ be the Dirichlet spectrum of the operator $-d^2/dx^2 + q(x)$, $q(x) \in C^2(0,b)$ acting in $(0,b)$. The asymptotic formula (1) $\mu_n = \pi^2 n^2/b^2 + C + O(n^{-2})$, $C = (1/b)\int_0^b q(x)\,dx$, is classic. Therefore, the series $s_\mu = \sum_{n=1}^\infty (\mu_n - \pi^2 n^2/b^2 - C)$ converges. In the early 1950s Gel′fand and Levitan proved the formula (2) $s_\mu = (q(0) + q(b))/4$. The author starts with an account of A. Douady's proof of formula (2). This proof is based on the investigation of the Green function of the corresponding heat equation.

2. The vector-valued function case: Put

$$\mathcal{L}_r^2(0,b) = \{u = (u_1, u_2, \cdots, u_r),\ u_j \in \mathcal{L}^2(0,b),\ 1 \le j \le r\},$$

$$(u,v) = \int_0^b u\cdot v\,dx = \int_0^b (u_1v_1 + u_2v_2 + \cdots + u_rv_r)\,dx.$$

Let (3) $L_0u = -d^2u/dx^2 + Q_0u$ and $Lu = L_0u + Q(x)u$, where Q_0 is an $r\times r$ symmetric matrix with constant entries and $Q(x) = \{q_{ij}(x)\}_{i,j=1}^r$ is a real, symmetric matrix and such that $q_{ij}(x) \in C^2(0,b)$ for all pairs i,j. The Dirichlet boundary conditions are added to expressions (3). Then L

is a selfadjoint operator in $\mathcal{L}_r^2(0,b)$.

Denote the eigenvalues of L by μ_n, $n \geq 1$, and the eigenvalues of L_0 by ν_n, $n \geq 1$. Adapting Douady's method to this case, the author obtains the regularized trace formula

$$\lim_{t\to 0} \frac{1}{t} \sum_n (e^{-t\mu_n} - e^{-t\nu_n}) = \frac{\operatorname{tr} Q(0) + \operatorname{tr} Q(b)}{4}, \tag{4}$$

under the assumption that $\int_0^b q_{ij}(x) = 0$, $1 \leq i,j \leq n$. If the (unproved) conjecture $\mu_n = \nu_n + O(1/n^2)$ holds, then from (4) it follows that

$$\sum_{n=1}^{\infty} (\mu_n - \nu_n) = \frac{\operatorname{tr} Q(0) + \operatorname{tr} Q(b)}{4}.$$

3. Case of rectangle: Let D be the domain $(0,b_1)\times(0,b_2)$, $L_0 u = \Delta u$, $Lu = L_0 u + q(x)u$, and Dirichlet boundary conditions hold. Let μ_n, $n \geq 1$, be the eigenvalues of L and ν_n the eigenvalues of L_0. We cite only one result of this section: Let $\int_0^{b_2} q(0,x_2)\,dx_2 = \int_0^{b_1} q(x_1,0)\,dx_1 = 0$. Then

$$\lim_{t\to 0} \frac{1}{t} \sum_n (e^{-\mu_n t} - e^{-\nu_n t}) = -\frac{q(0,0)+q(b_1,0)+q(0,b_2)+q(b_1,b_2)}{4} + \frac{1}{8\pi} \int_D q^2(x)\,dx.$$

4. Higher-order Schrödinger operator I: Let $L = (-\Delta)^m + q(x)$, $L_0 = (-\Delta)^m$, $x \in D \subset \mathbf{R}^d$, with Dirichlet boundary conditions ($D = (0,b_1)\times(0,b_2)\times\cdots\times(0,b_d)$). After some general useful remarks and preparations the author restricts his consideration to the one-dimensional case. The final result is the following: If $q(x) \in C^k[0,b]$, $k \geq 2$, then $\sum_n (\mu_n - \nu_n) = (q(0)+q(b))/4$.

5. Higher-order Schrödinger operators II: In this section the author proves the formula

$$\sum_n (\mu_n - \nu_n) = \tfrac{1}{16}[q(0,0)+q(b_1,0)+q(0,b_2)+q(b_1,b_2)]$$

if the number $\omega = b_1^2/b_2^2$ is Diophantine and m is sufficiently large.

{Reviewer's remark: In the beginning of the abstract and also on pages 219 and 220 the author attributes the asymptotic formula (1) to Gel′fand and Levitan. However, this formula was known long before Gel′fand and Levitan proved (2).}

Reviewed by *Boris M. Levitan*