An Inverse Spectral Result for the Periodic Euler-Bernoulli Equation

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Abstract

The Floquet (direct spectral) theory of the periodic Euler-Bernoulli equation has been developed by the author in [19], [21], and [20]. Here we begin a systematic study of the inverse periodic spectral theory, in the spirit of the corresponding theory of the second-order operator, namely the Hill's operator.

Our main result is that, if there are no pseudogaps (equivalently, if the Bloch-Floquet variety is reducible in a certain sense), then the Euler-Bernoulli operator is the square of a second-order (Hill-type) operator. This result had been conjectured by the author, in his earlier works.

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1 Introduction

The term "periodic Euler-Bernoulli equation" refers to the eigenvalue problem

$$[a(x)u''(x)]'' = \lambda\rho(x)u(x), \qquad -\infty < x < \infty, \tag{1}$$

where $a(x)$ and $\rho(x)$ are strictly positive and periodic with a common period b, satisfying the smoothness conditions $a \in C^2(\mathbb{R})$ and $\rho \in C(\mathbb{R})$. Furthermore, without loss of generality, $a(x)$ and $\rho(x)$ are normalized so that

$$\int_0^b \left[\frac{\rho(x)}{a(x)}\right]^{1/4} dx = b. \tag{2}$$

One advantage of this normalization is that the asymptotics of certain quantities, as $|\lambda| \to \infty$, become simpler, and this is the only reason that (2) is used in the present work.

The Floquet theory, i.e. the direct spectral theory, of (1) has been developed by the author in [19], [21], and [20] (in chronological order). There are theoretical as well as practical reasons for studying (1). The spectral theory of (1) is richer (analytically as well as algebraically) than its second-order counterpart (namely the Hill's equation). All the main second-order properties continue to hold, while new interesting (we can say surprising) phenomena arise which are nonexistent in the second-order case. In fact, we believe that (1) is a good "representative" of higher-order periodic spectral problems, hence the full understanding of (1) will give insight to the general higher-order case.

On the practical side, we notice that a typical application of (1) is that it models the transverse vibrations of a thin straight beam with periodic characteristics (see, e.g., [24] or [10]) and elastic structures consisting of many thin elements arranged periodically are quite common [16].

Recently there has been an increasing interest in higher order periodic eigenvalue problems (e.g. [1], [2]) and one expects that they will appear often in applied mathematics and mathematical physics as models of the physical world.

The present work initiates a systematic investigation of the inverse periodic spectral theory of (1). The goal is a theory in the spirit of [6] or [5] (see also [4], [7], [14], [15], [26]).

In Section 2 we review some facts and notions from our previous works, including the concept of the pseudospectrum, or ψ-spectrum, introduced in [21]. The section also contains some observations never published before.

In Section 3 we begin a systematic analysis of the periodic inverse spectral problem. The formulas of Section 3 are used in the next section (Section 4) in order to prove the main result of this work:

Theorem. If all pseudogaps of (1) are degenerate, then the Euler-Bernoulli operator is a perfect square of a second-order Hill-type operator, namely the product $a(x)\rho(x)$ is constant.

This statement appeared as a conjecture in our previous works [21] and [20] (and even [19] in a latent way).

2 Review of the Floquet/Spectral Theory

We start by recalling some general facts about (1) and some results established in [19], [21], and [20] (other references for Floquet or periodic spectral theory are, e.g., [3], [4], [8], Sec. XIII.7, [9], [11], [12], [13], [22]).

The problem (1) is self-adjoint (with no boundary conditions at $\pm\infty$). The underlying operator L (the "Euler-Bernoulli operator" or "beam operator") is given by

$$Lu = \rho^{-1}\left(au''\right)''.$$

The corresponding Hilbert space is the ρ-weighted space $L^2_\rho(\mathbb{R})$. Notice that L is a product of two second order differential operators, namely

$$L = L_2 L_1, \qquad \text{where} \quad L_1 u = -au'', \quad L_2 u = -\rho^{-1}u''.$$

If $a(x)\rho(x) \equiv$ const., the beam operator becomes the square of a second-order operator.

2.1 Floquet Multipliers and Floquet Solutions

Let $u_j(x) = u_j(x;\xi;\lambda)$, $j = 1,2,3,4$, be the fundamental solutions of (1) with respect to the reference point ξ, namely the solutions such that (primes refer to derivatives with respect to the first variable, namely x; δ_{jk} is the Kronecker delta)

$$u_j^{(k-1)}(\xi;\xi;\lambda) = \delta_{jk}, \quad k = 1,2, \qquad a(\xi)u_j''(\xi;\xi;\lambda) = \delta_{j3}, \qquad \left[au_j''\right]'(\xi;\xi;\lambda) = \delta_{j4}. \tag{3}$$

Each $u_j(x;\xi;\lambda)$ is entire in λ of order $1/4$. Here ξ is a given real. If $\xi = 0$, we write

$$u_j(x;\lambda) \stackrel{\text{def}}{=} u_j(x;0;\lambda).$$

The corresponding Floquet matrix $T = T(\xi;\lambda)$ is

$$T = \begin{bmatrix} u_1(\xi+b) & u_2(\xi+b) & u_3(\xi+b) & u_4(\xi+b) \\ u_1'(\xi+b) & u_2'(\xi+b) & u_3'(\xi+b) & u_4'(\xi+b) \\ a(\xi)u_1''(\xi+b) & a(\xi)u_2''(\xi+b) & a(\xi)u_3''(\xi+b) & a(\xi)u_4''(\xi+b) \\ \left[au_1''\right]'(\xi+b) & \left[au_2''\right]'(\xi+b) & \left[au_3''\right]'(\xi+b) & \left[au_4''\right]'(\xi+b) \end{bmatrix},$$

where the dependence in ξ and λ is suppressed for typographical convenience. In [19] it was shown that the eigenvalues r_1,r_2,r_3,r_4 of T (called Floquet multipliers) appear in pairs of inverses, namely

$$r_1 r_4 = r_2 r_3 = 1 \tag{4}$$

(in fact this is true for any self-adjoint ordinary differential operator with real, periodic coefficients). It follows that the characteristic equation of T has the (equivalent) forms

$$\begin{gathered} r^4 - A(\lambda)r^3 + [B(\lambda)+2]\,r^2 - A(\lambda)r + 1 = 0, \\ \left(r + r^{-1}\right)^2 - A(\lambda)\left(r + r^{-1}\right) + B(\lambda) = 0 \end{gathered} \tag{5}$$

(notice that the coefficients $A(\lambda)$ and $B(\lambda)$ do not depend on ξ). Hence, it is more appropriate to view the r_j's as the branches of the multivalued analytic function $r(\lambda)$, "living" (defined) on a four-sheeted Riemann surface which we denote by Γ. If we set

$$r = e^{ikb}, \tag{6}$$

then the characteristic equation of T becomes

$$F(\lambda; k) \overset{\text{def}}{=} B(\lambda) - 2A(\lambda)\cos(kb) + 4\cos^2(kb) = 0. \tag{7}$$

The function $F(\lambda; k)$ is entire in λ, k (it could be called the Akhiezer function). It is the analog of $\Delta(\lambda) - 2\cos(kb)$ of the Hill theory (where $\Delta(\lambda)$ is the discriminant). Equation (7), which sometimes is called the dispersion relation, defines a transcendental variety, called the Bloch-Floquet variety. A detailed study of the zeros of $F(\lambda; k)$, for any given $k \in \mathbb{C}$, can be found in [20].

The Floquet multipliers, being the roots of (5), satisfy

$$r_1 + r_4 = \frac{A(\lambda) + \sqrt{E(\lambda)}}{2} \overset{\text{def}}{=} \Delta_+(\lambda), \quad r_2 + r_3 = \frac{A(\lambda) - \sqrt{E(\lambda)}}{2} \overset{\text{def}}{=} \Delta_-(\lambda), \tag{8}$$

where $\sqrt{\cdot}$ is the principal branch of the square root function and

$$E(\lambda) \overset{\text{def}}{=} A(\lambda)^2 - 4B(\lambda). \tag{9}$$

Therefore

$$\begin{aligned} r_1 &= \frac{\Delta_+ + \sqrt{\Delta_+{}^2 - 4}}{2}, \qquad & r_4 &= \frac{\Delta_+ - \sqrt{\Delta_+{}^2 - 4}}{2}, \\ r_2 &= \frac{\Delta_- + \sqrt{\Delta_-{}^2 - 4}}{2}, \qquad & r_3 &= \frac{\Delta_- - \sqrt{\Delta_-{}^2 - 4}}{2} \end{aligned}$$

(notice that, by analytic continuation (4) holds for all λ, i.e. without permuting the indices of the r's).

Except for a discrete set of λ's, $T = T(\xi;\lambda)$ is similar to a diagonal matrix and its eigenvectors correspond to the (proper) Floquet solutions, namely to the solutions $f_j(x;\lambda)$, $j = 1, 2, 3, 4$, of (1) such that

$$f_j(x + b; \lambda) = r_j f_j(x; \lambda). \tag{10}$$

Thus, there are four linearly independent Floquet solutions if and only if T is similar to a diagonal matrix. These solutions are defined modulo a constant factor (i.e. a factor which is independent of x). To fix this ambiguity, for a given real number ξ, one introduces (see [6], [18]) the normalized Floquet solutions $\phi_j(x) = \phi_j(x;\xi;\lambda)$, $j = 1, 2, 3, 4$, for which

$$\phi_j(\xi;\xi;\lambda) = 1. \tag{11}$$

In case where $\xi = 0$, we write

$$\phi_j(x;\lambda) \stackrel{\text{def}}{=} \phi_j(x;0;\lambda),$$

and, hence, $\phi_j(0;\lambda) = 1$. In fact we have

$$\phi_j(x;\xi;\lambda) = \frac{f_j(x;\lambda)}{f_j(\xi;\lambda)} = \frac{\phi_j(x;0;\lambda)}{\phi_j(\xi;0;\lambda)} = \frac{\phi_j(x;\lambda)}{\phi_j(\xi;\lambda)} \tag{12}$$

(thus $\phi_j(x;\xi;\lambda)$ is proportional to $\phi_j(x;\lambda)$ by a factor which is independent of x).

Again it is more appropriate to view the ϕ_j's as the branches of a multivalued λ-analytic function $\phi(x;\xi;\lambda)$. In fact $\phi(x;\xi;\lambda)$ is a meromorphic function on Γ (the Riemann surface of $r(\lambda)$), whose set of poles is denoted by $\{\mu_n(\xi)\}$. For each $\mu_n(\xi)$ there is a $j \in \{1, 2, 3, 4\}$ such that

$$f_j(\xi;\mu_n(\xi)) = 0$$

(and, as one can see from (12), the normalization (11) is not possible when $\lambda = \mu_n(\xi)$).

2.2 Spectrum and Pseudospectrum

If $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\lambda'_n\}_{n=1}^{\infty}$ are respectively the periodic and antiperiodic eigenvalues of (1), then

$$0 = \lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 < \cdots.$$

The $L^2_\rho(\mathbb{R})$-spectrum $S_1(a,\rho)$ of (1) is

$$S_1(a,\rho) = \{\lambda \in \mathbb{C} : \ |r_j(\lambda)| = 1, \quad \text{for some } j\} = \bigcup_{n=1}^{\infty} B_n,$$

where $B_{2m+1} = \left[\lambda_{2m}, \lambda'_{2m+1}\right]$, $B_{2m+2} = \left[\lambda'_{2m+2}, \lambda_{2m+1}\right]$, $m = 0, 1, 2, \dots$, are the bands. Hence the gaps of the spectrum are

$$I_{2m-1} = \left(\lambda'_{2m-1}, \lambda'_{2m}\right), \qquad I_{2m} = \left(\lambda_{2m-1}, \lambda_{2m}\right), \qquad m = 1, 2, 3, \dots,$$

and empty gaps are traditionally called "closed" or degenerate. In the sequel the term "gap" will sometimes mean the closure of a gap, or a point, in case the gap is degenerate.

In [21] we proved that the zeros of $E(\lambda)$ of (9) are all real, negative, and simple or double; we denote them by

$$0 = \nu_0 > \nu'_1 \geq \nu'_2 > \nu_1 \geq \nu_2 > \nu'_3 \geq \nu'_4 > \cdots.$$

The pseudospectrum or ψ-spectrum $S_2(a, \rho)$ of (1) is

$$S_2(a, \rho) = \left\{\lambda \in \mathbb{C} : \, r_j(\lambda) = \overline{r_l(\lambda)}, \quad |r_j(\lambda)| \neq 1, \quad \text{for some } j \neq l\right\} = \bigcup_{n=1}^{\infty} \Psi_n,$$

where $\Psi_{2m+1} = \left[\nu'_{2m+1}, \nu_{2m}\right]$, $\Psi_{2m+2} = \left[\nu_{2m+1}, \nu'_{2m+2}\right]$, $m = 0, 1, 2, \dots$, are the ψ-bands.

If $a(x)\rho(x) \equiv$const. (the "perfect square" case), then all the nonzero zeros of $E(\lambda)$ are double, i.e. $E(\lambda)\lambda^{-1}$ is the square of an entire function (equivalently, all ψ-gaps are closed, i.e. empty). It follows that (see (8), (6), and (7)), if L is the square of a second-order operator, the Bloch-Floquet variety is reducible in the sense

$$F(\lambda; k) = \left[\Delta_+(\lambda) - 2\cos(kb)\right]\left[\Delta_-(\lambda) - 2\cos(kb)\right],$$

where

$$\Delta_\pm(\lambda) = \frac{A(\lambda) \pm \sqrt{E(\lambda)}}{2}$$

are entire function with respect to the variable $z = \sqrt{\lambda}$. Of course, in this case $\Delta_+(\lambda)$, as a function of $\sqrt{\lambda}$, is the discriminant of the Hill-type operator $L^{1/2}$.

2.3 A Multipoint Eigenvalue Problem

In [20] we introduced the following multipoint problem as the analog of Hill's Dirichlet problem for the Euler-Bernoulli case:

$$[a(x)u''(x)]'' = \lambda\rho(x)u(x), \qquad u(\xi) = u(\xi + b) = u(\xi + 2b) = u(\xi + 3b) = 0, \tag{13}$$

where $\xi \in \mathbb{R}$ is a given point. An eigenvalue of (13) is any value of λ for which (13) has a nontrivial solution. We call such a solution an eigenfunction of

(13). Physically the above problem describes the vibration of a beam fixed at the four points ξ, $\xi+b$, $\xi+2b$, $\xi+3b$.

Let $u_j(x;\xi;\lambda)$, $j=1,2,3,4$, be the fundamental solutions of (1) with respect to ξ. Since every solution of (1) is a linear combination of the fundamental solutions, it follows that λ is an eigenvalue of (13) if and only if λ is a zero of the function

$$H(\xi;\lambda) = \begin{vmatrix} u_2(\xi+b;\xi;\lambda) & u_3(\xi+b;\xi;\lambda) & u_4(\xi+b;\xi;\lambda) \\ u_2(\xi+2b;\xi;\lambda) & u_3(\xi+2b;\xi;\lambda) & u_4(\xi+2b;\xi;\lambda) \\ u_2(\xi+3b;\xi;\lambda) & u_3(\xi+3b;\xi;\lambda) & u_4(\xi+3b;\xi;\lambda) \end{vmatrix} \qquad (14)$$

(of course $H(\xi;\lambda)$ is entire in λ of order 1/4). We can, thus, say that the spectrum of (13) is the set of zeros of $H(\xi;\lambda)$. A simple calculation gives

$$H(\xi;0) = b^4\left(\int_0^b \frac{dx}{a(x)}\right)^2 > 0,$$

thus 0 is not an eigenvalue of (13) and

$$H(\xi;\lambda) = b^4\left(\int_0^b \frac{dx}{a(x)}\right)^2 \prod_m \left[1-\frac{\lambda}{\omega_m(\xi)}\right], \qquad (15)$$

where $\{\omega_m(\xi)\}_m$ is the spectrum of (13).

The theorem below should be compared with the property of the Hill operator stating that the Dirichlet eigenvalues are simple and their corresponding eigenfunctions are Floquet solutions.

Theorem A. Let ω be an eigenvalue of (13) and $V(\omega)$ the corresponding eigenspace, namely the vector space of all eigenfunctions of (13) associated to ω. Then $\dim V(\omega) = 1$ or 2. If $\dim V(\omega)=1$, then $V(\omega)$ is spanned by a (proper) Floquet solution; if $\dim V(\omega) = 2$, then $V(\omega)$ is spanned by two (proper) Floquet solutions, one belonging to $L^2(-\infty,0)$ and the other belonging to $L^2(0,\infty)$.

The main result regarding the spectrum of (14), i.e. the zeros of $H(\xi;\lambda)$, is the following:

Theorem B. All zeros of $H(\xi;\lambda)$, of (14), are real and they are located as follows: (a) $H(\xi;\lambda)$ has exactly one (simple) zero in the closure of each gap of the spectrum $S_1(a,\rho)$ (with the understanding that, if the gap is closed, i.e. collapses to a double periodic or antiperiodic eigenvalue, say λ^*, then the simple zero of $H(\xi;\lambda)$ is λ^*, and it is, of course, independent of ξ); (b) $H(\xi;\lambda)$ has exactly two zeros (counting multiplicities) in the closure of each ψ-gap of the pseudospectrum. In case (b), if the ψ-gap is closed, i.e. collapses to a point ν^*, where $\nu^* = \nu_{2n-1} = \nu_{2n}$, or $\nu^* = \nu'_{2n-1} = \nu'_{2n}$, for some $n = 1, 2, 3, ...$, then ν^* is a double zero of $H(\xi; \lambda)$, for any ξ. There are no other zeros of $H(\xi; \lambda)$.

The proofs of the above theorems can be found in [20].

Remark. A natural way to label the $\omega_m(\xi)$'s is the following. We take $m \in \mathbb{Z}' \stackrel{\text{def}}{=} \mathbb{Z} \setminus \{0\}$. If $m > 0$, then $\omega_m(\xi)$ is in the (closure of the) m-th spectral gap. If $m < 0$, then $\omega_m(\xi)$ is in the (closure of the) $\lfloor (1-m)/2 \rfloor$-th ψ-gap, where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. It follows from the analysis presented in [20] that each $\omega_m(\xi)$ changes continuously with ξ and:

(i) if $m > 0$, then $\omega_m(\xi)$ stays always in the (closure of the) m-th spectral gap, as ξ is moving;

(ii) if $m < 0$, then $\omega_{2m+1}(\xi)$ and $\omega_{2m}(\xi)$ stay always in the (closure of the) m-th ψ-gap.

In case (ii), given ξ, there is an index $j = j(\xi) \in \{1, 2, 3, 4\}$, and an index $l = l(\xi) \in \{1, 2, 3, 4\}$, such that the Floquet solutions $f_j(x; \xi; \lambda)$ and $f_l(x; \xi; \lambda)$ satisfy

$$f_j(\xi; \xi; \omega_{2m+1}(\xi)) = 0, \qquad f_l(\xi; \xi; \omega_{2m}(\xi)) = 0.$$

If, for the given ξ, $f_j(x; \xi; \omega_{2m+1}(\xi))$ belongs to $L^2(-\infty, 0)$ (resp. to $L^2(0, \infty)$), as a function of x, then $f_j(x; \xi; \omega_{2m+1}(\xi))$ belongs to $L^2(-\infty, 0)$ (resp. to $L^2(0, \infty)$), for all $\xi \in \mathbb{R}$, and $f_l(x; \xi; \omega_{2m}(\xi))$ belongs to $L^2(0, \infty)$ (resp. to $L^2(-\infty, 0)$), as a function of x, for all $\xi \in \mathbb{R}$. In particular $\{j(\xi) : \xi \in \mathbb{R}\} \cap \{l(\xi) : \xi \in \mathbb{R}\} = \emptyset$.

2.4 Branch Points of $r(\lambda)$ and Poles of $\phi(x; \xi; \lambda)$

The value $\lambda = \lambda_0 = 0$ is very special. The Floquet multiplier $r(\lambda)$ has a fourth root branch point there and there is only one proper Floquet solution, namely $\phi_j(x) = 1$, $j = 1, 2, 3, 4$.

The nonzero branch points of $r(\lambda)$ are of two types:

Type I branch points. These are the simple periodic and antiperiodic eigenvalues λ_n and λ'_n, $n = 1, 2, 3, \ldots$. The Type I branch points of $r(\lambda)$ are the endpoints of the nondegenerate gaps of the $L^2_\rho(\mathbb{R})$-spectrum, exactly as in the Hill case.

Type II branch points. These are the simple (nonzero) zeros ν_n and ν'_n, $n = 1, 2, 3, \ldots$, of $E(\lambda)$ of (9). The Type II branch points of $r(\lambda)$ are the endpoints of the nondegenerate ψ-gaps of the pseudospectrum.

As function of λ, $\phi(x; \xi; \lambda)$ is meromorphic on Γ (see Subsection 2.1). Theorems A and B imply that each pole $\mu_n(\xi)$ of $\phi(x; \xi; \lambda)$ is simple and its projection on $\mathbb{C}$ (since $\mu_n(\xi)$ lives on Γ) must be a zero of $H(\xi; \lambda)$. In fact, each zero of $H(\xi; \lambda)$ in a nondegenerate gap or ψ-gap is the projection of a $\mu_n(\xi)$. However, a zero ω of $H(\xi; \lambda)$ which is in a degenerate gap or ψ-gap is not the projection of a pole of $\phi(x;\xi;\lambda)$: If ω is such a zero, then ω is not a branch point of $r(\lambda)$ and there are four linearly independent Floquet solutions $f_j(x;\omega)$, $j=1,2,3,4$, which can be chosen so that $f_j(\xi;\omega)\neq 0$, for all j.

Thus each nondegenerate gap contains (the projection of) exactly one $\mu_n(\xi)$, while each nondegenerate ψ-gap contains (the projection of) exactly two $\mu_n(\xi)$'s. There is no other possibility for the poles of $\phi(x;\xi;\lambda)$. If we introduce the entire function

$$P(\xi;\lambda)=\prod_n\left[1-\frac{\lambda}{\mu_n(\xi)}\right], \tag{16}$$

then, the above discussion yields that

$$H(\xi;\lambda)=c\,(\lambda)\,P(\xi;\lambda), \tag{17}$$

where $c\,(\lambda)$ is the entire function (see (15))

$$c\,(\lambda)=b^4\left(\int_0^b\frac{dx}{a(x)}\right)^2\prod_{\lambda^*\in S_{\rm cl}}\left[1-\frac{\lambda}{\lambda^*}\right]\prod_{\nu^*\in\Psi_{\rm cl}}\left[1-\frac{\lambda}{\nu^*}\right]^2, \tag{18}$$

$S_{\rm cl}$ being the set of closed (i.e. degenerate) gaps (if any), and $\Psi_{\rm cl}$ the set of closed (i.e. degenerate) ψ-gaps (if any). The sets $S_{\rm cl}$ and $\Psi_{\rm cl}$ are, of course, independent of ξ, hence so is $c\,(\lambda)$.

In [6] it is suggested that the (*periodic*) *inverse spectral data for L* is the Riemann surface Γ (which is determined by the branch points of $r(\lambda)$) together with the set of poles $\{\mu_n=\mu_n(0)\}$ of $\phi(x;\lambda)$ (notice that each μ_n is a point on Γ, i.e. μ_n is not just a complex number, since it also contains the information: on which sheet of Γ does it lie). This is, of course, inspired by the inverse theory of the Hill's operator (see, e.g., [5], [6], [7], [14], [15], [26]).

For the Euler-Bernoulli operator, we need both the $L^2_\rho(\mathbb{R})$-spectrum and the pseudospectrum in order to determine the Riemann surface Γ and, hence, the intervals in which the μ_n's are confined. This is why the pseudospectrum plays an essential role in the Euler-Bernoulli inverse spectral theory. However, we notice that, if we know the periodic and antiperiodic eigenvalues $\{\lambda_n\}_{n\geq 1}$ and $\{\lambda'_n\}_{n\geq 1}$, then we can determine the entire functions $F(\lambda;0)$ (whose zeros are the λ_n's) and $F(\lambda;\pi/b)$ (whose zeros are the λ'_n's). Since (see (7))

$$F(\lambda;0)=B(\lambda)-2A(\lambda)+4\quad\text{and}\quad F(\lambda;\pi/b)=B(\lambda)+2A(\lambda)+4,$$

it follows $A(\lambda)$ and $B(\lambda)$, and therefore $E(\lambda)$ can be determined. Thus, if the spectrum has no degenerate gaps, then it determines the pseudospectrum. If there are degenerate gaps though, the spectrum (as a subset of $\mathbb{R}$) may not be enough to determine the pseudospectrum. If for example the spectrum is $[0,\infty)$, we believe that $a(x)$ and $\rho(x)$ are not necessarily constant and, hence, the pseudospectrum is not uniquely determined.

2.5 Asymptotics as $|\lambda|\rightarrow\infty$

We divide the complex λ-plane into 8 closed sectors S_l, $l=0,1,...,7$, defined by

$$\frac{l\pi}{4}\leq\arg(\lambda)\leq\frac{(l+1)\,\pi}{4}.$$

Then, for each S_l (see [17], Part I, Chap. II), there are four linearly independent solutions $v_{jl}(x;\lambda)$, $j=1,2,3,4$, of (1), analytic for $\lambda\in S_l$, such that, given $M>0$,

$$\left|v_{jl}(x;\lambda)-\frac{e^{\varepsilon_j\lambda^{1/4}S(x,0)}}{\rho(x)^{3/8}a(x)^{1/8}}\right|\leq K\frac{\left|e^{\varepsilon_j\lambda^{1/4}S(x,0)}\right|}{|\lambda|^{1/4}},\qquad 0\leq x\leq M, \tag{19}$$

where

$$S(x;\xi)=\int_{\xi}^{x}\left[\frac{\rho(y)}{a(y)}\right]^{1/4}dy \tag{20}$$

(in particular, due to normalization (2), $S(nb,0)=nb$, if $n\in\mathbb{Z}$). Here, $\lambda^{1/4}$ stands for the principal branch of the fourth root (so that $\Re\{\lambda^{1/4}\}\geq 0$, $\Im\{\lambda^{1/4}\}\geq 0$), $\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}=\{i,-1,-i,1\}$, and the (positive) constant K depends on $a(x)$, $\rho(x)$, and M. Similar formulas hold for the x-derivatives of $v_{jl}(x;\lambda)$.

We now present some consequences of the above formulas. If $r_j(\lambda)$, $j=1,2,3,4$, are the Floquet multipliers of (1), then

$$\left|\frac{r_j(\lambda)}{e^{\varepsilon_j\lambda^{1/4}b}}-1\right|\leq\frac{K}{|\lambda|^{1/4}},\qquad j=1,2,3,4, \tag{21}$$

where $\lambda^{1/4}$ is the principal branch of the fourth root, $\varepsilon_1=1$, $\varepsilon_2=i$, ε_3-i, $\varepsilon_4=-1$, and the constant $K>0$ depends on $a(x)$, $\rho(x)$.

If $\phi_j(x;\lambda)$, $j=1,2,3,4$, are the normalized Floquet solutions introduced in Section 2.1, then

$$\left|\phi_j(x;\lambda)-\frac{\rho(0)^{3/8}a(0)^{1/8}}{\rho(x)^{3/8}a(x)^{1/8}}e^{\varepsilon_j\lambda^{1/4}S(x,0)}\right|\leq K\frac{\left|e^{\varepsilon_j\lambda^{1/4}S(x,0)}\right|}{|\lambda|^{1/4}},\qquad 0\leq x\leq M, \tag{22}$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = i$, $\varepsilon_3 - i$, and $\varepsilon_4 = -1$. However here

$$\delta \leq \arg\lambda \leq \pi - \delta \qquad \text{or} \qquad \pi + \delta \leq \arg\lambda \leq 2\pi - \delta, \tag{23}$$

where $\delta > 0$ is any given constant. The quantities $S(x, 0)$, K, and M are as before, but K and M depend on δ. Under (23) similar formulas exist for the derivatives of $\phi_j(x;\lambda)$. Furthermore, under (23) we have

$$\frac{\phi_j'(x;\lambda)}{\phi_j(x;\lambda)} = \varepsilon_j \frac{\rho(x)^{1/4}}{a(x)^{1/4}} \lambda^{1/4} + \text{l.o.t.}, \tag{24}$$

where "l.o.t." stands for "lower order terms" and means terms whose magnitude is less than the magnitude of the leading term by a factor of (at least) $\lambda^{-1/4}$.

If $u_1(x;\xi;\lambda)$is the first fundamental solution of (1) with respect to ξ, then

$$u_1(x;\xi;\lambda) = \frac{a(\xi)^{1/8}\rho(\xi)^{3/8}}{2a(x)^{1/8}\rho(x)^{3/8}} \cdot \left\{\cosh\left[\lambda^{1/4} S(x;\xi)\right] + \cos\left[\lambda^{1/4} S(x;\xi)\right]\right\} + \text{l.o.t.},$$

$$u_1'(x;\xi;\lambda) = \frac{\lambda^{1/4} a(\xi)^{1/8}\rho(\xi)^{3/8}}{2a(x)^{3/8}\rho(x)^{1/8}} \cdot \left\{\sinh\left[\lambda^{1/4} S(x;\xi)\right] - \sin\left[\lambda^{1/4} S(x;\xi)\right]\right\} + \text{l.o.t.},$$

$$u_1''(x;\xi;\lambda) = \frac{\lambda^{1/2} a(\xi)^{1/8}\rho(\xi)^{3/8}}{2a(x)^{5/8}\rho(x)^{-1/8}} \cdot \left\{\cosh\left[\lambda^{1/4} S(x;\xi)\right] - \cos\left[\lambda^{1/4} S(x;\xi)\right]\right\} + \text{l.o.t.},$$

$$a(x)u_1'''(x;\xi;\lambda) = \frac{\lambda^{3/4} a(\xi)^{1/8}\rho(\xi)^{3/8}}{2a(x)^{-1/8}\rho(x)^{-3/8}} \cdot \left\{\sinh\left[\lambda^{1/4} S(x;\xi)\right] + \sin\left[\lambda^{1/4} S(x;\xi)\right]\right\} + \text{l.o.t.},$$

as $|\lambda| \to \infty$, where primes denote derivatives with respect to x, as usual; x and ξ are in a bounded interval and $S(x;\xi)$ is given by (20). Furthermore λ satisfies

$$\pi + \delta \leq \arg\lambda \leq 2\pi - \delta,$$

which is a little weaker condition than (23). Similar relations hold for the other fundamental solutions.

From the above it follows that, if $E(\lambda)$ and $H(\xi;\lambda)$ are the functions defined by (9) and (14) respectively, then, under (23),

$$E(\lambda) = 4\left[\cosh\left(\lambda^{1/4} b\right) - \cos\left(\lambda^{1/4} b\right)\right]^2 + \text{l.o.t.} \tag{25}$$

and

$$H(\xi;\lambda) = \frac{\sinh\left(\lambda^{1/4} b\right)\sin\left(\lambda^{1/4} b\right)}{\lambda^{3/2} a(\xi)^{1/2}\rho(\xi)^{3/2}} \cdot \left\{\cosh\left(\lambda^{1/4} b\right) - \cos\left(\lambda^{1/4} b\right)\right\}^2 + \text{l.o.t.}, \tag{26}$$

where, as usual "l.o.t." stands for "lower order terms" as explained above.

3 A Deeper Study of the Floquet Solutions $\phi_j(x;\xi;\lambda)$

The analysis presented here is expected to play an important role in the investigation of the periodic inverse spectral problem. The first application will be found in the next section.

Let $\phi_j(x;\xi;\lambda)$, $j=1,2,3,4,$ be the normalized Floquet solutions (see Subsection 2.1) and $u_j(x;\xi;\lambda)$, $j=1,2,3,4,$ the fundamental solutions of (1) with respect to ξ. Then

$$\phi_j(x;\xi;\lambda)=u_1(x;\xi;\lambda)+c_{2j}(\xi;\lambda)u_2(x;\xi;\lambda)+c_{3j}(\xi;\lambda)u_3(x;\xi;\lambda)+c_{4j}(\xi;\lambda)u_4(x;\xi;\lambda), \tag{27}$$

or, due to (12),

$$\frac{\phi_j(x;\lambda)}{\phi_j(\xi;\lambda)}=u_1(x;\xi;\lambda)+c_{2j}(\xi;\lambda)u_2(x;\xi;\lambda)+c_{3j}(\xi;\lambda)u_3(x;\xi;\lambda)+c_{4j}(\xi;\lambda)u_4(x;\xi;\lambda).$$

The coefficients $c_{2j}(\xi;\lambda)$, $c_{3j}(\xi;\lambda)$, and $c_{4j}(\xi;\lambda)$ are the Weyl-Kodaira m-functions. They, too, are branches of meromorphic functions living on Γ. Differentiating with respect to x yields

$$\frac{\phi_j'(x;\lambda)}{\phi_j(\xi;\lambda)}=u_1'(x;\xi;\lambda)+c_{2j}(\xi;\lambda)u_2'(x;\xi;\lambda)+c_{3j}(\xi;\lambda)u_3'(x;\xi;\lambda)+c_{4j}(\xi;\lambda)u_4'(x;\xi;\lambda). \tag{28}$$

By setting $x=\xi$ in the above formula one obtains

$$\phi_j(x;\xi;\lambda)=\frac{\phi_j(x;\lambda)}{\phi_j(\xi;\lambda)}=e^{\int_\xi^x c_{2j}(\eta;\lambda)d\eta}. \tag{29}$$

Since $[1,c_{2j}(\xi;\lambda),c_{3j}(\xi;\lambda),c_{4j}(\xi;\lambda)]^\top$ is the eigenvector of $T(\xi;\lambda)$ corresponding to the eigenvalue $r_j(\lambda)$, we must have

$$c_{2j}(\xi;\lambda)=\frac{\begin{vmatrix} -u_1'(\xi+b) & u_3'(\xi+b) & u_4'(\xi+b) \\ -a(\xi)u_1''(\xi+b) & a(\xi)u_3''(\xi+b)-r_j(\lambda) & a(\xi)u_4''(\xi+b) \\ -\left[au_1''\right]'(\xi+b) & \left[au_3''\right]'(\xi+b) & \left[au_4''\right]'(\xi+b)-r_j(\lambda)\end{vmatrix}}{\begin{vmatrix} u_2'(\xi+b)-r_j(\lambda) & u_3'(\xi+b) & u_4'(\xi+b) \\ a(\xi)u_2''(\xi+b) & a(\xi)u_3''(\xi+b)-r_j(\lambda) & a(\xi)u_4''(\xi+b) \\ \left[au_2''\right]'(\xi+b) & \left[au_3''\right]'(\xi+b) & \left[au_4''\right]'(\xi+b)-r_j(\lambda)\end{vmatrix}},$$

where the dependence of $u_j^{(l)}$ in ξ and λ is suppressed for typographical convenience. If λ is in a degenerate gap or ψ-gap, then this formula gives $0/0$ and $c_{2j}(\xi;\lambda)$ has a removable singularity.

Starting from (27), there are many other ways to express $c_{2j}(\xi;\lambda)$. We mention one:

If n is an integer, then, of course,

$$\phi_j(\xi + nb; \xi; \lambda) = r_j(\lambda)^n,$$

thus we can set $x = \xi + nb$ in (27) and get

$$r_j{}^n = u_1(\xi+nb;\xi)+c_{2j}(\xi)u_2(\xi+nb;\xi)+c_{3j}(\xi)u_3(\xi+nb;\xi)+c_{4j}(\xi)u_4(\xi+nb;\xi)$$

(here the dependence in λ is suppressed for typographical convenience). Writing this formula for $n = 1, 2, 3$ and then solving for $c_{2j}(\xi; \lambda)$ yields

$$c_{2j}(\xi;\lambda) = \frac{\begin{vmatrix} r_j(\lambda) - u_1(\xi+b;\xi;\lambda) & u_3(\xi+b;\xi;\lambda) & u_4(\xi+b;\xi;\lambda) \\ r_j(\lambda)^2 - u_1(\xi+2b;\xi;\lambda) & u_3(\xi+2b;\xi;\lambda) & u_4(\xi+2b;\xi;\lambda) \\ r_j(\lambda)^3 - u_1(\xi+3b;\xi;\lambda) & u_3(\xi+3b;\xi;\lambda) & u_4(\xi+3b;\xi;\lambda) \end{vmatrix}}{\begin{vmatrix} u_2(\xi+b;\xi;\lambda) & u_3(\xi+b;\xi;\lambda) & u_4(\xi+b;\xi;\lambda) \\ u_2(\xi+2b;\xi;\lambda) & u_3(\xi+2b;\xi;\lambda) & u_4(\xi+2b;\xi;\lambda) \\ u_2(\xi+3b;\xi;\lambda) & u_3(\xi+3b;\xi;\lambda) & u_4(\xi+3b;\xi;\lambda) \end{vmatrix}},$$

where the denominator is $H(\xi; \lambda)$ of (14) (which is entire in λ). Thus the above formula can be written (see (29)) as

$$c_{2j}(\xi;\lambda) = \frac{\phi_j'(\xi;\lambda)}{\phi_j(\xi;\lambda)} = \frac{\gamma_0(\xi;\lambda) + \gamma_1(\xi;\lambda)\,r_j(\lambda) + \gamma_2(\xi;\lambda)\,r_j(\lambda)^2 + \gamma_3(\xi;\lambda)\,r_j(\lambda)^3}{H(\xi;\lambda)}, \tag{30}$$

where $\gamma_j(\xi; \lambda)$, $j = 0, 1, 2, 3$, are entire in λ.

We continue with the observation that (see (16) and (17))

$$\phi_1(x;\lambda)\phi_2(x;\lambda)\phi_3(x;\lambda)\phi_4(x;\lambda) = \frac{P(x;\lambda)}{P(0;\lambda)} = \frac{H(x;\lambda)}{H(0;\lambda)}. \tag{31}$$

Hence (primes denote derivatives with respect to x)

$$\frac{[\phi_1(x;\lambda)\phi_2(x;\lambda)\phi_3(x;\lambda)\phi_4(x;\lambda)]'}{\phi_1(x;\lambda)\phi_2(x;\lambda)\phi_3(x;\lambda)\phi_4(x;\lambda)} = \frac{P'(x;\lambda)}{P(x;\lambda)} = \frac{H'(x;\lambda)}{H(x;\lambda)}$$

or

$$\frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)} + \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)} + \frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} + \frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} = \frac{P'(x;\lambda)}{P(x;\lambda)} = \frac{H'(x;\lambda)}{H(x;\lambda)}. \tag{32}$$

Next we use (30), (4), and (8) to obtain

$$\frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)} + \frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} = \frac{\alpha_0(x;\lambda) + \alpha_1(x;\lambda)\sqrt{E(\lambda)}}{H(x;\lambda)}, \tag{33}$$

where $\alpha_0(x;\lambda)$ and $\alpha_1(x;\lambda)$ are entire in λ. Likewise

$$\frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)} + \frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} = \frac{\alpha_0(x;\lambda) - \alpha_1(x;\lambda)\sqrt{E(\lambda)}}{H(x;\lambda)}. \tag{34}$$

Adding up the last two equalities and then invoking (32) yields

$$2\alpha_0(x;\lambda) = H'(x;\lambda).$$

Finally using again (30), (4), and (8) one gets

$$\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)} = \frac{[r_3(\lambda) - r_2(\lambda)]\left[\beta_0(x;\lambda) - \beta_1(x;\lambda)\sqrt{E(\lambda)}\right]}{H(x;\lambda)} \tag{35}$$

and

$$\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)} = \frac{[r_4(\lambda) - r_1(\lambda)]\left[\beta_0(x;\lambda) + \beta_1(x;\lambda)\sqrt{E(\lambda)}\right]}{H(x;\lambda)} \tag{36}$$

where $\beta_0(x;\lambda)$ and $\beta_1(x;\lambda)$ are entire in λ.

4 The Absence of Nondegenerate Pseudogaps

We consider the following inverse periodic spectral problem: If there are no open ψ-gaps, i.e. if the pseudospectrum is $(-\infty, 0]$, what can we say about the operator L? The assumption of the nonexistence of open ψ-gaps has some equivalent versions. We mention few of them:

(i) The Floquet multiplier $r(\lambda)$ has no Type II branch points (see Subsection 2.4);

(ii) The entire function $E(\lambda)$ of (9) has the form

$$E(\lambda) = \lambda E_1(\lambda)^2, \tag{37}$$

where $E_1(\lambda)$ is entire;

(iii) The function $F(\lambda;k)$ of (7) can be factored as

$$F(\lambda;k) = [\Delta_+(\lambda) - 2\cos(kb)]\,[\Delta_-(\lambda) - 2\cos(kb)]\,,$$

where $\Delta_\pm(\lambda)$ are entire functions with respect to the variable $z = \sqrt{\lambda}$. Thus, the Bloch-Floquet variety is reducible in this sense.

Version (iii) can be also expressed by saying that the function

$$\widetilde{F}(\lambda;\zeta) = B(\lambda) - 2A(\lambda)\zeta + 4\zeta^2,$$

can be factored in a nontrivial way as

$$\widetilde{F}(\lambda;\zeta) = F_1(\lambda;\zeta)F_2(\lambda;\zeta),$$

where $F_1(\lambda;\zeta)$ and $F_2(\lambda;\zeta)$ are entire functions with respect to the variables $z = \sqrt{\lambda}$ and ζ. Here "nontrivial" means that none of the above factors is constant nor it has the form $e^{g(\sqrt{\lambda},\zeta)}$, with $g(z,\zeta)$ entire in z and ζ.

We believe that (iii) is equivalent to the statement: The function $F(\lambda;k)$ of (7) can be factored in a nontrivial way as

$$F(\lambda;k) = F_1(\lambda;k)F_2(\lambda;k),$$

where $F_1(\lambda;k)$ and $F_2(\lambda;k)$ are entire functions with respect to the variables $z = \sqrt{\lambda}$ and k.

In this section we will prove the following theorem that had appeared as a conjecture in our earlier works:

Theorem. If all pseudogaps of (1) are degenerate, i.e. if (37) holds, then the Euler-Bernoulli operator L is a perfect square of a second-order Hill-type operator, namely the product $a(x)\rho(x)$ is constant.

We first give two lemmas.

Lemma 1. Let $g(z)$ be entire of order $\leq 1/2$. If there are two angles θ_1 and θ_2, $0 \leq \theta_1 < \theta_2 < 2\pi$, such that $g(z)$ is bounded on the half-lines $\arg z = \theta_1$ and $\arg z = \theta_2$, then $g(z)$ is constant.

Proof. The statement is an immediate consequence of the Phragmén-Lindelöf Theorem(s) for sectors (see, e.g. [25]). ■

If the function $g(z)$ above is bounded on the half-lines by a power of z, then it must be a polynomial. Also a statement similar to Lemma 1 holds for an entire function $g(z)$ whose order is bounded by some number M, but, in general the function must be bounded on more than two half-lines (the number of the half-lines and the maximum angle that two consecutive half-lines can have depend on M), in order to conclude that $g(z)$ is constant.

Lemma 2. If (37) holds, then

$$\begin{vmatrix} \phi_2(x;\lambda) & \phi_3(x;\lambda) \\ \phi_2'(x;\lambda) & \phi_3'(x;\lambda) \end{vmatrix} \begin{vmatrix} \phi_1(x;\lambda) & \phi_4(x;\lambda) \\ \phi_1'(x;\lambda) & \phi_4'(x;\lambda) \end{vmatrix} = \frac{w(\lambda)}{a(x)\rho(x)}, \tag{38}$$

where $w(\lambda)$ is (of course independent of x and) meromorphic on Γ.

Proof. Under the assumption (37), (35) becomes

$$\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)} = \frac{[r_3(\lambda) - r_2(\lambda)]\left[\beta_0(x;\lambda) - \widehat{\beta}_1(x;\lambda)\sqrt{\lambda}\right]}{H(x;\lambda)}, \tag{39}$$

where $\widehat{\beta}_1(x;\lambda) = \beta_1(x;\lambda)\, E_1(\lambda)$ is entire in λ.

Let

$$\widetilde{H}(x;\lambda) = \prod_{m\geq 1}\left[1 - \frac{\lambda}{\omega_m(x)}\right],$$

which means that $\widetilde{H}(x;\lambda)$ is entire in λ (of order 1/4) whose zeros are the positive zeros of $H(x;\lambda)$. Then (37) and Theorem B imply that

$$H(x;\lambda) = c_0\lambda^{-1}E(\lambda)\widetilde{H}(x;\lambda),$$

where c_0 is a constant. Hence (39) becomes

$$\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)} = \frac{\left[r_3(\lambda) - r_2(\lambda)\right]\left[\beta_0(x;\lambda) - \widehat{\beta}_1(x;\lambda)\sqrt{\lambda}\right]}{c_0\lambda^{-1}E(\lambda)\widetilde{H}(x;\lambda)} \tag{40}$$

and in the same way (35) becomes

$$\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)} = \frac{\left[r_4(\lambda) - r_1(\lambda)\right]\left[\beta_0(x;\lambda) + \widehat{\beta}_1(x;\lambda)\sqrt{\lambda}\right]}{c_0\lambda^{-1}E(\lambda)\widetilde{H}(x;\lambda)}. \tag{41}$$

Now let $\nu \neq 0$ be a zero of $E(\lambda)$. Then (37) implies (see Subsection 2.4) that $\phi_j'(x;\nu)/\phi_j(x;\nu)$ is finite for all $j = 1,2,3,4$ (the poles of $\phi_j'(x;\lambda)/\phi_j(x;\lambda)$ are included in the zeros of $\widetilde{H}(x;\lambda)$). Also $r_3(\nu) - r_2(\nu) \neq 0$ and $r_4(\nu) - r_1(\nu) \neq 0$. Thus (40) and (41) imply that

$$\frac{\beta_0(x;\lambda) - \widehat{\beta}_1(x;\lambda)\sqrt{\lambda}}{\lambda^{-1}E(\lambda)} \qquad \text{and} \qquad \frac{\beta_0(x;\lambda) + \widehat{\beta}_1(x;\lambda)\sqrt{\lambda}}{\lambda^{-1}E(\lambda)}$$

are finite for all $\lambda \in \mathbb{C}$. Therefore

$$\widetilde{\beta}_0(x;\lambda) = \frac{\beta_0(x;\lambda)}{\lambda^{-1}E(\lambda)} \qquad \text{and} \qquad \widetilde{\beta}_1(x;\lambda) = \frac{\widehat{\beta}_1(x;\lambda)}{\lambda^{-1}E(\lambda)}$$

are entire in λ (whose order has to be at most 1/4) and (40) and (41) can be written as

$$\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)} = \frac{\left[r_3(\lambda) - r_2(\lambda)\right]E(\lambda)\left[\widetilde{\beta}_0(x;\lambda) - \widetilde{\beta}_1(x;\lambda)\sqrt{\lambda}\right]}{\lambda H(x;\lambda)} \tag{42}$$

and

$$\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)} = \frac{\left[r_4(\lambda) - r_1(\lambda)\right]E(\lambda)\left[\widetilde{\beta}_0(x;\lambda) + \widetilde{\beta}_1(x;\lambda)\sqrt{\lambda}\right]}{\lambda H(x;\lambda)}$$

respectively. Multiplying these formulas one gets

$$\left[\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)}\right]\left[\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)}\right] =$$

$$\frac{\left[r_3(\lambda) - r_2(\lambda)\right]\left[r_4(\lambda) - r_1(\lambda)\right]E(\lambda)}{\lambda H(x;\lambda)} \cdot \frac{E(\lambda)\left[\widetilde{\beta}_0(x;\lambda)^2 - \widetilde{\beta}_1(x;\lambda)^2\lambda\right]}{\lambda H(x;\lambda)}. \tag{43}$$

Now (24) gives

$$\left[\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)}\right]\left[\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)}\right] = 4i\frac{\rho(x)^{1/2}}{a(x)^{1/2}}\lambda^{1/2} + \text{l.o.t.}, \tag{44}$$

while (21), (25), and (26) yield

$$\frac{\left[r_3(\lambda) - r_2(\lambda)\right]\left[r_4(\lambda) - r_1(\lambda)\right]E(\lambda)}{\lambda H(x;\lambda)} = 16i\lambda^{1/2}a(x)^{1/2}\rho(x)^{3/2} + \text{l.o.t.}, \tag{45}$$

where in both cases λ must obey the condition (23).

Next we notice that (31) implies that the functions

$$\left[\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)}\right]\left[\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)}\right]$$

and

$$\frac{\left[r_3(\lambda) - r_2(\lambda)\right]\left[r_4(\lambda) - r_1(\lambda)\right]E(\lambda)}{\lambda H(x;\lambda)}$$

have the same poles (viewed as meromorphic functions of λ, living on Γ). Using this observation in (43) we get the important conclusion that

$$\frac{E(\lambda)\left[\widetilde{\beta}_0(x;\lambda)^2 - \widetilde{\beta}_1(x;\lambda)^2\lambda\right]}{\lambda H(x;\lambda)}$$

must be <u>entire</u> in λ(!). Furthermore, if we use the asymptotic formulas (44) and (45) in (43) we get (under (23)) that

$$\frac{E(\lambda)\left[\widetilde{\beta}_0(x;\lambda)^2 - \widetilde{\beta}_1(x;\lambda)^2\lambda\right]}{\lambda H(x;\lambda)} = \frac{1}{4a(x)\rho(x)} + \text{l.o.t.},$$

but, then by Lemma 1 this entire function must be constant (with respect to λ), thus

$$\frac{E(\lambda)\left[\widetilde{\beta}_0(x;\lambda)^2 - \widetilde{\beta}_1(x;\lambda)^2\lambda\right]}{\lambda H(x;\lambda)} = \frac{1}{4a(x)\rho(x)}.$$

It follows that (43) becomes

$$\left[\frac{\phi_3'(x;\lambda)}{\phi_3(x;\lambda)} - \frac{\phi_2'(x;\lambda)}{\phi_2(x;\lambda)}\right]\left[\frac{\phi_4'(x;\lambda)}{\phi_4(x;\lambda)} - \frac{\phi_1'(x;\lambda)}{\phi_1(x;\lambda)}\right] = \frac{(r_3 - r_2)(r_4 - r_1)E(\lambda)}{\lambda H(x;\lambda)} \cdot \frac{1}{4a(x)\rho(x)},$$

where the dependence of r_j in λ is suppressed for typographical economy. Finally we invoke again (31) and get

$$\begin{vmatrix} \phi_2(x;\lambda) & \phi_3(x;\lambda) \\ \phi_2'(x;\lambda) & \phi_3'(x;\lambda) \end{vmatrix}\begin{vmatrix} \phi_1(x;\lambda) & \phi_4(x;\lambda) \\ \phi_1'(x;\lambda) & \phi_4'(x;\lambda) \end{vmatrix} = \frac{(r_3 - r_2)(r_4 - r_1)H(0;\lambda)\lambda^{-1}E(\lambda)}{4a(x)\rho(x)},$$

which is (38). ■

Proof of the Theorem. Having established (38), to finish the proof of the theorem we can just look at the behavior of the left hand side of (38) near $\lambda = 0$. We have

$$r_j(\lambda) = 1 + \varepsilon_j a_0 b \lambda^{1/4} + O\left(\lambda^{1/2}\right), \qquad j = 1, 2, 3, 4,$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = i$, $\varepsilon_3 - i$, $\varepsilon_4 = -1$ and a_0 is a positive constant (b is the period of $a(x)$ and $\rho(x)$, as usual). Also

$$\phi_j(x;\lambda) = 1 + \varepsilon_j a_0 \lambda^{1/4} x + O\left(\lambda^{1/2}\right), \qquad j = 1, 2, 3, 4,$$

and

$$\phi_j'(x;\lambda) = \varepsilon_j a_0 \lambda^{1/4} + O\left(\lambda^{1/2}\right), \qquad j = 1, 2, 3, 4.$$

Therefore, as $\lambda \to 0$,

$$\begin{vmatrix} \phi_2(x;\lambda) & \phi_3(x;\lambda) \\ \phi_2'(x;\lambda) & \phi_3'(x;\lambda) \end{vmatrix}\begin{vmatrix} \phi_1(x;\lambda) & \phi_4(x;\lambda) \\ \phi_1'(x;\lambda) & \phi_4'(x;\lambda) \end{vmatrix} = 4ia_0^2\lambda^{1/2} + o\left(\lambda^{1/2}\right),$$

and hence, by comparing with (38) we conclude that $a(x)\rho(x)$ must be a constant. ■

Final Remarks. As we have already mentioned, it is straightforward to see that the converse of this theorem is true. Notice that the theorem implies that if the (periodic) inverse spectral data for L is given (see Subsection 2.4) and from this data it follows that the Riemann surface Γ has no Type II branch points, then $L^{1/2}$ can be recovered (being a Hill-type operator) and hence the inverse problem for L can be solved.

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