Some Initial Value Problems Containing a Large Parameter

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Abstract

We derive the asymptotic behavior of the solution of the problem.

$$w''(t) + \frac{\alpha}{t} w'(t) + B^2 \rho(t)^2 w(t) = 0, \quad t > 0,$$

$$w(0) = 1, \qquad w'(0) = 0,$$

as $B \to \infty$. Here $\alpha > 0$ and $\rho(t) > 0$. We also discuss the asymptotics of the nonlinear Schrödinger-type problem

$$u'' + \frac{\alpha}{t} u' + u^{2p+1} = 0, \qquad t > 0,$$

$$u(0) = \gamma, \qquad u'(0) = 0,$$

as $\gamma \to \infty$.

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1 INTRODUCTION

Consider the initial value problem

$$w''(t) + \frac{\alpha}{t} w'(t) + B^2 \rho(t)^2 w(t) = 0, \quad t > 0, \tag{1}$$

$$w(0) = 1, \qquad w'(0) = 0, \tag{2}$$

where $\rho(t)$ is twice continuously differentiable and strictly positive on a given interval $[0, b]$, α is a fixed strictly positive real number, and B is a large parameter.

Such problems arise, e.g., when we are interested in the behavior of the radially symmetric and bounded solution of the multidimensional equation

$$\Delta w + B^2 \rho(r)^2 w = 0$$

(where $r = \sqrt{x_1^2 + \cdots + x_n^2}$), as $B \to \infty$.

In Section 2 of this note we compute the asymptotics of $w(t)$, as $B \to \infty$, using special Liouville-type transformations, asymptotic matching, and the WKB approximation.

In Section 3 we discuss the asymptotics of the solution $u(t)$ of the nonlinear problem

$$u'' + \frac{\alpha}{t} u' + u^{2p+1} = 0, \qquad t > 0,$$

$$u(0) = \gamma, \qquad u'(0) = 0,$$

as $\gamma \to \infty$. In the case where α is a positive integer, the above equation is equivalent to a multidimensional nonlinear Schrödinger equation with radial symmetry. Using the results of Section 2 we derive heuristically the behavior of the amplitude of the solution, as $\gamma \to \infty$.

The case $\alpha = 2$ which is special and somehow easier to handle has been analyzed in [3].

2 THE LINEAR PROBLEM

Let $b > 0$ be a fixed number and consider the problem (1)–(2), where $\rho(t)$ is a strictly positive function in $C^2[0, b]$. We are interested in the asymptotic behavior of the solution $w(t)$, $t \in [0, b]$, as $B \to \infty$.

2.1 The Case $0 < \alpha < 1$

In this case we introduce the change of variables

$$t = z^\lambda, \qquad \text{where} \quad \lambda = \frac{1}{1 - \alpha} > 1, \tag{3}$$

and to make things clear we set

$$v(z) = w(t). \tag{4}$$

In view of the above transformation, a straightforward calculation yields that (1)–(2) is equivalent to

$$v''(z) \; + \; \lambda^2 B^2 z^{2\lambda-2} \rho(z^\lambda)^2 v(z) = 0, \quad z > 0, \tag{5}$$

$$v(0) = 1, \qquad v'(0) = 0. \tag{6}$$

The WKB theory together with asymptotic matching (see, e.g., [1]) implies that in some region I of the form

$$z \gg (1/B)^\sigma, \qquad \sigma > 0 \tag{7}$$

the so-called physical optics approximation to $v(z)$ is

$$v_{\mathrm{I}}(z) \sim [Q(z)]^{-1/4} \; \{C_1 \cos[BS_0(z)] \; + \; C_2 \sin[BS_0(z)]\}, \quad \text{as } B \to \infty, \tag{8}$$

where

$$Q(z) = \lambda^2 z^{2\lambda-2} \rho(z^\lambda)^2, \tag{9}$$

$$S_0(z) = \int_0^z \sqrt{Q(\tau)}\, d\tau = \int_0^z \lambda \tau^{\lambda-1} \rho(\tau^\lambda)\, d\tau = \int_0^{z^\lambda} \rho(\tau)\, d\tau \tag{10}$$

and C_1, C_2 are constants (to be determined). It will be convenient for the sequel to rewrite the approximate formula (8) as follows:

$$\begin{aligned} v_{\mathrm{I}}(z) \quad \sim \quad & \frac{C_1^*}{\sqrt{\lambda z^{\lambda-1} \rho(z^\lambda)}} \cos\left[B \int_0^{z^\lambda} \rho(\tau)\, d\tau + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right] \; + \\ & + \frac{C_2^*}{\sqrt{\lambda z^{\lambda-1} \rho(z^\lambda)}} \sin\left[B \int_0^{z^\lambda} \rho(\tau)\, d\tau + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right], \end{aligned} \tag{11}$$

where z is in region I and C_1^*, C_2^* are constants. The fact that $\rho(t) > 0$, for $t > 0$, guarantees that, for z bounded away from zero, the difference between the exact solution $v(z)$ of (1)–(2) and $v_{\mathrm{I}}(z)$ is of order $1/B$, as $B \to \infty$.

In order to determine region I (i.e. to estimate σ of (7)), we have to check (see [1]) the validity of the following two criteria

$$BS_0 \gg S_1 \gg S_2/B, \quad S_2/B \ll 1, \quad \text{as } B \to \infty, \tag{12}$$

where

$$S_1(z) = -\frac{1}{4} \ln[Q(z)], \quad S_2(z) = \pm \int^z \left[\frac{Q''(\tau)}{8(Q(\tau))^{3/2}} - \frac{5(Q'(\tau))^2}{32(Q(\tau))^{5/2}} \right] d\tau. \tag{13}$$

For typographical convenience let us set

$$\rho_0 = \rho(0) > 0.$$

Then, as $z \to 0^+$, i.e. when $z \ll 1$ (see (9), (10), and (13)), we have

$$Q(z) \sim \lambda^2 \rho_0^2 z^{2\lambda-2}, \quad S_0(z) \sim \rho_0 z^\lambda, \quad S_1(z) \sim -\frac{1}{2}\ln(\lambda\rho_0) - \frac{\lambda-1}{2}\ln z$$

and

$$S_2(z) \sim c_1 z^{-\lambda} + c_2,$$

where c_1 and c_2 are constants. Taking into account the above approximations, we infer that the criteria (12) are satisfied for

$$\frac{z}{(\ln z)^{1/\lambda}} \gg (1/B)^{1/\lambda}, \qquad \text{as } B \to \infty.$$

It follows from the above that the region I can be taken as in (7) where σ is any number satisfying

$$\sigma < \frac{1}{\lambda}. \tag{14}$$

Next, we turn to the analysis of the problem (1)–(2) in region II, i.e. for $z > 0$, $z \ll 1$. In this region, the WKB approximation is not valid because Q has a (multiple) zero at 0. Nevertheless, we may solve the approximate problem

$$v''(z) \; + \; B^2\lambda^2\rho_0^2 z^{2\lambda-2} v(z) = 0, \tag{15}$$

$$v(0) = 1, \qquad v'(0) = 0, \tag{16}$$

in terms of Bessel functions. Indeed, the general solution of (15) is (see, e.g., [1])

$$v(z) = \sqrt{z}\left[C J_{1/(2\lambda)}(\rho_0 B z^\lambda) + D J_{-1/(2\lambda)}(\rho_0 B z^\lambda)\right], \qquad z > 0,$$

where C, D are constants.The series expansions of Bessel functions imply that for $z > 0$,

$$J_{1/(2\lambda)}(\rho_0 B z^\lambda) = (\rho_0 B/2)^{1/(2\lambda)}\, z^{1/2} F(z), \quad J_{-1/(2\lambda)}(\rho_0 B z^\lambda) = (\rho_0 B/2)^{-1/(2\lambda)}\, z^{-1/2} G(z),$$

where

$$F(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\rho_0^2 B^2 z^{2\lambda}\right)^n}{\Gamma\left(n + \frac{1}{2\lambda} + 1\right)}, \qquad F(0) = \frac{1}{\Gamma\left(1 + \frac{1}{2\lambda}\right)},$$

$\Gamma(\cdot)$ being the Gamma function, and

$$G(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\rho_0^2 B^2 z^{2\lambda}\right)^n}{\Gamma\left(n - \frac{1}{2\lambda} + 1\right)}, \qquad G(0) = \frac{1}{\Gamma\left(1 - \frac{1}{2\lambda}\right)}, \quad G'(0) = 0.$$

Then

$$v(z) = \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} C z F(z) \; + \; (\rho_0 B/2)^{-1/(2\lambda)}\, D G(z), \qquad z \geq 0,$$

and, hence, by using the initial condition (16) we get

$$C = 0, \qquad D = \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1 - \frac{1}{2\lambda}\right).$$

Consequently,

$$v_{\mathrm{II}}(z) = \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1 - \frac{1}{2\lambda}\right) \sqrt{z} J_{-1/(2\lambda)}(\rho_0 B z^{\lambda}), \qquad z > 0, \quad z \ll 1. \tag{17}$$

It is clear from the above mentioned work that $v_{\mathrm{I}}, v_{\mathrm{II}}$ given by (11), (17) respectively, have a common region of validity (overlap region), namely,

$$(1/B)^{1/\lambda} \ll z \ll 1, \quad B \to \infty.$$

In order to to match these two approximate solutions, we must further approximate them in the overlap region.

First, we consider $v_{\mathrm{I}}(z)$. For $(1/B)^{1/\lambda} \ll z \ll 1$ $(B \to \infty)$, z is "small", so we have

$$\rho(t) \sim \rho_0 t, \qquad \text{as} \quad t \to 0^+,$$

and thus (see (11)),

$$v_{\mathrm{I}}(z) \sim \frac{1}{\sqrt{\rho_0 \lambda z^{\lambda-1}}} \left[C_1^* \cos\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right) + C_2^* \sin\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right)\right] \tag{18}$$

Next, we consider $v_{\mathrm{II}}(z)$. In the overlap region we have $Bz^{\lambda} \to \infty$, as $B \to \infty$, so it is necessary to approximate the Bessel function $J_{-1/(2\lambda)}(\cdot)$ by its leading asymptotic behavior for "large" positive argument. The appropriate formula is (see, e.g., [1])

$$J_{-1/(2\lambda)}(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \qquad \text{as} \quad x \to \infty, \tag{19}$$

which implies

$$J_{-1/(2\lambda)}(\rho_0 B z^{\lambda}) \sim \sqrt{\frac{2}{\pi \rho_0 B z^{\lambda}}} \cos\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \qquad z \gg (1/B)^{1/\lambda}, \ B \to \infty.$$

Now (17) combined with the above asymptotics gives

$$v_{\mathrm{II}}(z) \sim \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1 - \frac{1}{2\lambda}\right) \sqrt{\frac{2}{\pi \rho_0 B z^{\lambda-1}}} \cos\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \tag{20}$$

for $1 \gg z \gg (1/B)^{1/\lambda}$.

Requiring that (18), (20) match on the overlap region we obtain

$$C_2^* = 0, \qquad C_1^* = B^{\frac{1}{2\lambda} - \frac{1}{2}} \sqrt{\frac{2\lambda}{\pi} \left(\frac{\rho_0}{2}\right)^{1/\lambda}} \Gamma\left(1 - \frac{1}{2\lambda}\right).$$

In summary, the approximations to $v(z)$ in each of the regions I, II are the following:

$$v_{\mathrm{I}}(z) \sim B^{\frac{1}{2\lambda}-\frac{1}{2}}\sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{1/\lambda}}\Gamma\left(1-\frac{1}{2\lambda}\right)\frac{1}{\sqrt{z^{\lambda-1}\rho(z^\lambda)}}\cos\left(B\int_0^{z^\lambda}\rho(\tau)\,d\tau+\frac{\pi}{4\lambda}-\frac{\pi}{4}\right),$$

for $z \gg (1/B)^{1/\lambda}$, $B \to \infty$,

and

$$v_{\mathrm{II}}(z) \sim \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)}\Gamma\left(1-\frac{1}{2\lambda}\right)\sqrt{z}J_{-1/(2\lambda)}(\rho_0 B z^\lambda), \qquad z>0, \quad z \ll 1, \quad B\to\infty.$$

Accordingly, in the case $0<\alpha<1$ the approximations to the solution $w(t)$ of the original problem (1), (2) are the following:

$$w_{\mathrm{I}}(t) \sim B^{\frac{1}{2\lambda}-\frac{1}{2}}\sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{1/\lambda}}\Gamma\left(1-\frac{1}{2\lambda}\right)\frac{t^{1/(2\lambda)}}{\sqrt{t\rho(t)}}\cos\left(B\int_0^{t}\rho(\tau)\,d\tau+\frac{\pi}{4\lambda}-\frac{\pi}{4}\right), \tag{21}$$

for $t \gg 1/B$, $B \to \infty$,

and

$$w_{\mathrm{II}}(t) \sim \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)}\Gamma\left(1-\frac{1}{2\lambda}\right)t^{1/(2\lambda)}J_{-1/(2\lambda)}(\rho_0 B t), \qquad t>0, \quad t \ll 1, \quad B\to\infty, \tag{22}$$

where λ is given by (3).

2.2 The Case $\alpha > 1$

If $\alpha > 1$, we use the transformation

$$w(t) = z^{-1}v(z), \qquad t = z^\mu,$$

where

$$\mu = \frac{1}{\alpha-1} > 1. \tag{23}$$

A straightforward calculation yields that (1)–(2) is equivalent to

$$v''(z) + \mu^2 B^2 z^{2\mu-2}\rho(z^\mu)^2 v(z) = 0, \quad z>0, \tag{24}$$

$$v(0)=0, \qquad v'(0)=1. \tag{25}$$

By using similar arguments as in the case $\alpha<1$ we get

$$v_{\mathrm{I}}(z) \sim B^{-\frac{1}{2\mu}-\frac{1}{2}}\sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{-1/\mu}}\Gamma\left(1+\frac{1}{2\mu}\right)\frac{1}{\sqrt{z^{\mu-1}\rho(z^\mu)}}\cos\left(B\int_0^{z^\mu}\rho(\tau)\,d\tau-\frac{\pi}{4\mu}-\frac{\pi}{4}\right),$$

for $z \gg (1/B)^{1/\mu}$, $B \to \infty$,

and

$$v_{\mathrm{II}}(z) \sim \left(\frac{\rho_0 B}{2}\right)^{-1/(2\mu)} \Gamma\left(1+\frac{1}{2\mu}\right)\sqrt{z}J_{1/(2\mu)}(\rho_0 B z^{\mu}), \qquad z>0, \quad z \ll 1, \quad B \to \infty.$$

Thus,

$$w_{\mathrm{I}}(t) \sim B^{-\frac{1}{2\mu}-\frac{1}{2}}\sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{-1/\mu}}\Gamma\left(1+\frac{1}{2\mu}\right)\frac{t^{-1/(2\mu)}}{\sqrt{t\rho(t)}}\cos\left(B\int_0^t \rho(\tau)\,d\tau - \frac{\pi}{4\mu}-\frac{\pi}{4}\right),$$

$t>0$, $t \gg 1/B$, $B \to \infty$,

$$w_{\mathrm{II}}(t) \sim \left(\frac{\rho_0 B}{2}\right)^{-1/(2\mu)} \Gamma\left(1+\frac{1}{2\mu}\right)t^{-1/(2\mu)}J_{1/(2\mu)}(\rho_0 B t), \qquad t>0, \quad t \ll 1, \quad B \to \infty,$$

where μ is given by (23).

Since (3) and (23) give

$$-\frac{1}{2\lambda}=\frac{\alpha-1}{2} \quad \text{(when } 0<\alpha<1\text{)} \qquad \text{and} \qquad \frac{1}{2\mu}=\frac{\alpha-1}{2} \quad \text{(when } \alpha>1\text{)},$$

we observe that the last two approximations and (21),(22) have exactly the same form. Consequently, (21), (22) are valid for every $\alpha \neq 1$. We summarize our results in the following theorem.

Theorem. Let $w(t)$ be the solution of the problem (1)–(2), where $t \in [0,b]$ and $\alpha \neq 1$. Then, in region I, i.e. when $t \gg 1/B$, as $B \to \infty$,

$$w_{\mathrm{I}}(t) \sim \frac{1}{B^{\alpha/2}}\sqrt{\frac{2}{\pi}}\left(\frac{\rho_0}{2}\right)^{(1-\alpha)/2}\Gamma\left(\frac{\alpha+1}{2}\right)\frac{1}{t^{\alpha/2}\sqrt{\rho(t)}}\cos\left(B\int_0^t \rho(\tau)\,d\tau - \frac{\pi\alpha}{4}\right), \tag{26}$$

while in region II, i.e. when $t>0$, $t \ll 1$, as $B \to \infty$,

$$w_{\mathrm{II}}(t) \sim \left(\frac{\rho_0 B}{2}\right)^{(1-\alpha)/2}\Gamma\left(\frac{\alpha+1}{2}\right)t^{(1-\alpha)/2}J_{(\alpha-1)/2}(\rho_0 B t). \tag{27}$$

Remarks. (i) We believe that the above formulas are valid even for $\alpha=1$.

(ii) Since

$$J_{1/2}(x)=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sin x}{\sqrt{x}},$$

if we set $\alpha=2$ in (26) and (27), the formulas reduce to

$$w(t) \sim \frac{\sin\left[B\int_0^t \rho(\tau)d\tau\right]}{B\rho_0^{1/2}\rho(t)^{1/2}t},$$

valid for all $t \in [0, b]$. This agrees with the formula given in [3].

(iii) It would be nice to have a Langer-type formula for $w(t)$, namely an asymptotic formula which is uniformly valid for all $t \in [0, b]$, as $B \to \infty$. In fact, if

$$\rho'(0) = 0,$$

one can check that the solution $w(t)$ of (1)–(2) satisfies

$$w(t) \sim \left(\frac{\rho_0 B}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) t^{(1-\alpha)/2} \sqrt{\frac{S(t)}{t\rho(t)}} J_{(\alpha-1)/2}\left[BS(t)\right], \qquad \text{for all } t \in [0, b], \tag{28}$$

where

$$S(t) = \int_0^t \rho(\tau)\, d\tau.$$

Unless $\alpha = 2$, one needs the condition $\rho'(0) = 0$ in order for the expression shown in (28) to satisfy the initial condition $w'(0) = 0$.

3 THE NONLINEAR PROBLEM

Let us now consider the nonlinear initial value problem

$$u'' + \frac{\alpha}{t} u' + u^{2p+1} = 0, \qquad t > 0, \tag{29}$$

$$u(0) = \gamma, \qquad u'(0) = 0, \tag{30}$$

where $p \geq 1$ is a positive integer and $\alpha > 0$ (notice that, if α is an integer, then $u'' + (\alpha/t)u'$ is the $(\alpha+1)$-dimensional radial Laplacian of u, hence (29) is a radially symmetric multidimensional nonlinear Schrödinger equation). Again the boundary conditions must be interpreted in the right way, i.e. as limits when $t \to 0^+$.

Proposition. The problem (29)–(30) has a unique solution for all $t > 0$.

Proof. We first notice that (29)–(30) are equivalent to the integral equation

$$u(t) = \gamma - \int_0^t \frac{t^{1-\alpha}\tau^\alpha - \tau}{1-\alpha} u(\tau)^{2p+1} d\tau, \tag{31}$$

where the integrand makes sense even for $\alpha = 1$, since in this case it becomes

$$u(t) = \gamma - \int_0^t \tau \left(\ln t - \ln \tau\right) u(\tau)^{2p+1} d\tau.$$

We must, therefore, look at the map

$$\mathcal{F}[u](t) = \gamma - \int_0^t \frac{t^{1-\alpha}\tau^\alpha - \tau}{1-\alpha} u(\tau)^{2p+1} d\tau,$$

mapping $C[0,\varepsilon]$ into itself, for any given $\varepsilon > 0$. It is easy to see that, if ε is chosen sufficiently small, then $\mathcal{F}$ is a contraction, namely

$$\|\mathcal{F}[u] - \mathcal{F}[v]\|_\infty \leq c\,\|u - v\|_\infty\,,$$

where $c < 1$. Hence $\mathcal{F}$ has a unique fixed point u in $C[0,\varepsilon]$ which is the unique solution of (31) in $[0,\varepsilon]$ (and it is automatically smooth). Then the global existence and uniqueness follows by the fact that the energy

$$E(t) = u(t)^{2p+2} + (p+1)u'(t)^2 \tag{32}$$

is decreasing. ■

The solution $u(t)$ of (29)–(30) is highly oscillatory, due to the term u^{2p+1}, but with a decreasing amplitude of oscillation, due to the dissipative term $(\alpha/t)u'$. Using the expression (32) for the energy, we can define the amplitude of oscillation as (the same definition was used in [3])

$$A(t) = E(t)^{1/(2p+2)}. \tag{33}$$

Let $0 = t_0' < t_1' < t_2' < \cdots$ be the (positive) zeros of $u'(t)$. Then

$$A(t_j') = \left|u(t_j')\right| = (-1)^j u(t_j'). \tag{34}$$

In [2], it was shown that for a fixed $j \geq 0$,

$$t_{j+1}' - t_j' = \frac{c_p}{2\gamma^p} + O\left(\gamma^{-2p}\right), \qquad \text{as } \gamma \to \infty, \tag{35}$$

where the constant c_p is given by

$$c_p = 4\sqrt{p+1}\int_0^1 \frac{dx}{\sqrt{1 - x^{2p+2}}} = \frac{2\sqrt{\pi}}{\sqrt{(p+1)}}\frac{\Gamma\left(\frac{1}{2p+2}\right)}{\Gamma\left(\frac{p+2}{2p+2}\right)}$$

The problem we want to discuss here is: For a given $b > 0$ determine the (leading) asymptotic behavior of $A(b)$, as $\gamma \to \infty$.

As it was shown in [3], for any $\gamma > 0$ there is an $n = n(\gamma) \geq 0$ such that

$$t_{2n}' \leq b < t_{2n+2}'.$$

We set

$$b^* = t_{2n}' \tag{36}$$

(b^* depends on γ and b; in particular $b^* \leq b$). Thus $u(b^*)$ is a local maximum of $u(t)$ and

$$u(b^*) = A\left(b^*\right). \tag{37}$$

Notice that (35) implies that, as $\gamma \to \infty$,

$$b - b^* = O\left(\gamma^{-p}\right),$$

which, in turn gives (see (33))

$$A(b) - A\left(b^*\right) = O\left(\gamma^{1-p}\right), \tag{38}$$

hence, in order to estimate $A(b)$, it suffices, thanks to (37) and (38), to estimate $u(b^*)$.

By setting

$$u(t) = \gamma u_1(t), \tag{39}$$

(29)–(30) can be written as

$$u_1'' + \frac{\alpha}{t} u_1' + \gamma^{2p} u_1^{2p+1} = 0, \qquad t > 0, \tag{40}$$

$$u_1(0) = 1, \qquad\qquad u'(0) = 0. \tag{41}$$

We propose the following heuristic way to estimate $u_1(b^*)$ as $\gamma \to \infty$. Applying (26) to (40), (41) for $\rho(t) = |u_1(t)|^p$ (hence $\rho_0 = 1$), $B = \gamma^p$, we obtain that for $t > 0$, $t \gg 1/\gamma^p$,

$$u_1(t) \sim \frac{1}{\gamma^{p\alpha/2}} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2} |u_1(t)|^{p/2}} \cos\left(\gamma^p \int_0^t |u_1(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right),$$

as $\gamma \to \infty$, or, due to (39),

$$u(t) \sim \gamma^{1+[p(1-\alpha)/2]} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2} |u(t)|^{p/2}} \cos\left(\int_0^t |u(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right). \tag{42}$$

It should be kept in mind that (42) **is valid as long as**

$$A(t) \to \infty, \qquad \text{as } \gamma \to \infty. \tag{43}$$

Formula (42) implies that, under (43),

$$|u(t)|^{(p+2)/2} \sim \gamma^{1+[p(1-\alpha)/2]} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2}} \left|\cos\left(\int_0^t |u(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right)\right|,$$

or

$$|u(t)| \sim \gamma^{(p+2-p\alpha)/(p+2)} \left(\frac{1}{\pi}\right)^{1/(p+2)} 2^{\alpha/(p+2)} \Gamma\left(\frac{\alpha+1}{2}\right)^{2/(p+2)} \frac{1}{t^{\alpha/(p+2)}} \left|\cos\left(\int_0^t |u(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right)\right|^{2/(p+2)},$$

as $\gamma \to \infty$.

Therefore, as $\gamma \to \infty$,

$$A(b) \sim \gamma^{(p+2-p\alpha)/(p+2)} \left(\frac{1}{\pi}\right)^{1/(p+2)} 2^{\alpha/(p+2)} \Gamma\left(\frac{\alpha+1}{2}\right)^{2/(p+2)} \frac{1}{b^{\alpha/(p+2)}},$$

as long as $A(b) \to \infty$, as $\gamma \to \infty$, i.e. when

$$p(\alpha - 1) < 2.$$

If, on the other hand, $p(\alpha - 1) \geq 2$, then

$$A(b) = O(1), \qquad \text{as } \gamma \to \infty.$$

The case $\alpha = 2$ reduces to the statement appeared in [3].

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