Birkhoff-James ϵ-orthogonality sets and numerical ranges

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(Joint work with Christos Chorianopoulos)

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The Birkhoff-James ϵ-Orthogonality

For two elements χ and ψ of a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, χ is called **Birkhoff-James orthogonal** to ψ, denoted by $\chi \perp_{BJ} \psi$, if

$$\|\chi + \lambda\psi\| \geq \|\chi\|, \quad \forall\, \lambda \in \mathbb{C}.$$

Furthermore, for any $\epsilon \in [0, 1)$, we say that χ is **Birkhoff-James** ϵ**-orthogonal** to ψ, denoted by $\chi \perp_{BJ}^{\epsilon} \psi$, if

$$\|\chi + \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\chi\|, \quad \forall\, \lambda \in \mathbb{C}.$$

In an inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, a $\chi \in \mathcal{X}$ is called ϵ**-orthogonal** to a $\psi \in \mathcal{X}$, denoted by $\chi \perp^{\epsilon} \psi$, if

$$|\langle\chi, \psi\rangle| \leq \epsilon\,\|\chi\|\,\|\psi\|.$$

Furthermore, $\chi \perp^{\epsilon} \psi$ if and only if $\chi \perp_{BJ}^{\epsilon} \psi$.

The Classical Numerical Range

The **(standard) numerical range** of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$F(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n,\, x^*x = 1\}.$$

It is a **compact** and **convex** subset of $\mathbb{C}$ that contains the **spectrum** of A, $\sigma(A)$, and has a rich structure. Stampfli and Williams (1968), and later Bonsall and Duncan (1973), observed that

$$\begin{aligned} F(A) &= \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda|,\, \forall\, \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I_n\|_2\}. \end{aligned}$$

Hence, $F(A)$ is an infinite intersection of closed disks

$$\mathcal{D}(\lambda, \|A - \lambda I_n\|_2) = \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I_n\|_2\} \quad (\lambda \in \mathbb{C}).$$

A Primer Work

Inspired by the previous intersection property of the numerical range, Chorianopoulos, Karanasios and Ps. (2009) proposed a definition of numerical range for rectangular complex matrices. In particular, for any $A, B \in \mathbb{C}^{n\times m}$ with $\|B\| \geq 1$, and any matrix norm $\|\cdot\|$, the **numerical range of** A **with respect to** B is defined as

$$\begin{aligned} F_{\|\cdot\|}(A;B) &= \{\mu \in \mathbb{C} : \|A - \lambda B\| \geq |\mu - \lambda|, \forall\, \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \|A - \lambda B\|\right) \\ &= \left\{\mu \in \mathbb{C} : B \perp_{BJ}^{\epsilon_B} (A - \mu B)\right\} \quad \left(\text{where } \epsilon_B = \frac{\sqrt{\|B\|^2 - 1}}{\|B\|}\right). \end{aligned}$$

The New Definition

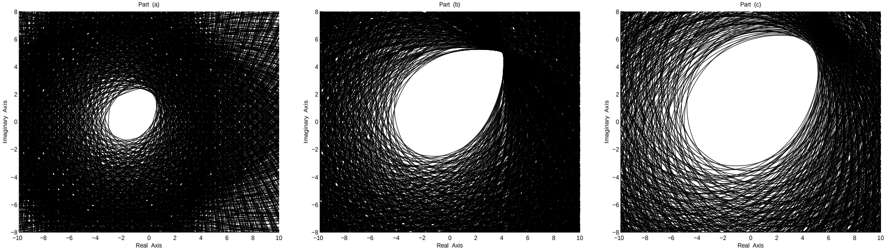
For any $A, B \in \mathbb{C}^{n\times m}$ with $B \neq 0$, any matrix norm $\|\cdot\|$, and any $\epsilon \in [0,1)$, the **Birkhoff-James ϵ-orthogonality set of A with respect to B** is defined and denoted by

$$\begin{aligned} F^{\epsilon}_{\|\cdot\|}(A;B) &= \{\mu \in \mathbb{C} : \; B \perp^{\epsilon}_{BJ} (A - \mu B)\} \\ &= \left\{\mu \in \mathbb{C} : \; \|A - \lambda B\| \geq \sqrt{1-\epsilon^2}\,\|B\|\,|\mu - \lambda|, \; \forall \, \lambda \in \mathbb{C}\right\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|A - \lambda B\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right). \end{aligned}$$

The Birkhoff-James ϵ-orthogonality set $F^{\epsilon}_{\|\cdot\|}(A;B)$ is a nonempty **compact** and **convex** subset of $\mathbb{C}$ that lies in $\mathcal{D}\left(0, \frac{\|A\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right)$.

Clearly, for $n = m$, we have $F^{0}_{\|\cdot\|_2}(A; I_n) \equiv F(A)$.

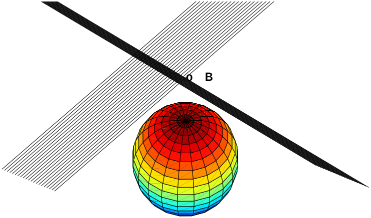
An Example



The sets $F^{0.5}_{\|\cdot\|_2}(A;B)$, $F^{\sqrt{0.5}}_{\|\cdot\|_2}(A;B)$ and $F^{\sqrt{0.6}}_{\|\cdot\|_2}(A;B)$ for

$$A = \begin{bmatrix} 4+\mathrm{i}\,5 & 0 & \mathrm{i} & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -\mathrm{i}\,2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \end{bmatrix}.$$

A Geometric Interpretation



A scalar $\mu \in \mathbb{C}$ lies in $F^{\epsilon}_{\|\cdot\|}(A;B)$ if and only if the affine space $\{B + \lambda(A - \mu B) : \lambda \in \mathbb{C}\}$ does not intersect the open ball

$$\mathcal{B}^{\circ}(0, \sqrt{1-\epsilon^2}\,\|B\|) = \{M \in \mathbb{C}^{n\times m} : \|M\| < \sqrt{1-\epsilon^2}\,\|B\|\}.$$

Some Basic Properties

- $A = bB$ for a $b \in \mathbb{C}$ if and only if $F^{\epsilon}_{\|\cdot\|}(A; B) = \{b\}, \ \forall\, \epsilon \in [0, 1)$.

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- Suppose matrix norm $\|\cdot\|$ is induced by a vector norm and $n \geq m$, and let $\mu_0 \in \mathbb{C}$ be an **eigenvalue of** A **with respect to** B, with associate unit **eigenvector** $x_0 \in \mathbb{C}^m$, i.e., $(A - \mu_0 B)x_0 = 0$. Then $\mu_0 \in F^{\epsilon}_{\|\cdot\|}(A; B), \; \forall \, \epsilon \in \left[\frac{\sqrt{\|B\|^2 - \|Bx_0\|^2}}{\|B\|} \, , \, 1 \right)$.

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- If the matrix norm $\|\cdot\|$ is induced by the inner product of matrices $\langle\cdot,\cdot\rangle$ (this is the case of the **Frobenius norm**), then

$$F^{\epsilon}_{\|\cdot\|}(A;B) = \mathcal{D}\left(\frac{\langle A,B\rangle}{\|B\|^2}, \left\|A - \frac{\langle A,B\rangle}{\|B\|^2}B\right\| \frac{\epsilon}{\sqrt{1-\epsilon^2}\,\|B\|}\right).$$

The Growth

Suppose $A, B \in \mathbb{C}^{n\times m}$ such that $B \neq 0$ and A is not a scalar multiple of B. Then the following hold:

- If $0 \leq \epsilon_1 < \epsilon_2 < 1$, then $F^{\epsilon_1}_{\|\cdot\|}(A;B)$ lies in the interior of $F^{\epsilon_2}_{\|\cdot\|}(A;B)$.

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- For any bounded region $\Omega \subset \mathbb{C}$, there is an $\epsilon_\Omega \in [0,1)$ such that $\Omega \subseteq F^{\epsilon_\Omega}_{\|\cdot\|}(A;B)$.
- The mappings $A \mapsto F^{\epsilon}_{\|\cdot\|}(A;B)$ and $\epsilon \mapsto F^{\epsilon}_{\|\cdot\|}(A;B)$ are continuous.

The Boundary

Suppose $A, B \in \mathbb{C}^{n\times m}$ such that $B \neq 0$ and A is not a scalar multiple of B. Then the following hold:

- A point μ_0 lies on $\partial F^{\epsilon}_{\|\cdot\|}(A;B)$ if and only if

$$\inf_{\lambda\in\mathbb{C}} \left\{ \|A-\lambda B\| - \sqrt{1-\epsilon^2}\,\|B\|\,|\mu_0-\lambda| \right\} = 0.$$

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- If $\epsilon > 0$, then μ_0 lies on $\partial F^{\epsilon}_{\|\cdot\|}(A;B)$ if and only if

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- If $0 < \epsilon < 1$, then $\partial F^{\epsilon}_{\|\cdot\|}(A;B)$ has no flat portions.

Matrix Polynomials

Consider an $n \times m$ matrix polynomial

$$P(z) = A_l z^l + A_{l-1} z^{l-1} + \cdots + A_1 z + A_0,$$

where z is a complex variable and $A_j \in \mathbb{C}^{n\times m}$ with $A_l \neq 0$.

If $n \geq m$, then a scalar $\mu_0 \in \mathbb{C}$ is called an **eigenvalue of** $P(z)$ if

$$P(\mu_0)x_0 = 0$$

for some $0 \neq x_0 \in \mathbb{C}^m$ known as an associated **eigenvector**.

For $n = m$, the **(standard) numerical range of** $P(z)$ is defined as

$$\begin{aligned} W(P(z)) &= \{\mu \in \mathbb{C} : x^* P(\mu) x = 0,\, x \in \mathbb{C}^n,\, x \neq 0\} \\ &= \{\mu \in \mathbb{C} : 0 \in F(P(\mu))\}. \end{aligned}$$

The New Definition

For an $n \times m$ matrix polynomial $P(z)$ as above, any nonzero matrix $B \in \mathbb{C}^{n\times m}$, any matrix norm $\|\cdot\|$, and any $\epsilon \in [0,1)$, we define the **Birkhoff-James ϵ-orthogonality set of $P(z)$ with respect to B**

$$\begin{aligned} W^{\epsilon}_{\|\cdot\|}(P(z);B) &= \left\{\mu \in \mathbb{C} :\ 0 \in F^{\epsilon}_{\|\cdot\|}(P(\mu);B)\right\} \\ &= \left\{\mu \in \mathbb{C} :\ \|P(\mu) - \lambda B\| \geq \sqrt{1-\epsilon^2}\, \|B\|\, |\lambda|,\ \forall\, \lambda \in \mathbb{C}\right\} \\ &= \{\mu \in \mathbb{C} :\ B \perp^{\epsilon}_{BJ} P(\mu)\}\,. \end{aligned}$$

The continuity of norms yields the closeness of $W^{\epsilon}_{\|\cdot\|}(P(z);B)$.

Some Basic Properties

- For any $\alpha \in \mathbb{C}\backslash\{0\}$, $W^{\epsilon}_{\|\cdot\|}(P(\alpha z); B) = \alpha^{-1} W^{\epsilon}_{\|\cdot\|}(P(z); B)$ and $W^{\epsilon}_{\|\cdot\|}(P(z+\alpha); B) = W^{\epsilon}_{\|\cdot\|}(P(z); B) - \alpha$.

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- If $R(z) = A_0 z^l + \cdots + A_{l-1} z + A_l$, then $W^{\epsilon}_{\|\cdot\|}(R(z);B)\backslash\{0\} = \left\{\mu \in \mathbb{C} : \mu^{-1} \in W^{\epsilon}_{\|\cdot\|}(P(z);B)\backslash\{0\}\right\}$.

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- If the norm $\|\cdot\|$ is invariant under the conjugate operation $\overline{\cdot}$, and the coefficients of $P(z)$ and B are all real matrices, then $W^{\epsilon}_{\|\cdot\|}(P(z); B)$ is symmetric with respect to the real axis.

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- If the matrix norm $\|\cdot\|$ is induced by a vector norm, and μ_0 is an eigenvalue of $P(z)$ with an associated unit eigenvector $x_0 \in \mathbb{C}^n$, then $\mu_0 \in W^{\epsilon}_{\|\cdot\|}(P(z); B)$, $\forall \epsilon \in \left[\frac{\sqrt{\|B\|^2 - \|Bx_0\|^2}}{\|B\|}, 1\right)$.

Boundedness and Boundary

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- If μ_0 lies on $\partial W^{\epsilon}_{\|\cdot\|}(P(z);B)$, then $0 \in \partial F^{\epsilon}_{\|\cdot\|}(P(\mu_0);B)$.
- If $0 \in \partial F^{\epsilon}_{\|\cdot\|}(P(\mu_0);B) \backslash F^{\epsilon}_{\|\cdot\|}(P'(\mu_0);B)$ and $P(\mu_0) \neq 0$, then $\mu_0 \in \partial W^{\epsilon}_{\|\cdot\|}(P(z);B)$.

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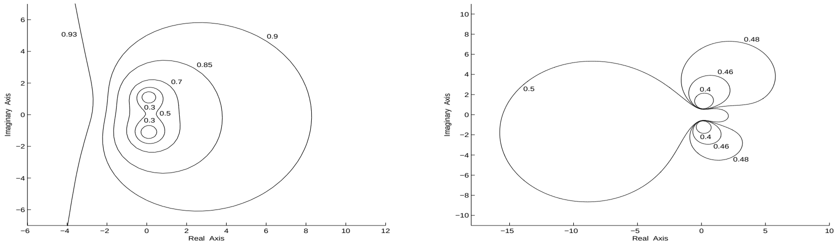
- If $0 \leq \epsilon < 1$, and there is a $\mu_0 \in \mathbb{C}$ such that $P(\mu_0) = 0$ and $0 \notin F^{\epsilon}_{\|\cdot\|}(P'(\mu_0);B)$, then μ_0 is an isolated point of $W^{\epsilon}_{\|\cdot\|}(P(z);B)$.

Matrix Norms Induced by Inner Products



If the norm $\|\cdot\|$ is induced by the inner product of matrices $\langle\cdot,\cdot\rangle$, then

$$W^{\epsilon}_{\|\cdot\|}(P(z);B) = \left\{\mu : \sum_{i,j=0}^{l} \langle A_i, B\rangle\langle B, A_j\rangle \mu^i \overline{\mu}^j - \epsilon^2 \|B\|^2 \sum_{i,j=0}^{l} \langle A_i, A_j\rangle \mu^i \overline{\mu}^j \leq 0\right\}.$$

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