An envelope for the spectrum of a matrix

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Abstract

We introduce and study an envelope-type region $\mathcal{E}(A)$ in the complex plane that contains the eigenvalues of a given $n \times n$ complex matrix A. $\mathcal{E}(A)$ is the intersection of an infinite number of regions defined by cubic curves. The notion and method of construction of $\mathcal{E}(A)$ extend the notion of the numerical range of A, $F(A)$, which is known to be an intersection of an infinite number of half-planes; as a consequence, $\mathcal{E}(A)$ is contained in $F(A)$ and represents an improvement in localizing the spectrum of A.

Keywords: eigenvalue bounds, numerical range, cubic curve.

AMS Subject Classifications: 15A18, 15A60.

1 Introduction

It is well known that the real part of each eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$ is bounded above by the largest eigenvalue, say $\delta_1(A)$, of the hermitian part of A. As a consequence, the spectrum of A lies in the intersection of all half-planes of the form $\left\{e^{-\mathrm{i}\,\theta}(s+\mathrm{i}\,t) : s,t \in \mathbb{R} \text{ with } s \le \delta_1(e^{\mathrm{i}\,\theta}A)\right\}$, $\theta \in [0, 2\pi]$. In fact, this infinite intersection of half-planes coincides with the numerical range of A, $F(A)$.

The purpose of this paper is to improve upon the above spectrum localization result. We will achieve this by replacing the infinite intersection of half-planes by an infinite intersection of regions in the complex plane, which are defined by cubic curves. These cubic curves are obtained by an inequality that all eigenvalues of A must satisfy [1]. The outcome is a localization region for the spectrum of A that is contained in $F(A)$, and it can in fact be quite smaller. We will refer to this region as the cubic envelope (or, for simplicity, just envelope) of A, study its basic properties and compare it to the numerical range $F(A)$.

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In Section 2 we describe some of the basic properties of the numerical range needed in our analysis. Section 3 contains a discussion about the cubic curve studied in [1] that bounds the spectrum, along with an answer (see Theorem 3.2) to a question arising in [1]. In Sections 4 and 5, we define and develop the properties of the envelope of A as a containment region of the spectrum of A. Our results and concepts are illustrated with several examples and figures.

2 Preliminaries on the numerical range

Let $A \in \mathbb{C}^{n\times n}$ be an $n \times n$ complex matrix with *spectrum*

$$\sigma(A) \,=\, \{\lambda \in \mathbb{C} : \det(\lambda I_n - A) = 0\}\,,$$

where I_n denotes the $n \times n$ identity matrix. Consider also the *hermitian* and *skew-hermitian parts* of A, $H(A) = (A + A^*)/2$ and $S(A) = (A - A^*)/2$, respectively, and let $\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$ denote the eigenvalues of $H(A)$. The *numerical range* (also known as the *field of values*) of A is defined as

$$F(A) \,=\, \{v^*Av \in \mathbb{C} : \, v \in \mathbb{C}^n \text{ with } v^*v = 1\}\,;$$

it is a *compact* and *convex* subset of $\mathbb{C}$ that contains $\sigma(A)$ (see [3] and the references therein). Moreover, the following well-known properties of $F(A)$ hold [3].

($\mathbf{P}_1$) For any $A \in \mathbb{C}^{n\times n}$, $F(A^T) = F(A)$ and $F(A^*) = F(\overline{A}) = \overline{F(A)}$. In particular, if A is a real matrix, then $F(A)$ is symmetric with respect to the real axis.

($\mathbf{P}_2$) For any $a, b \in \mathbb{C}$, $F(aA + bI_n) = aF(A) + b$.

($\mathbf{P}_3$) For any unitary matrix $U \in \mathbb{C}^{n\times n}$, $F(U^*AU) = F(A)$.

($\mathbf{P}_4$) The numerical ranges of the hermitian and skew-hermitian parts of A satisfy $F(H(A)) = \mathrm{Re}F(A)$ and $F(S(A)) = \mathrm{i}\,\mathrm{Im}F(A)$.

($\mathbf{P}_5$) If A is normal, then $F(A)$ coincides with the convex hull of $\sigma(A)$, $\mathrm{Co}(\sigma(A))$. Moreover, A is hermitian if and only if $F(A) = F(H(A)) = [\delta_n(A), \delta_1(A)]$.

($\mathbf{P}_6$) For any unit vector $v_0 \in \mathbb{C}^n$ (i.e., $v_0^*v_0 = 1$), the following are equivalent:

- **(i)** $\mathrm{Re}\,\{v_0^*Av_0\} = \max\{\mathrm{Re}\,z : \, z \in F(A)\}$,
- **(ii)** $v_0^*H(A)v_0 = \max\{h : \, h \in F(H(A))\}$, and
- **(iii)** $H(A)v_0 = \delta_1(A)v_0$.

By Property (P_6), we have that

$$\max\{\mathrm{Re}\,z : \, z \in F(A)\} \,=\, \max\{s : \, s \in F(H(A))\} \,=\, \delta_1(A),$$

that is, the largest eigenvalue of $H(A)$ coincides with the real part of the right most point of $F(A)$. Furthermore, if $y_1 \in \mathbb{C}^n$ is a unit eigenvector of $H(A)$ corresponding

to $\delta_1(A)$, then the right most point of $F(A)$ is $y_1^*Ay_1$ and the vertical line $\mathcal{L}_0 = \{z \in \mathbb{C} : \operatorname{Re} z = \delta_1(A)\}$ is tangential to $F(A)$ at $y_1^*Ay_1$.

Based on the latter observation, Johnson [4] (see also [3, Section 1.5]) proposed an algorithm for the estimation of the numerical range by computing (and plotting) its boundary points. Specifically, for each angle $\theta \in [0, 2\pi]$, we consider the largest eigenvalue $\delta_1(e^{\mathrm{i}\,\theta}A)$ and an associated unit eigenvector $y_1(\theta)$ of the hermitian matrix $H(e^{\mathrm{i}\,\theta}A)$. Then the point $z_\theta = y_1(\theta)^*Ay_1(\theta)$ lies on the boundary of $F(A)$, denoted by $\partial F(A)$, and the line

$$\begin{aligned}\mathcal{L}_\theta &= \{e^{-\mathrm{i}\,\theta}(\delta_1(e^{\mathrm{i}\,\theta}A) + \mathrm{i}\,t) : t \in \mathbb{R}\} \\ &= \{t\sin\theta + \delta_1(e^{\mathrm{i}\,\theta}A)\cos\theta + \mathrm{i}\,(t\cos\theta - \delta_1(e^{\mathrm{i}\,\theta}A)\sin\theta) : t \in \mathbb{R}\}\end{aligned}$$

is tangential to $F(A)$ at z_θ. Moreover, line $\mathcal{L}_\theta$ separates the complex plane into the closed half-plane

$$\mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A) = \{e^{-\mathrm{i}\,\theta}(s + \mathrm{i}\,t) : s, t \in \mathbb{R} \text{ with } s \le \delta_1(e^{\mathrm{i}\,\theta}A)\},$$

which contains $F(A)$, and the open half-plane

$$\mathcal{H}_{out}(e^{\mathrm{i}\,\theta}A) = \{e^{-\mathrm{i}\,\theta}(s + \mathrm{i}\,t) : s, t \in \mathbb{R} \text{ with } s > \delta_1(e^{\mathrm{i}\,\theta}A)\}.$$

Thus, we may represent $F(A)$ as an infinite intersection of closed half-planes (see Theorem 1.5.12 of [3]), namely,

$$F(A) = \bigcap_{\theta\in[0,2\pi]} \mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A). \tag{1}$$

In particular, we can estimate $F(A)$ simply by drawing the tangent lines $\mathcal{L}_{\theta_j}$, $j = 1, 2, \ldots, k$, for a partition $0 = \theta_1 < \theta_1 < \cdots < \theta_{k-1} < \theta_k = 2\pi$ of the interval $[0, 2\pi]$. This is illustrated in our first example below.

Example 2.1. Consider the complex Toeplitz matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & \mathrm{i} \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

The boundary of the numerical range $F(A)$ is sketched in the left part of Figure 1. In the right part of the figure, $F(A)$ is illustrated as an envelope of 120 tangent lines. Here, and in all figures of the paper, the eigenvalues are marked with +'s.

In the sequel, our goal is to replace the tangent lines by cubic curves, introducing an envelope-type set that contains the spectrum and lies in the numerical range. We note in passing that another subset of $F(A)$ that contains the eigenvalues, known as the *block numerical range*, has been studied extensively; see [9] and [10]. There is no tractable relation known to us between the block numerical range and the envelope.

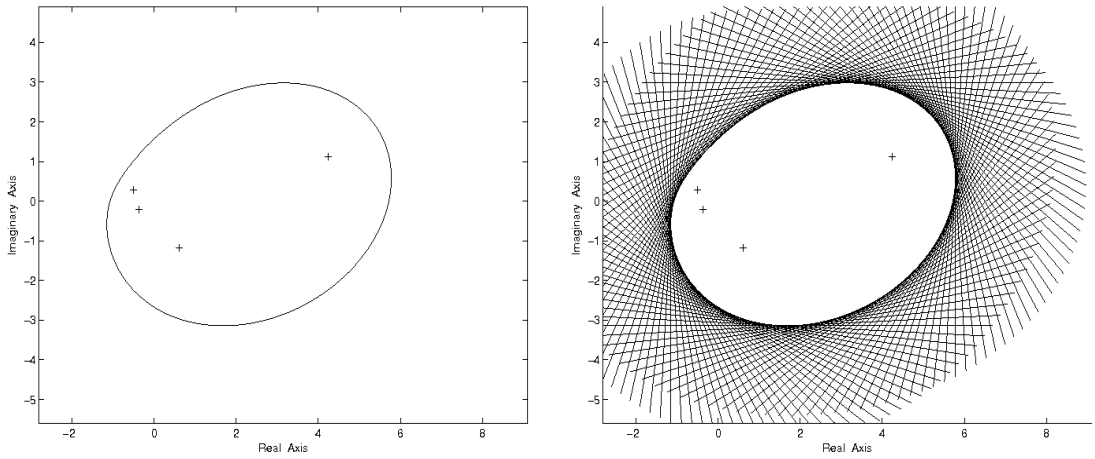

Figure 1: The numerical range of a 4×4 Toeplitz matrix.

3 A cubic curve that bounds the spectrum

Let A be an $n \times n$ complex matrix, and recall that by $y_1 \in \mathbb{C}^n$ we denote a unit eigenvector of the hermitian matrix $H(A)$ corresponding to its largest eigenvalue $\delta_1(A)$. Define also the quantities

$$\mathrm{v}(A) \;=\; \|S(A)y_1\|_2^2 \quad \text{and} \quad \mathrm{u}(A) \;=\; \mathrm{Im}(y_1^* S(A) y_1) \;\le\; \|S(A)y_1\|_2 \;=\; \sqrt{\mathrm{v}(A)}.$$

Adam and Tsatsomeros [1, Theorem 3.1], extending the methodology of [6], obtained the following theorem.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$. Then for every eigenvalue $\lambda \in \sigma(A)$,*

$$\begin{aligned} &(\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Im}\lambda - \mathrm{u}(A))^2 \;\le \\ &(\delta_1(A) - \mathrm{Re}\lambda)[\mathrm{v}(A) - \mathrm{u}(A)^2 + (\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Re}\lambda - \delta_1(A))]. \end{aligned} \tag{2}$$

Motivated by the above result, the authors of [1] also introduced and studied the algebraic curve

$$\begin{aligned} \Gamma(A) \;&=\; \big\{ s + \mathrm{i}\, t :\, s,t \in \mathbb{R},\; (\delta_2(A) - s)[(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2] \\ &\qquad + (\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2) = 0 \big\} \\ &=\; \{\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)\} \;\cup \\ &\qquad \left\{ s + \mathrm{i}\, t \neq \delta_1(A) + \mathrm{i}\,\mathrm{u}(A) :\, s,t \in \mathbb{R},\; \delta_2(A) - s + \frac{(\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2)}{(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2} = 0 \right\}. \end{aligned}$$

This is a cubic algebraic curve in $s, t \in \mathbb{R}$, which separates the complex plane into the regions

$$\begin{aligned} \Gamma_{in}(A) \;&=\; \big\{ s + \mathrm{i}\, t :\, s,t \in \mathbb{R},\; (\delta_2(A) - s)[(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2] \\ &\qquad + (\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2) \ge 0 \big\}. \end{aligned}$$

and

$$\begin{aligned}\Gamma_{out}(A) &= \big\{s+\mathrm{i}\,t:\ s,t\in\mathbb{R},\ (\delta_2(A)-s)[(\delta_1(A)-s)^2+(\mathrm{u}(A)-t)^2]\\ &\quad +(\delta_1(A)-s)(\mathrm{v}(A)-\mathrm{u}(A)^2)<0\big\}.\end{aligned}$$

These types of cubic curves have been extensively studied; a suggested general reference is [7]. Below we analyze some of the geometric features of $\Gamma(A)$.

By Theorem 3.1, it is apparent that $\sigma(A)\subset\Gamma_{in}(A)$. Furthermore, if $s>\delta_1(A)$ or $s<\delta_2(A)$, then $s+\mathrm{i}\,t$ ($t\in\mathbb{R}$) cannot satisfy the defining equation of $\Gamma(A)$. Thus, the curve $\Gamma(A)$ lies in the vertical zone $\{z\in\mathbb{C}:\ \delta_2(A)\le\operatorname{Re}z\le\delta_1(A)\}$. It is also clear that (always for $s,t\in\mathbb{R}$)

$$s+\mathrm{i}\,t\ \in\ \Gamma(A)\quad\text{if and only if}\quad s+\mathrm{i}\,(2\mathrm{u}(A)-t)\ \in\ \Gamma(A),$$

and

$$s\ \longrightarrow\ \delta_2(A)\,,\quad\text{as}\quad t\ \longrightarrow\ \pm\infty.$$

Thus, the curve $\Gamma(A)$ is symmetric with respect to the horizontal line

$$\mathcal{L}\ =\ \{z\in\mathbb{C}:\ \operatorname{Im}z=\mathrm{u}(A)\}$$

which it intercepts at the point $\delta_1(A)+\mathrm{i}\,\mathrm{u}(A)$, and is asymptotic to the vertical line $\{z\in\mathbb{C}:\ \operatorname{Re}z=\delta_2(A)\}$. Note that point $\delta_1(A)+\mathrm{i}\,\mathrm{u}(A)$ is a right most point of the numerical range $F(A)$.

Recall that a *flat portion* of the boundary of $F(A)$ is a maximal (in the sense of set inclusion) non-degenerate line segment that belongs entirely to $\partial F(A)$ [2]. By Property (P_6), it follows that if $\partial F(A)$ has a flat portion on the vertical line $\mathcal{L}_0=\{z\in\mathbb{C}:\ \operatorname{Re}z=\delta_1(A)\}$, then $\delta_1(A)=\delta_2(A)$, the curve $\Gamma(A)$ reduces to the line $\mathcal{L}_0$, and the region $\Gamma_{in}(A)$ coincides with the half-plane $\mathcal{H}_{in}(A)$.

When $\delta_1(A)>\delta_2(A)$, it is apparent that $\delta_1(A)+\mathrm{i}\,\mathrm{u}(A)$ is the unique right most point of $\Gamma(A)$, i.e., the only point of the curve with real part equal to $\delta_1(A)$. Moreover, the vertical line $\mathcal{L}_0$ is tangential to $\Gamma(A)$ at $\delta_1(A)+\mathrm{i}\,\mathrm{u}(A)$. This means that $\mathcal{L}_0$ is a common tangent to the curve $\Gamma(A)$ and the numerical range $F(A)$ at $\delta_1(A)+\mathrm{i}\,\mathrm{u}(A)$.

For $t=\mathrm{u}(A)$ and $\delta_2(A)<s<\delta_1(A)$, the defining equation of $\Gamma(A)$ is

$$(\delta_1(A)-s)[s^2-(\delta_1(A)+\delta_2(A))s+\delta_1(A)\delta_2(A)+\mathrm{v}(A)-\mathrm{u}(A)^2]\ =\ 0.$$

The discriminant of the quadratic factor in this expression is

$$\Delta\ =\ (\delta_1(A)-\delta_2(A))^2-4(\mathrm{v}(A)-\mathrm{u}(A)^2).$$

Thus, we have the following cases, which are illustrated in Figure 2.

(a) If $\Delta<0$, then $\Gamma(A)$ intercepts the horizontal line $\mathcal{L}$ only once, at $\delta_1(A)+\mathrm{i}\,\mathrm{u}(A)$, and is an unbounded simple open curve.

(b) If $\Delta = 0$, then $\Gamma(A)$ is a folium of Descartes and intercepts $\mathcal{L}$ at

$$\frac{\delta_1(A) + \delta_2(A)}{2} + \mathrm{i}\,\mathrm{u}(A) \quad \text{and} \quad \delta_1(A) + \mathrm{i}\,\mathrm{u}(A),$$

where the first point (double root) is the node point (cusp) of $\Gamma(A)$.

(c) If $\Delta > 0$, then $\Gamma(A)$ comprises two branches, an open unbounded branch and a closed bounded branch, and it intercepts line $\mathcal{L}$ at

$$\frac{\delta_1(A) + \delta_2(A) - \sqrt{\Delta}}{2} + \mathrm{i}\,\mathrm{u}(A), \quad \frac{\delta_1(A) + \delta_2(A) + \sqrt{\Delta}}{2} + \mathrm{i}\,\mathrm{u}(A), \text{ and } \delta_1(A) + \mathrm{i}\,\mathrm{u}(A).$$

In cases (a) and (c), $\Gamma(A)$ is a nonsingular elliptic curve [7]. When $\Delta < 0$ and $\Gamma(A)$ is an open simple curve, all eigenvalues of A lie to the left of $\Gamma(A)$. When $\Delta > 0$, then, as shown in the next theorem, the closed branch of $\Gamma(A)$ encompasses exactly one eigenvalue of A, while the remaining eigenvalues of A must lie to the left of the (unbounded) open branch of $\Gamma(A)$.

Theorem 3.2. *Suppose that $\delta_1(A)$ is a simple eigenvalue of $H(A)$ and $\Delta > 0$, i.e., $\Gamma(A)$ is not connected and has a closed branch. Then exactly one eigenvalue of A, which is simple, lies inside or on the closed branch of $\Gamma(A)$.*

Proof. Consider the Levinger transformation of A (see [5, 8])

$$L(A, r) \;=\; (1 - r)A + rA^* \;=\; H(A) + (1 - 2\,r)S(A)\;; \quad r \in [0, 1/2].$$

Observe that $\mathrm{v}(L(A,r)) = (1-2\,r)^2\mathrm{v}(A)$ and $\mathrm{u}(L(A,r)) = (1-2\,r)\mathrm{u}(A)$, and hence, $\mathrm{v}(L(A,r)) - \mathrm{u}(L(A,r))^2 = (1-2\,r)^2(\mathrm{v}(A) - \mathrm{u}(A)^2)$. Furthermore, all matrices $L(A,r)$, $r \in [0, 1/2]$, have hermitian part $H(A)$ and so, $\delta_1(L(A,r))$ and $\delta_2(L(A,r))$ are constant as functions of r. As a consequence, for every $r \in [0, 1/2]$, the curve $\Gamma(L(A,r))$ is not connected, consisting of an (unbounded) open branch and a (bounded) closed branch; see case (c) above.

Since $\delta_1(A)$ is a simple eigenvalue of $H(A)$, the point $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is the unique right most point of $F(A)$. By continuity of the eigenvalues, there is an $r_1 \in [0, 1/2)$ such that the right most eigenvalue of $L(A,r)$, as a function of $r \in [r_1, 1/2]$, say $\lambda(r)$, is simple and forms a continuous path in $\mathbb{C}$ with endpoints $\lambda(r_1)$ and $\lambda(1/2) = \delta_1(A)$. By the above discussion (and keeping in mind that the nonnegative quantity $\mathrm{v}(L(A,r)) - \mathrm{u}(L(A,r))^2$ can be arbitrarily small as $r \longrightarrow 1/2$), it follows that for some $r_2 \in [r_1, 1/2)$ sufficiently close to $1/2$, the curve $\Gamma(L(A,r_2))$ is not connected and the simple eigenvalue $\lambda(r_2)$ of $L(A,r_2)$ is the only eigenvalue of $L(A,r_2)$ that lies inside or on the closed branch of $\Gamma(L(A,r_2))$.

Let now $r \in [0, r_2]$ and $r \longrightarrow 0$. Since $\Gamma(L(A,r))$, $r \in [0, r_2]$, remains disconnected, the continuity of the eigenvalues implies that there is exactly one (simple) eigenvalue of $L(A,r)$ lying inside or on the closed branch of $\Gamma(L(A,r))$. □

The three situations described in (a)–(c) above and Theorem 3.2 are illustrated in Figure 2 for three appropriately chosen 3×3 matrices.

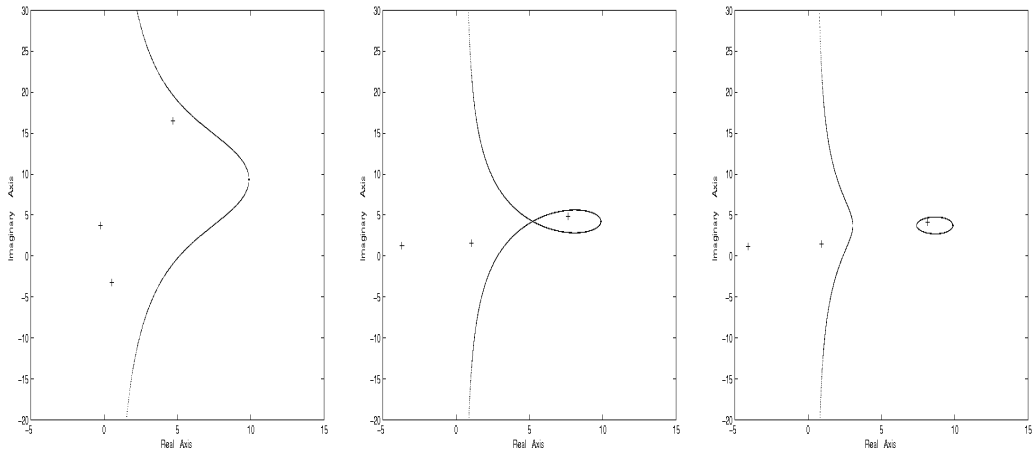

Figure 2: The cases of $\Gamma(A)$ with $\Delta < 0$ (left), $\Delta = 0$ (middle), and $\Delta > 0$ (right).

4 An envelope of the spectrum

In the previous section, we saw that the spectrum of an $n \times n$ complex matrix A lies in the region $\Gamma_{in}(A)$. We will take advantage of this fact by considering all rotations $e^{\mathrm{i}\,\theta}A$ ($\theta \in [0, 2\pi]$) in order to define a region in $\mathbb{C}$ that contains the spectrum of A. Indeed, for any $\theta \in [0, 2\pi]$, we have

$$\sigma(A) \;=\; e^{-\mathrm{i}\,\theta}\sigma(e^{\mathrm{i}\,\theta}A) \;\subseteq\; e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A) \;\subseteq\; \mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A). \tag{3}$$

As a consequence, we define the *cubic envelope* (or simply, the *envelope*) of A, as

$$\mathcal{E}(A) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A).$$

Then (1) and (3) yield the following.

Theorem 4.1. *For any matrix $A \in \mathbb{C}^{n\times n}$,*

$$\sigma(A) \;\subseteq\; \mathcal{E}(A) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A) \;\subseteq\; \bigcap_{\theta\in[0,2\pi]} \mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A) \;=\; F(A).$$

Example 4.2. Recall the 4×4 Toeplitz matrix A of Example 2.1. The numerical range of A is drawn as an envelope of 120 tangent lines on the left of Figure 3. In the right part of the figure, $\mathcal{E}(A)$ is the unshaded region resulting from having drawn 120 curves $e^{-\mathrm{i}\,\theta}\Gamma(e^{\mathrm{i}\,\theta}A)$. Notice that the cubic envelope $\mathcal{E}(A)$ consists of two connected components, and is a significantly improved localization of the spectrum of A as compare to $F(A)$.

We remark that the numerical range $F(A)$ appears, as a by-product, in all of our plots of the envelope $\mathcal{E}(A)$; specifically, $F(A)$ is depicted as the outer outlined region.

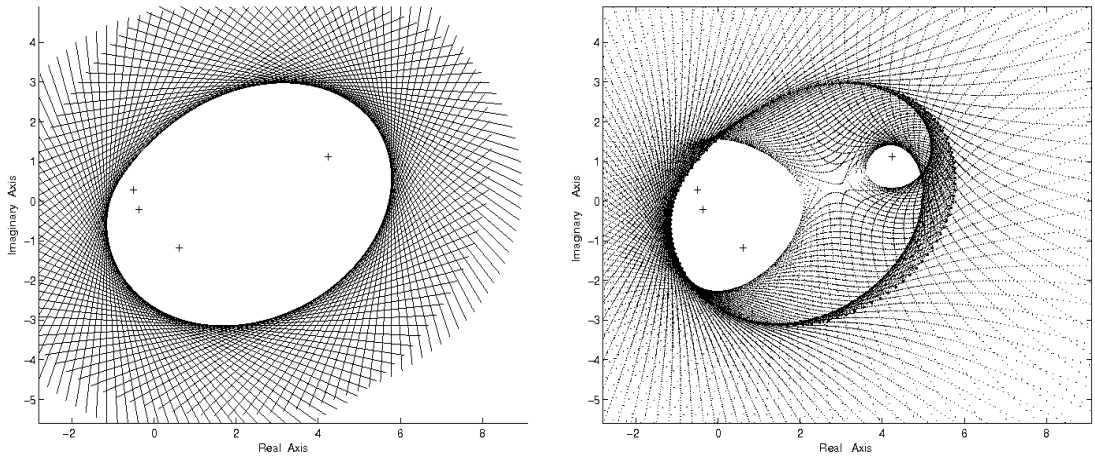

Figure 3: The numerical range $F(A)$ (left), and the cubic envelope $\mathcal{E}(A)$ (right).

In light of Theorem 4.1, we will subsequently study the properties of $\mathcal{E}(A)$. To begin, the envelope $\mathcal{E}(A)$ is compact, since it is a closed subset of the compact numerical range $F(A)$. It is not (necessarily) convex or connected, as illustrated in Example 4.2, but it satisfies some of the other basic properties of $F(A)$ and, more importantly, of $\sigma(A)$.

Proposition 4.3. *For any $A \in \mathbb{C}^{n\times n}$, the following hold:*

(i) $\Gamma(A^T) = \Gamma(A)$ *and* $\mathcal{E}(A^T) = \mathcal{E}(A)$*;*

(ii) $\Gamma(A^*) = \Gamma(\overline{A}) = \overline{\Gamma(A)}$ *and* $\mathcal{E}(A^*) = \mathcal{E}(\overline{A}) = \overline{\mathcal{E}(A)}$.

Proof. (i) Since $A^T = H(A)^T + S(A)^T$, it follows that $\delta_1(A^T) = \delta_1(A)$, $\delta_2(A^T) = \delta_2(A)$ and $\overline{y}_1$ is a unit eigenvector of $H(A)^T$ corresponding to $\delta_1(A^T)$. Hence, $\mathrm{v}(A^T) = \mathrm{v}(A)$ and $\mathrm{u}(A^T) = \mathrm{u}(A)$, and consequently, $\Gamma(A^T) = \Gamma(A)$.

For the envelope of A^T, we have

$$\mathcal{E}(A^T) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A^T) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}((e^{\mathrm{i}\,\theta}A)^T) \;=\; \mathcal{E}(A).$$

(ii) Since $A^* = H(A)^* + S(A)^* = H(A) - S(A)$, it is apparent that $\delta_1(A^*) = \delta_1(A)$, $\delta_2(A^*) = \delta_2(A)$, $\mathrm{v}(A^*) = \mathrm{v}(A)$ and $\mathrm{u}(A^*) = -\mathrm{u}(A)$. As a consequence, $\Gamma(A^*) = \overline{\Gamma(A)}$.

For the envelope of A^*, it holds that

$$\begin{aligned}\mathcal{E}(A^*) \;&=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A^*) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{\mathrm{i}\,\theta}\Gamma_{in}((e^{\mathrm{i}\,\theta}A)^*) \\ &=\; \overline{\bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A)} \;=\; \mathcal{E}(A).\end{aligned}$$

Finally, $\Gamma(\overline{A}) = \Gamma((A^*)^T) = \Gamma(A^*) = \overline{\Gamma(A)}$ and $\mathcal{E}(\overline{A}) = \mathcal{E}((A^*)^T) = \mathcal{E}(A) = \overline{\mathcal{E}(A)}$. □

Corollary 4.4. *If $A \in \mathbb{R}^{n\times n}$, then the curve $\Gamma(A)$ and the envelope $\mathcal{E}(A)$ are symmetric with respect to the real axis.*

Next we show that the curve $\Gamma(A)$ and the envelope $\mathcal{E}(A)$ are invariant under unitary similarity.

Proposition 4.5. *For every unitary matrix $U \in \mathbb{C}^{n\times n}$, $\Gamma(U^*AU) = \Gamma(A)$ and $\mathcal{E}(U^*AU) = \mathcal{E}(A)$.*

Proof. It is enough to prove the first assertion. We observe that the largest eigenvalue of $H(U^*AU)$ is $\delta_1(U^*AU) = \delta_1(A)$ with U^*y_1 as an associated unit eigenvector. Thus,

$$\mathrm{v}(U^*AU) \;=\; \|(U^*S(A)U)(U^*y_1)\|_2^2 \;=\; \|(U^*(S(A)y_1)\|_2^2 \;=\; \mathrm{v}(A)$$

and

$$\mathrm{u}(U^*AU) \;=\; \mathrm{Im}((U^*y_1)^*U^*S(A)U(U^*y_1)) \;=\; \mathrm{Im}(y_1^*UU^*S(A)UU^*y_1) \;=\; \mathrm{u}(A).$$

Since the second largest eigenvalue of $H(U^*AU)$ is $\delta_2(U^*AU) = \delta_2(A)$, the proof follows readily from the definition of $\Gamma(U^*AU)$. □

Proposition 4.6. *For any $b \in \mathbb{C}$, $\Gamma(A + bI_n) = \Gamma(A) + b$ and $\mathcal{E}(A + bI_n) = \mathcal{E}(A) + b$.*

Proof. Let $b = s_b + \mathrm{i}\, t_b$ ($s_b, t_b \in \mathbb{R}$) and $B = A + bI_n$. The hermitian and skew-hermitian parts of B are $H(B) = H(A) + s_bI_n$ and $S(B) = S(A) + \mathrm{i}\, t_bI_n$. It follows that $\delta_1(B) = \delta_1(A) + s_b$, $\delta_2(B) = \delta_2(A) + s_b$, $\mathrm{u}(B) = \mathrm{u}(A) + t_b$ and $\mathrm{v}(B) = \mathrm{v}(A) + 2\, t_b\mathrm{u}(A) + t_b^2$. As a consequence,

$$\begin{aligned} \mathrm{v}(B) - \mathrm{u}(B)^2 &= \mathrm{v}(A) + 2\, t_b\mathrm{u}(A) + t_b^2 - (\mathrm{u}(A)^2 + 2\, t_b\mathrm{u}(A) + t_b^2) \\ &= \mathrm{v}(A) - \mathrm{u}(A)^2 \end{aligned}$$

and

$$\Gamma(B) \;=\; \{s + \mathrm{i}\, t :\; s, t \in \mathbb{R} \text{ with } s - s_b + \mathrm{i}\,(t - t_b) \in \Gamma(A)\} \;=\; \Gamma(A) + \{b\}.$$

Moreover,

$$\begin{aligned} \mathcal{E}(B) &= \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}B) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}(A + bI_n)) \\ &= \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}(\Gamma_{in}(e^{\mathrm{i}\,\theta}A) + e^{\mathrm{i}\,\theta}b) \;=\; b + \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A), \end{aligned}$$

and the proof is complete. □

Proposition 4.7. *For every real $r > 0$ and $a \in \mathbb{C}$, $\Gamma(rA) = r\,\Gamma(A)$ and $\mathcal{E}(aA) = a\,\mathcal{E}(A)$.*

Proof. Let r be a positive real number. Then $\delta_1(rA) = r\delta_1(A)$, $\delta_2(rA) = r\delta_2(A)$, $\mathrm{u}(rA) = r^2\mathrm{u}(A)$ and $\mathrm{v}(rA) = r^2\mathrm{v}(A)$. Consequently, $\mathrm{v}(rA)-\mathrm{u}(rA) = r^2(\mathrm{v}(A)-\mathrm{u}(A))$, and

$$\Gamma(rA) \;=\; \left\{ s + \mathrm{i}\, t :\; s,t \in \mathbb{R} \text{ with } \frac{s}{r} + \mathrm{i}\,\frac{t}{r} \in \Gamma(A) \right\} \;=\; r\,\Gamma(A).$$

For the envelope of A, we have

$$\mathcal{E}(rA) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta} rA) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta} r\,\Gamma_{in}(e^{\mathrm{i}\,\theta}A) \;=\; r\,\mathcal{E}(A).$$

One can also see that for each $\phi \in [0, 2\pi]$,

$$\begin{aligned}
\mathcal{E}(e^{\mathrm{i}\,\phi}A) \;&=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,(\theta+\phi)}A) \\
&=\; \bigcap_{\theta+\,\phi\,\in[0,2\pi]} e^{\mathrm{i}\,\phi}\left(e^{-\mathrm{i}\,(\theta+\phi)}\Gamma_{in}(e^{\mathrm{i}\,(\theta+\phi)}A)\right) \\
&=\; e^{\mathrm{i}\,\phi} \bigcap_{\theta\in[0,2\pi]} \Gamma_{in}(e^{\mathrm{i}\,\theta}A) \;=\; e^{\mathrm{i}\,\phi}\,\mathcal{E}(A),
\end{aligned}$$

and the proof is complete. □

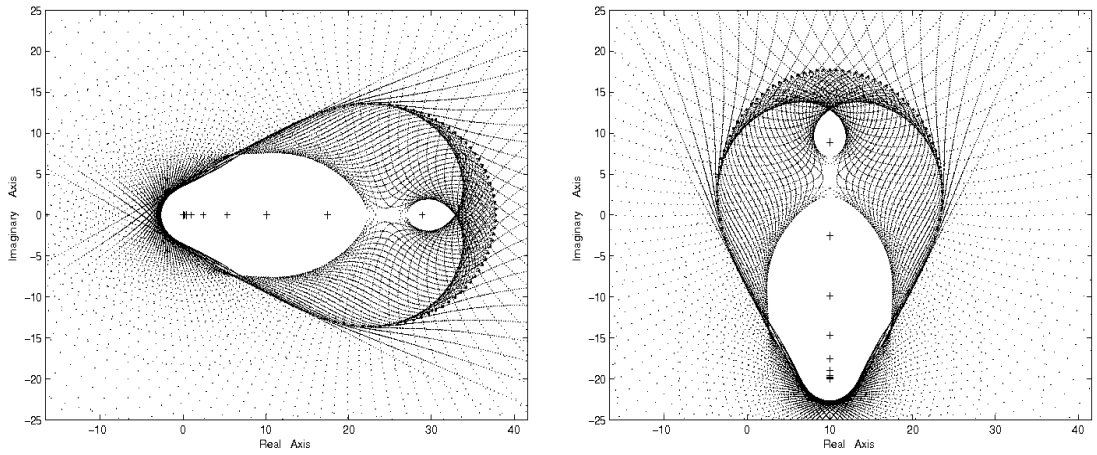

Figure 4: The envelopes of a Frank matrix (left) and its rotation-translation (right).

Example 4.8. Consider the 11×11 real Frank matrix A (that can be generated by the Matlab command `frank(11)`). The envelope $\mathcal{E}(A)$ is illustrated in the left part of Figure 4. Clearly, $\mathcal{E}(A)$ is symmetric with respect to the real axis, in accordance to Corollary 4.4. In the right part of the figure, is the envelope $\mathcal{E}(\mathrm{i}\,A + (10 - \mathrm{i}\,20)I_{11}) = \mathrm{i}\,\mathcal{E}(A) + 10 - \mathrm{i}\,20$, verifying Propositions 4.6 and 4.7.

5 Comparison of $\mathcal{E}(A)$ and $F(A)$

We will conclude with some observations regarding $\mathcal{E}(A)$ as compared to $F(A)$ in localizing the spectrum of A. Referring to the notation used in the previous sections, in computing the bounding curve $\Gamma(e^{\mathrm{i}\,\theta}A)$, $\theta \in [0, 2\pi]$, the additional computational effort required is for $\delta_2(A)$ and the quantities $\mathrm{v}(A)$ and $\mathrm{u}(A)$ which depend on y_1. However, as is evident in Examples 4.2 and 4.8, the envelope $\mathcal{E}(A)$ can represent a dramatic improvement over $F(A)$ in localizing the eigenvalues of A. In what follows, we will quantify when such an improvement should be expected and when not. As we will see, the geometry of the eigenvalues is the deciding factor.

Recall that an eigenvalue λ_0 of A is called *normal* if its algebraic and geometric multiplicities are equal and every eigenvector of A corresponding to λ_0 is orthogonal to every eigenvector of A corresponding to each eigenvalue different from λ_0. By [3, Theorem 1.6.6], every eigenvalue of A that lies on the boundary of $F(A)$ is a normal eigenvalue of A. Moreover, the non-differentiable points (corners) of $\partial F(A)$ are necessarily eigenvalues of A [3, Theorem 1.6.3].

If $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is an eigenvalue of A, then it is clear that $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a normal eigenvalue of A that lies on $\partial F(A)$, and $\mathrm{i}\,\mathrm{u}(A)$ is an eigenvalue of $S(A)$. Furthermore, the eigenvalues $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$, $\delta_1(A)$ and $\mathrm{i}\,\mathrm{u}(A)$ of A, $H(A)$ and $S(A)$, respectively, have the same eigenspaces. If, in addition, $\delta_1(A)$ is a simple eigenvalue of $H(A)$, then $\mathrm{v}(A) - \mathrm{u}(A)^2 = 0$, and the cubic curve $\Gamma(A)$ reduces to the union of the point $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ and the vertical line $\{z \in \mathbb{C} : \operatorname{Re} z = \delta_2(A)\}$. Otherwise, i.e., when $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a normal eigenvalue of A on $\partial F(A)$ and $\delta_1(A)$ is a multiple eigenvalue of $H(A)$, then (as already has been mentioned) $\Gamma(A)$ reduces to the vertical line $\{z \in \mathbb{C} : \operatorname{Re} z = \delta_1(A)\}$, and $\Gamma_{in}(A)$ coincides with the half-plane $\mathcal{H}_{in}(A)$. As a consequence, we have the following result.

Proposition 5.1. *Let λ_0 be a simple eigenvalue of A on the boundary of the numerical range $F(A)$. If λ_0 does not lie on a flat portion of $\partial F(A)$, or it is a non-differentiable point of $\partial F(A)$, then λ_0 is an isolated point of the envelope $\mathcal{E}(A)$.*

Proof. Suppose that λ_0 does not lie on a flat portion of $\partial F(A)$, or it is a non-differentiable point of $\partial F(A)$. Then, by Properties (P_2) and (P_6), there is a $\theta_0 \in [0, 2\pi]$ such that $e^{\mathrm{i}\,\theta_0}\lambda_0$ is the unique right most point of the numerical range $F(e^{\mathrm{i}\,\theta_0}A) = e^{\mathrm{i}\,\theta_0}F(A)$ (see also [2]). Note that $e^{\mathrm{i}\,\theta_0}\lambda_0$ is a simple normal eigenvalue of $e^{\mathrm{i}\,\theta_0}A$. Hence, $\delta_1(e^{\mathrm{i}\,\theta_0}A)$ is a simple eigenvalue of the hermitian matrix $H(e^{\mathrm{i}\,\theta_0}A)$, and the proposition follows from the above discussion. □

Suppose now that an eigenvalue $\lambda_0 \in \sigma(A)$ is close to being normal and also λ_0 lies relatively far from the other eigenvalues of $A \in \mathbb{C}^{n\times n}$. As a consequence of continuity and Proposition 5.1, the envelope may have a connected component $\mathcal{G}_0$ such that $\mathcal{G}_0 \cap \sigma(A) = \{\lambda_0\}$. We can construct examples (see below) where $\mathcal{E}(A)$ has n connected components. Eluding us at this stage are upper or lower bounds for the number of connected components of $\mathcal{E}(A)$, and for the number of eigenvalues in each component. Moreover, the proximity of $\mathcal{E}(A)$ to the spectrum $\sigma(A)$ is related to the

distance of A from the set of $n \times n$ normal matrices, as it is illustrated in the following example.

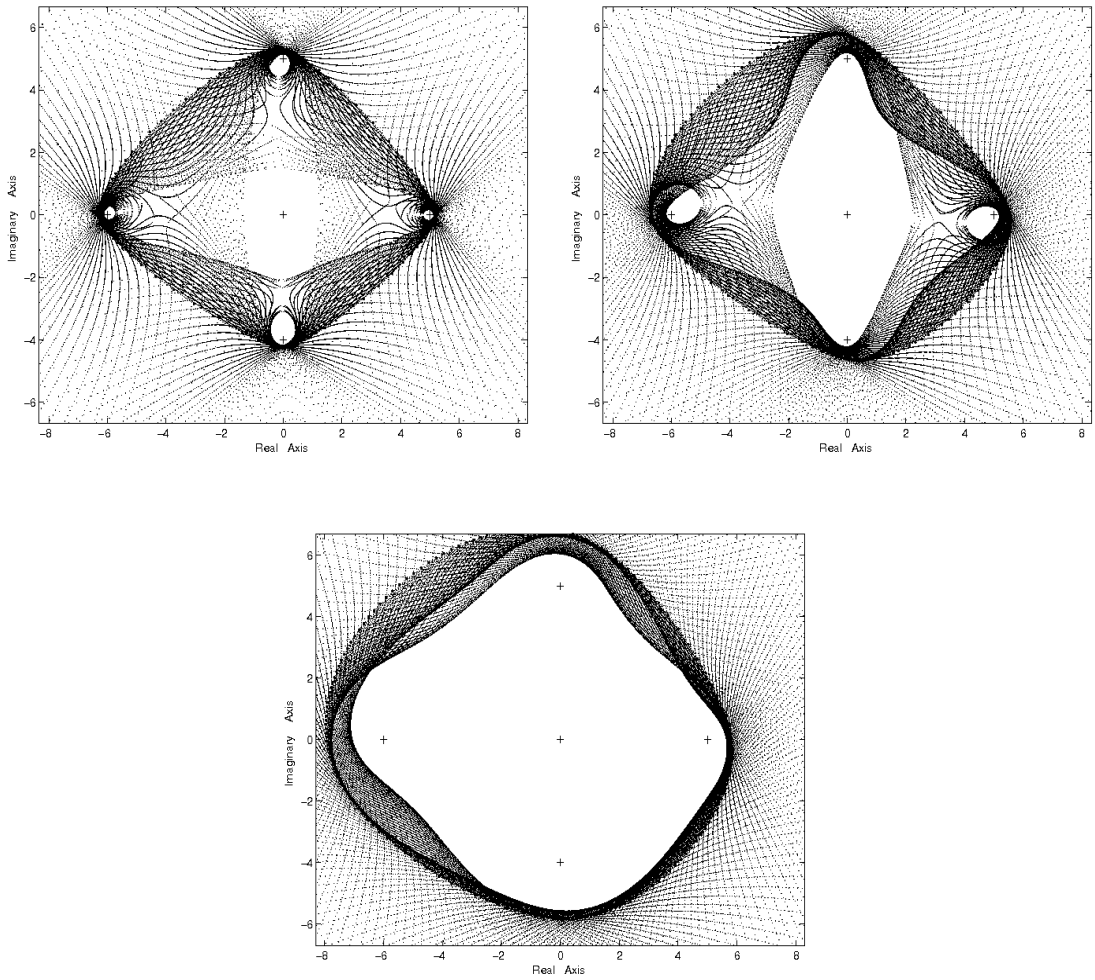

Figure 5: Envelopes of three isospectral matrices with varying distances to normality.

Example 5.2. The 5×5 triangular complex matrices

$$A = \begin{bmatrix} 5 & 1 & -1 & -\mathrm{i} & \mathrm{i} \\ 0 & -6 & 1 & -1 & -\mathrm{i}\,2 \\ 0 & 0 & -\mathrm{i}\,4 & \mathrm{i} & -\mathrm{i} \\ 0 & 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 & \mathrm{i}\,5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 & -3 & -\mathrm{i} & \mathrm{i} \\ 0 & -6 & 1 & -1 & -\mathrm{i}\,4 \\ 0 & 0 & -\mathrm{i}\,4 & \mathrm{i} & -\mathrm{i} \\ 0 & 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 & \mathrm{i}\,5 \end{bmatrix}$$

$$\text{and} \quad C = \begin{bmatrix} 5 & 1 & -4 & -\mathrm{i} & \mathrm{i} \\ 0 & -6 & 2 & -5 & -\mathrm{i}\,4 \\ 0 & 0 & -\mathrm{i}\,4 & -3 & \mathrm{i}\,4 \\ 0 & 0 & 0 & 0 & -\mathrm{i}\,4 \\ 0 & 0 & 0 & 0 & \mathrm{i}\,5 \end{bmatrix}$$

have the same spectrum. Let D be the 5×5 diagonal matrix with diagonal the common diagonal of A, B and C, and let $\|\cdot\|_F$ denote the Frobenius norm. We have

$$\frac{\|A-D\|_F}{\|A\|_F} = 0.3362, \quad \frac{\|B-D\|_F}{\|B\|_F} = 0.4944 \quad \text{and} \quad \frac{\|C-D\|_F}{\|C\|_F} = 0.7122,$$

which can be viewed as (relative) distances of A, B and C to normality, respectively. The envelopes $\mathcal{E}(A)$, $\mathcal{E}(B)$ and $\mathcal{E}(C)$ are depicted in order in Figure 5. Notice that the smaller the distance from normality, the more connected components the envelope has. In the case of A, the envelope consists of five connected components surrounding its eigenvalues, and $\mathcal{E}(A)$ represents a significant improvement over $F(A)$ (recall that the numerical range appears in the plots of the envelope as the outer outlined region). In the case of C, $\mathcal{E}(C)$ is barely smaller than $F(A)$, while $\mathcal{E}(B)$ represents an intermediate situation.

References

[1] M. Adam and M. Tsatsomeros, An eigenvalue inequality and spectrum localization for complex matrices, *Electronic Journal of Linear Algebra*, **15** (2006), 239–250.

[2] E.S. Brown and I.M. Spitkovsky, On flat portions on the boundary of numerical range, *Linear Algebra and its Applications*, **390** (2004), 75–109.

[3] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[4] C.R. Johnson, Numerical determination of the field of values of a general complex matrix, *Linear Algebra and its Applications*, **15** (1978), 595–602.

[5] B.W. Levinger, An inequality for nonnegative matrices, *Notices of the American Mathematical Society*, **17** (1970), 260.

[6] J.J. McDonald, P.J. Psarrakos and M.J. Tsatsomeros, Almost skew-symmetric matrices, *Rocky Mountain Journal of Mathematics*, **34** (2004), 269–288.

[7] J.S. Milne, *Elliptic Curves*, BookSurge Publishers, Charleston, 2006.

[8] P.J. Psarrakos and M.J. Tsatsomeros, Bounds for Levinger's function of nonnegative almost skew-symmetric matrices, *Linear Algebra and its Applications*, **416** (2006), 759–772.

[9] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, London, 2008.

[10] C. Tretter and M. Wagenhofer, The block numerical range of an $n \times n$ block operator matrix, *SIAM Journal on Matrix Analysis and Applications*, **24** (2003), 1003–1017.