A note on the controllability of higher order linear systems

G. Kalogeropoulos[1] and P. Psarrakos[2]

February 17, 2005

Abstract

In this paper, a new condition for the controllability of higher order linear dynamical systems is obtained. The suggested test contains rank conditions of suitably defined matrices and is based on the notion of compound matrices and the Binet-Cauchy formula.

Key words: companion matrix, compound matrix, closed loop system, eigenvalue, input vector, matrix polynomial, state

AMS subject classifications: 93C05, 93B05, 93B52

1 Introduction and preliminaries

Consider the higher order linear system

$$A_l q^{(l)}(t) + A_{l-1} q^{(l-1)}(t) + \cdots + A_1 q^{(1)}(t) + A_0 q(t) \;=\; Bu(t), \tag{1}$$

where $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, l)$, t is the independent time variable, $q(t) \in \mathbb{C}^n$ is the unknown vector function, $u(t) \in \mathbb{C}^m$ is the piecewise continuous *input (control) vector* and $B \in \mathbb{C}^{n\times m}$ is the *input matrix.* (The indices on $q(t)$ denote derivatives with respect to t.) Applying the Laplace transformation to (1) yields the *matrix polynomial*

$$L(\lambda) \;=\; A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_1\lambda + A_0, \tag{2}$$

where λ is a complex variable. As a consequence, the spectral analysis of $L(\lambda)$ leads to solutions of (1). The suggested references on matrix polynomials and their applications to differential equations are [1, 2].

A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $L(\lambda)$ in (2) if the system $L(\lambda_0)y = 0$ has a nonzero solution $y_0 \in \mathbb{C}^n$. This solution y_0 is known as an

[1]Department of Mathematics, University of Athens, Panepistimioupolis 15784, Athens, Greece (E-mail: gkaloger@math.uoa.gr).

[2]Department of Mathematics, National Technical University, Zografou Campus 15780, Athens, Greece (E-mail: ppsarr@math.ntua.gr).

eigenvector of $L(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $L(\lambda)$ is the *spectrum* of $L(\lambda)$, namely, $\sigma(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}$. At this point and for the remainder of this paper, we shall assume that the matrix polynomial $L(\lambda)$ in (2) has a nonsingular leading coefficient A_l, and thus, $L(\lambda)$ has exactly nl eigenvalues, counting multiplicities.

The dynamical system (1) is equivalent to the first order system

$$x^{(1)}(t) \;=\; C_L x(t) + \tilde{B} u(t), \tag{3}$$

where

$$C_L \;=\; \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_l^{-1}A_0 & -A_l^{-1}A_1 & -A_l^{-1}A_2 & \cdots & -A_l^{-1}A_{l-1} \end{bmatrix} \in \mathbb{C}^{nl\times nl}, \tag{4}$$

$$\tilde{B} \;=\; \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_l^{-1}B \end{bmatrix} \in \mathbb{C}^{nl\times m} \quad \text{and} \quad x(t) = \begin{bmatrix} q(t) \\ q^{(1)}(t) \\ \vdots \\ q^{(l-1)}(t) \end{bmatrix} \in \mathbb{C}^{nl}. \tag{5}$$

The $nl \times nl$ matrix C_L in (4) is known as the *(block) companion matrix* of $L(\lambda)$ and its spectrum coincides with $\sigma(L)$, and the vector $x(t)$ in (5) is called the *state vector* of the system (3) [1, 2, 3]. As a consequence, for a given initial condition $x_0 = x(0)$, the general solution of (1) is given by [1, Theorem 1.5]

$$q(t) \;=\; Z_r\, e^{tC_L} x_0 + Z_r \int_0^t e^{(t-s)C_L} Z_c B\, u(s)\, ds,$$

where $Z_r = [\, I \;\; 0 \;\; \cdots \;\; 0\,] \in \mathbb{C}^{n\times nl}$ and $Z_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_l^{-1} \end{bmatrix} \in \mathbb{C}^{nl\times n}$.

The notion of the *controllability* of dynamical systems has attracted attention for some years. It refers to the ability of a system to transfer the state vector from one specified vector value to another in finite time. In particular, the systems (1) and (3) are called *controllable* if for every $x_0, \omega \in \mathbb{C}^{nl}$, there exist an input vector $u(t)$ and a real $t_0 > 0$ such that $x(0) = x_0$ and $x(t_0) = \omega$.

In this article, we obtain an alternative test for the controllability of higher order linear dynamical systems. The important feature of the new rank condition (Theorem 2) is that it is independent of λ and requires no computation of the eigenvalues of $L(\lambda)$ (see statements (ii) and (iii) in Theorem 1) or the inverse of the leading coefficient A_l (see statement (v) in Theorem 1). It is assumed that the reader is familiar with the notion of compound matrices and the Binet-Cauchy formula. The suggested reference is [4].

2 The new test

One of the major concerns for a control engineer is to maintain the stability of certain systems. For this reason, in many cases, the behavior of the dynamical system (1) is modified by applying state feedback (i.e., input vector) of the form

$$u(t) \;=\; v(t) - F_{l-1}q^{(l-1)}(t) - \cdots - F_1 q^{(1)}(t) - F_0 q(t),$$

where $F_j \in \mathbb{C}^{m\times n}$ $(j = 1, 2, \ldots, l-1)$ [3, 5, 6, 7]. The new *closed loop* system

$$A_l q^{(l)}(t) + (A_{l-1} + BF_{l-1})q^{(l-1)}(t) + \cdots + (A_0 + BF_0)q(t) \;=\; Bv(t) \qquad (6)$$

is associated with the matrix polynomial

$$L_F(\lambda) \;=\; L(\lambda) + BF(\lambda), \qquad (7)$$

where $F(\lambda) = F_{l-1}\lambda^{l-1} + \cdots + F_1\lambda + F_0$.

Classical results on the controllability of first order dynamical systems (see for example [2, 8, 9] and the references therein) have been generalized to the higher order systems (1) and (6) in a natural way.

Theorem 1 ([3, Theorem 2.5 and Proposition 2.1], [7, Theorem V.2])
The following statements are equivalent:

(i) *The system (1) is controllable.*

(ii) $\mathrm{Ker}B^* \cap \mathrm{Ker}L(\lambda)^* = \{0\}$ *for all* $\lambda \in \sigma(L)$.

(iii) $\mathrm{rank}[\, L(\lambda) \;\; B \,] = n$ *for all* $\lambda \in \sigma(L)$.

(iv) *The system (6) is controllable.*

(v) $\mathrm{rank}[\, \tilde{B} \;\; C_L\tilde{B} \;\; \cdots \;\; C_L^{nl-1}\tilde{B} \,] = nl$.

Observe now that the matrix polynomial $L_F(\lambda)$ in (7) is written

$$L_F(\lambda) \;=\; \begin{bmatrix} L(\lambda) & B\lambda^{l-1} & \cdots & B\lambda & B \end{bmatrix} \begin{bmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{bmatrix}.$$

By taking the n-th compound matrix $\mathcal{C}_n(\cdot)$ of both sides in the above equation, and using the Binet-Cauchy formula for compound matrices [4], it follows

$$\begin{aligned} \det L_F(\lambda) \;&=\; \mathcal{C}_n(L_F(\lambda)) \\ &=\; \mathcal{C}_n\left(\begin{bmatrix} L(\lambda) & B\lambda^{l-1} & \cdots & B\lambda & B \end{bmatrix}\right) \mathcal{C}_n\left(\begin{bmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{bmatrix}\right). \end{aligned} \qquad (8)$$

Moreover, it is clear that

$$\mathcal{C}_n\left(\begin{bmatrix} L(\lambda) & B\lambda^{l-1} & \cdots & B\lambda & B\end{bmatrix}\right) = \left[\, p_1(\lambda) \;\; p_2(\lambda) \;\; \cdots \;\; p_\xi(\lambda) \,\right],$$

where $\xi = \left(\begin{array}{c} n+lm \\ n \end{array}\right)$, and $p_1(\lambda), p_2(\lambda), \ldots, p_\xi(\lambda)$ are scalar polynomials of degree no more than nl. For every $j = 1, 2, \ldots, \xi$,

$$p_j(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{nl}\end{bmatrix} r_j,$$

where $r_j \in \mathbb{C}^{nl+1}$ is the vector of the (corresponding) coefficients of $p_j(\lambda)$. Hence, for the $(nl+1)\times\xi$ complex matrix

$$P(L(\lambda), B) = \left[\, r_1 \;\; r_2 \;\; \cdots \;\; r_\xi \,\right], \tag{9}$$

which is known as the *Plücker matrix* of the system (1) [10, 11], it follows

$$\mathcal{C}_n\left(\begin{bmatrix} L(\lambda) & B\lambda^{l-1} & \cdots & B\lambda & B\end{bmatrix}\right) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{nl}\end{bmatrix} P(L(\lambda), B). \tag{10}$$

Theorem 2 *The higher order dynamical system (1) is controllable if and only if the Plücker matrix $P(L(\lambda), B)$ in (9) has full (row) rank, i.e.,*

$$\mathrm{rank} P(L(\lambda), B) = nl + 1.$$

Proof Suppose that the system (1) is controllable. Then by applying [9, Theorem 2.1] to the first order system (3), we have that the system (1) is controllable if and only if the spectrum of the closed loop system (6) can be assigned arbitrarily by suitable choice of $F_0, F_1, \ldots, F_{l-1}$. Hence, for every monic scalar polynomial $d(\lambda)$ of degree nl, there exist $m\times n$ matrices $F_0, F_1, \ldots, F_{l-1}$ such that the matrix polynomial $L_F(\lambda)$ in (7) satisfies (recall that $\det A_l \neq 0$)

$$\det L_F(\lambda) = \det A_l \, d(\lambda).$$

Hence, equation (8) yields

$$\mathcal{C}_n\left(\begin{bmatrix} L(\lambda) & B\lambda^{l-1} & \cdots & B\lambda & B\end{bmatrix}\right) \mathcal{C}_n\left(\begin{bmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{bmatrix}\right) = \det A_l \begin{bmatrix} 1 & \lambda & \cdots & \lambda^{nl}\end{bmatrix} z_d,$$

where z_d is the vector of the (corresponding) coefficients of $d(\lambda)$. Denoting

$$g_F = \mathcal{C}_n\left(\begin{bmatrix} I \\ F_{l-1} \\ \vdots \\ F_1 \\ F_0 \end{bmatrix}\right),$$

by the above discussion, it follows that

$$\left[\, 1 \;\; \lambda \;\; \lambda^2 \;\; \cdots \;\; \lambda^{nl} \,\right] P(L(\lambda), B)\, g_F \;=\; \det A_l \left[\, 1 \;\; \lambda \;\; \lambda^2 \;\; \cdots \;\; \lambda^{nl} \,\right] z_d$$

for every $\lambda \in \mathbb{C}$, which implies that

$$P(L(\lambda), B)\, g_F \;=\; \det A_l\, z_d. \tag{11}$$

This system has $nl+1$ equations and $\xi = \begin{pmatrix} n+lm \\ n \end{pmatrix}$ unknowns, and since $n, m, l \geq 1$, one can see that $\xi \geq nl+1$. As a consequence, (11) has solutions for every vector $z_d \in \mathbb{C}^{nl+1}$ (with its first coordinate equal to 1) if and only if the Plücker matrix $P(L(\lambda), B)$ has full (row) rank.

Conversely, assume that $\operatorname{rank} P(L(\lambda), B) = nl+1$ and that the dynamical system (1) is not controllable. Then by Theorem 1, there is a $\lambda_0 \in \mathbb{C}$ such that $\operatorname{rank}[\, L(\lambda_0) \;\; B \,] < n$. Moreover,

$$\operatorname{rank} \left[\, L(\lambda_0) \;\; B\lambda_0^{l-1} \;\; \cdots \;\; B\lambda_0 \;\; B \,\right] \;<\; n,$$

which means that all the $n \times n$ minors of the matrix $\left[\, L(\lambda_0) \;\; B\lambda_0^{l-1} \;\; \cdots \;\; B\lambda_0 \;\; B \,\right]$ are zero. Hence,

$$\mathcal{C}_n \left(\left[\, L(\lambda_0) \;\; B\lambda_0^{l-1} \;\; \cdots \;\; B\lambda_0 \;\; B \,\right] \right) \;=\; 0$$

and by (10),

$$\left[\, 1 \;\; \lambda_0 \;\; \lambda_0^2 \;\; \cdots \;\; \lambda_0^{nl} \,\right] P(L(\lambda), B) \;=\; 0.$$

Since $\xi \geq nl+1$ and $\left[\, 1 \;\; \lambda_0 \;\; \lambda_0^2 \;\; \cdots \;\; \lambda_0^{nl} \,\right] \neq 0$, it is clear that

$$\operatorname{rank} P(L(\lambda), B) \;<\; nl+1,$$

that is a contradiction. Thus, the system (1) is controllable. $\square$

Notice that the above method involves no computation of the spectrum $\sigma(L)$ or the matrix A_l^{-1}. Our result is illustrated in the following example.

Example Let $L(\lambda)$ be the 2×2 matrix polynomial

$$L(\lambda) \;=\; I\lambda^2 + A_1\lambda + A_0 \;=\; \begin{bmatrix} \lambda^2+1 & \lambda-1 \\ \lambda-1 & \lambda^2-1 \end{bmatrix},$$

and let $B_1 = [\, 0 \;\; 1\,]^T$ and $B_2 = [\, 1 \;\; 0\,]^T$. Consider the second order linear systems

$$q^{(2)}(t) + A_1 q^{(1)}(t) + A_0 q(t) \;=\; B_1 u(t) \tag{12}$$

and

$$q^{(2)}(t) + A_1 q^{(1)}(t) + A_0 q(t) \;=\; B_2 u(t), \tag{13}$$

where $u(t)$ is the 1×1 input vector. For every $\lambda \in \mathbb{C}$,

$$\operatorname{rank}[\, L(\lambda) \;\; B_1 \,] \;=\; \operatorname{rank} \begin{bmatrix} \lambda^2+1 & \lambda-1 & 0 \\ \lambda-1 & \lambda^2-1 & 1 \end{bmatrix} \;=\; 2.$$

On the other hand, for $\lambda_0 = 1$,

$$\operatorname{rank}[\,L(\lambda_0)\ \ B_2\,] \;=\; \operatorname{rank}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \;=\; 1 \;<\; 2.$$

Thus, by Theorem 1, the system (12) is controllable but the system (13) is not. Furthermore, we can see that

$$\begin{aligned}
\mathcal{C}_2([\,L(\lambda)\ \ B_1\lambda\ \ B_1\,]) \;&=\; \mathcal{C}_2\left(\begin{bmatrix} \lambda^2+1 & \lambda-1 & 0 & 0 \\ \lambda-1 & \lambda^2-1 & \lambda & 1 \end{bmatrix}\right) \\
&=\; [\,\lambda^4-\lambda^2+2\lambda-2,\ \lambda^3+\lambda,\ \lambda^2+1,\ \lambda^2-\lambda,\ \lambda-1,\ 0\,]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_2([\,L(\lambda)\ \ B_2\lambda\ \ B_2\,]) \;&=\; \mathcal{C}_2\left(\begin{bmatrix} \lambda^2+1 & \lambda-1 & \lambda & 1 \\ \lambda-1 & \lambda^2-1 & 0 & 0 \end{bmatrix}\right) \\
&=\; [\,\lambda^4-\lambda^2+2\lambda-2,\ -\lambda^2+\lambda,\ -\lambda+1,\ -\lambda^3+\lambda,\ -\lambda^2+1,\ 0\,].
\end{aligned}$$

As a consequence,

$$\operatorname{rank}P(L(\lambda),B_1) \;=\; \operatorname{rank}\begin{bmatrix} -2 & 0 & 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \;=\; 5$$

and

$$\operatorname{rank}P(L(\lambda),B_2) \;=\; \operatorname{rank}\begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \;=\; 4 \;<\; 5,$$

confirming Theorem 2. $\square$

References

[1] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York, (1982).

[2] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd edition, Academic Press, Orlando, (1985).

[3] P. Lancaster, *Lectures on Linear Algebra, Control and Stability*, Research Paper No 801, University of Calgary, Calgary, (1998).

[4] F.R. Gantmacher, *The Theory of Matrices*, 2nd edition, Chelsea Publishing Company, New York, (1959).

[5] B.N. Datta and F. Rincón, Feedback stabilization of a second order system: A nonmodal approach, *Linear Algebra Appl.* **188/189** 135-161, (1991).

[6] B.N. Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Academic Press, New York, (to appear).

[7] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Trans. on Automatic Control* **AC-36** 259-294, (1991).

[8] P. Antsaklis and A. Michel, *Linear Systems*, McGraw-Hill Companies, New York, (1997).

[9] W.M. Wonham, *Linear Multivariable Control*, Springer Verlag, Berlin, (1979).

[10] G. Kalogeropoulos, D. Kytagias and K.Arvanitis, On the computation of a reduced set of quadratic Plücker relations and their use in the solution of the determinental assignement problem, *Systems Sci.* **26** (2) 2-25, (2000).

[11] N. Karcanias and C. Giannakopoulos, Grassman invariants, almost zeros and the determinantal zero, pole assignemment problems of linear systems, *Internat. J. Control* **40** 673-698, (1984).