

Ενδεικτικές Λύσεις Φυλλαδίου 1

1. **i)** Είναι $[XX^{-1}]^T = I^T = I \Rightarrow (X^{-1})^T X^T = I \Rightarrow (X^T)^{-1} = (X^{-1})^T$. **ii)** Είναι: $[(XY)^{-1}]^T = [Y^{-1}X^{-1}]^T = (X^{-1})^T(Y^{-1})^T \stackrel{(i)}{=} (X^T)^{-1}(Y^T)^{-1}$.

2. **i)** Είναι $A^2 = I$ και $A^3 = A^2 A = IA = A$, οπότε για κάθε $\nu \in \mathbb{N}$ έχουμε: $A^{2\nu} = (A^2)^\nu = I^\nu = I$ και $A^{2\nu+1} = A^{2\nu} A = IA = A$.

ii) $I + A + A^2 + \dots + A^{2\nu} = (I + A^2 + A^4 + \dots + A^{2\nu}) + (A + A^3 + \dots + A^{2\nu-1}) = (\nu+1)I + \nu A$.

3. Είναι:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

και $A^5 = \mathbb{O}$, οπότε $A^\nu = \mathbb{O}$ για κάθε $\nu \geq 5$.

4. Αν $A = (a_{ij})$ και $B = (b_{ij})$ τότε

$$\text{i)} \quad \text{tr}(A+B) = \sum_{i=1}^n (a_{ii} + b_{ii}) = \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} = \text{tr}(A) + \text{tr}(B).$$

$$\text{ii)} \quad \text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji} a_{ij} = \text{tr}(BA).$$

iii) Από i) και ii) έχουμε $n = \text{tr}(I) = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$, αδύνατο.

iv) Έστω $A = (a_{ij})$, $A^* = (\overline{b_{ij}})$, οπότε $b_{ij} = \overline{a_{ji}}$, και

$$\text{tr}(AA^*) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji} = \sum_{j=1}^n \sum_{i=1}^n a_{ij} \overline{a_{ij}} = \sum_{j=1}^n \sum_{i=1}^n |a_{ij}|^2 = 0$$

$\Rightarrow a_{ij} = 0$ για κάθε $i, j = 1, 2, \dots, n \Rightarrow A = \mathbb{O}$.

5. Είναι: $-A - A^2 - \dots - A^\kappa = I \Rightarrow A(-I - A + \dots + A^{\kappa-1}) = I$
 $\Rightarrow A^{-1} = -I - A - \dots - A^{\kappa-1} = A^\kappa$.

6. $(A + 3I)^2 = \mathbb{O} \Rightarrow A^2 + 6A + 9I = \mathbb{O} \Rightarrow A(A + 6I) = -9I \Rightarrow A(-\frac{1}{9}(A + 6I)) = (-\frac{1}{9}(A + 6I))A = I \Rightarrow A^{-1} = -\frac{1}{9}(A + 6I)$.

$\mathbb{O} = (A + 3I)^2 = ((A + I) + 2I)^2 = (A + I)^2 + 4(A + I) + 4I = \mathbb{O} \Rightarrow (A + I)((A + I) + 4I) = ((A + I) + 4I)(A + I) = -4I \Rightarrow (A + I)^{-1} = -\frac{1}{4}(A + 5I)$.

7. $(A + B)^2 = A + B \Rightarrow A^2 + AB + BA + B^2 = A + B \Rightarrow A + AB + BA + B = A + B \Rightarrow AB + BA = \mathbb{O} \quad (1)$.

Από (1) παίρνουμε: $\left\{ \begin{array}{l} A^2 B + ABA = \mathbb{O} \\ ABA + BA^2 = \mathbb{O} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} AB = -ABA \\ BA = -ABA \end{array} \right\} \Rightarrow AB = BA \quad (2)$.

Από (1),(2) προκύπτει $AB = BA = \mathbb{O}$.

8.

$$\begin{aligned}
 A = \left[\begin{array}{cc|cc|c} 2 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \Rightarrow A^3 = \left[\begin{array}{ccc} \left[\begin{array}{cc} 2 & 0 \\ -1 & 3 \end{array} \right]^3 & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} 0 & 0 \end{array} \right] & [2]^3 & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] & \left[\begin{array}{cc} 2 & 0 \\ 1 & -1 \end{array} \right]^3 \end{array} \right] = \\
 \left[\begin{array}{ccc} \left[\begin{array}{cc} 8 & 0 \\ -19 & 27 \end{array} \right] & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} 0 & 0 \end{array} \right] & [8] & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{c} 0 \\ 0 \end{array} \right] & \left[\begin{array}{cc} 8 & 0 \\ 3 & -1 \end{array} \right] \end{array} \right] \Rightarrow A^3 = \left[\begin{array}{ccccc} 8 & 0 & 0 & 0 & 0 \\ -19 & 27 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 3 & -1 \end{array} \right].
 \end{aligned}$$