APPROXIMABILITY OF THE GENERALIZED INVERSE OF AN OPERATOR

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator with closed range. It is proved that the generalized inverse T^+, of T, is a polynomial of T if and only if T is algebraic and commutes with T^+. It is also given sufficient conditions so that the generalized inverse T^+, belongs to the weakly closed algebra generated by T and the identity operator. Finally it is shown that T^+ always belongs to the weakly closed star algebra generated by T and the identity operator.

Key words. Linear operator, generalized inverse, algebra of operators, approximability.

AMS subject classifications. 15A15, 15F10.

1. Introduction. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is finite dimensional and T is invertible then there is a polynomial p such that $T^{-1} = p(T)$. The infinite dimensional analogue of this fact is generally false even in the weak operator topology. Many authors have studied this problem and have given sufficient and necessary and sufficient conditions to ensure that T^{-1} is a limit of polynomials of T in different operator topologies, e.g. [2],[3],[4],[5],[6]. In

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this paper we study the analogue problem concerning the generalized inverse T^+ (the unique Moore-Penrose inverse) of an operator T with closed range. Note that the generalized inverse T^+ is a bounded operator if and only if the range of T is closed. We give necessary and sufficient conditions so that T^+ to be a polynomial of T and sufficient conditions so that the generalized inverse T^+ to be in the weakly closed algebra generated by T and the identity operator. We also show that T^+ is always in the weakly closed star algebra generated by T and the identity operator.

2. Preliminaries and notation. Let $\mathcal{H}$ be a complex Hilbert space. $\mathcal{B}(\mathcal{H})$ will denote the algebra of bounded linear operators on $\mathcal{H}$, and for $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(T)$ will denote the range of T, $\mathcal{N}(T)$ will denote the kernel of T and $\operatorname{Lat} T$ will denote the set of all closed invariant subspaces of T. The weakly closed algebra of operators generated by T and the identity operator I is denoted by $\mathcal{A}(T, I)$ and the weakly closed star algebra generated by T and I is denoted by $\mathcal{A}(T, T^\star)$. For each natural number n, $\mathcal{H}^{(n)}$ stands for the orthogonal direct sum of n copies of $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $T^{(n)}$ will denote the operator on $\mathcal{H}^{(n)}$ defined by $T^{(n)}(x_1, x_2, \ldots, x_n) = (Tx_1, Tx_2, \ldots, Tx_n)$. Then, from a result in [8], we have $\mathcal{A}(T, I) = \{A \in \mathcal{B}(\mathcal{H}) : \operatorname{Lat} T^{(n)} \subseteq \operatorname{Lat} A^{(n)},$ for any $n \in \mathbb{N}\}$.

If T is normal then from Sarason [9], we have that $\mathcal{A}(T, I) = \{A \in \mathcal{B}(\mathcal{H}) : \operatorname{Lat} T \subseteq \operatorname{Lat} A\}$.

The generalized inverse (Moore- Penrose inverse) of an operator $T \in \mathcal{B}(\mathcal{H})$ with closed range, is the unique operator satisfying the following equations:

$$TT^+ = (TT^+)^\star, \qquad T^+T = (T^+T)^\star, \qquad TT^+T = T, \qquad T^+TT^+ = T^+,$$

in which $T^\star$ denotes the adjoint operator of T.

It's easy to see that $\mathcal{R}(T^+) = \mathcal{N}(T)^\perp$, TT^+ is the orthogonal projection of $\mathcal{H}$ onto

$\mathcal{R}(T)$ and that T^+T is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(T)^\perp$.

3. Weak approximations. Note that if T is normal then T^+ is also normal and $T^+T = TT^+$. Pearl in [7] proved that if the Hilbert space is finite dimensional then T commutes with T^+ if and only if T^+ can be expressed as a polynomial of T. In the infinite dimensional case we prove that this is true only for algebraic operators.

THEOREM 3.1. *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ with closed range and let T^+ be its generalized inverse. Then T^+ is a polynomial of T if and only if $T^+T = TT^+$ and T is algebraic.*

Proof. Let $T^+ = p(T)$, where p is a polynomial. Then $T^+T = p(T)T = Tp(T) = TT^+$ and $T = TT^+T = Tp(T)T$. Therefore T commutes with T^+ and T is algebraic. Conversely let T be algebraic and $T^+T = TT^+$. Then $m(T) = T^s + a_1T^{s+1} + \cdots + a_kT^{s+k} = 0$, where $m(t)$ is the minimal polynomial of T. We will show that $s = 1$. Since T is not invertible we have $s \neq 0$. Let $s \geq 2$. Then

$$T^+m(T) = 0 \Rightarrow TT^+T(T^{s-2} + T^{s-1} + \cdots + T^{s+k-2}) = 0$$

$$\Rightarrow T^{s-1} + T^s + \cdots + T^{s+k-1} = 0, \quad \text{a contradiction. Therefore} \quad s = 1$$

and

$$T + a_1T^2 + \cdots + T^{k+1} = 0 \Rightarrow T^+T = -a_1T^+T^2 - \cdots - a_kT^+T^{k+1} = 0.$$

Since $T^+T = TT^+$, we have

$$TT^+ = -a_1T - a_2T^2 - \cdots - a_kT^k. \tag{3.1}$$

Hence multiplying equation (3.1) by T^+ from the left, we get

$$\begin{aligned} T &= T^+TT^+ = -a_1T^+T - a_2T^+T^2 - \cdots - a_kT^+T^k \\ &= -a_1T^+T - a_2T - \cdots - a_kT^{k-1} = Q(T). \end{aligned} \tag{3.2}$$

Hence T^+ is a polynomial of T. □

REMARK 3.2. Since in the infinite dimensional case most of the operators are not algebraic, a natural question to rise is:
When T^+ belongs to the algebra $\mathcal{A}(T,I)$, or more strongly, when T^+ is a weak limit of a sequence of polynomials in T?

In the sequel we will give sufficient conditions for T so that T^+ be in the weakly closed algebra generated by T and the identity. Note that if T^+ belongs to $\mathcal{A}(T,I)$ then the projections TT^+, T^+T belong too. More over since $\mathcal{A}(T,I)$ is commutative we must have $TT^+ = T^+T$. So the conditions $TT^+ = T^+T$ and $TT^+, T^+T \in \mathcal{A}(T,I)$ are necessary for T^+ to be in $\mathcal{A}(T,I)$.

DEFINITION 3.3. Let $T \in \mathcal{B}(\mathcal{H})$. T is called almost full if for every invariant subspace M, different of $\mathcal{N}(T)$ and $\mathcal{H}$, is $\overline{TM} = M$.

In the finite dimensional case every operator with eigenvectors that do not span $\mathcal{H}$ is almost full.

PROPOSITION 3.4. *If $T \in \mathcal{B}(\mathcal{H})$ is an almost full and normal operator with closed range, then T^+ belongs to $\mathcal{A}(T,I)$.*

Proof. Let $M \in \operatorname{Lat} T$. Then, if $M = \mathcal{N}(T)$, since T is normal, we have $\mathcal{N}(T) = \mathcal{N}(T^+)$ and hence $\mathcal{N}(T) \in \operatorname{Lat} T^+$. Let now $M \neq \mathcal{N}(T)$. Then $M = \overline{TM} \subseteq \mathcal{R}(T)$ and

$$T^+M = T^+(\overline{TM}) \subseteq \overline{T^+(TM)} = \overline{(TT^+)M} = M.$$

Hence $M \in \operatorname{Lat} T^+$ and so $\operatorname{Lat} T \subseteq \operatorname{Lat} T^+$. Since T is normal, from [9], we have $T^+ \in \mathcal{A}(T,I)$. □

LEMMA 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with closed range, commuting with T^+. Then $\mathcal{N}(T) = \mathcal{N}(T^+)$, the operators T, T^+ have the same eigenvectors and $\lambda \neq 0$ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^+.*

Proof. Let $x \in \mathcal{N}(T)$. Then $Tx = 0 \Rightarrow TT^+x = T^+Tx = 0 \Rightarrow T^+TT^+x = 0 \Rightarrow T^+x = 0$. Similarly $T^+x = 0 \Rightarrow Tx = 0$.

Let x be an eigenvector of T. Then $x \in \mathcal{R}(T)$ and there exists a scalar λ such that $Tx = \lambda x$. Since TT^+ is the orthogonal projection with range $\mathcal{R}(T)$, we have $\lambda T^+x = T^+Tx = TT^+x = x$ and hence $T^+x = \frac{1}{\lambda}x$. The converse is proved similarly. □

From the proof of Proposition 3.4 and the above Lemma we get the following

COROLLARY 3.6. *If $T \in \mathcal{B}(\mathcal{H})$ is an almost full operator which commutes with T^+ then $Lat T \subseteq Lat T^+$.*

Next proposition is a rephrase of a result of Wermer's paper [10]

PROPOSITION 3.7. *If T is a linear bounded normal operator on a Hilbert space $\mathcal{H}$, with a non empty point spectrum, then each of the following conditions is equivalent to spectral synthesis*

(i) *Every closed invariant subspace of T contains at least one eigenvector*

(ii) *Every closed invariant subspace of T is also invariant under $T^\star$*

PROPOSITION 3.8. *If T is a linear bounded operator on a Hilbert space $\mathcal{H}$ with closed range and non empty point spectrum, then each of the following conditions implies $T^+ \in \mathcal{A}(T, I)$*

(i) *T is normal and admits spectral synthesis*

(ii) *T is normal and there is a sequence of polynomials p_n such that $p_n(T) \to T^\star$ weakly.*

(iii) *T is completely normal*

(iv) *T is polynomially compact normal operator.*

Proof. **(i)** Since T is normal and hence it commutes with T^+, from Lemma 3.5, we have that T^+ has the same eigenvectors with T. Spectral synthesis now for T

implies that $\mathrm{Lat}\, T \subseteq \mathrm{Lat}\, T^+$. Therefore, from Sarason [9], $T^+ \in \mathcal{A}(T, I)$.

(ii) The hypothesis implies that T admits spectral synthesis (see Theorem 5, [10]).

(iii) If T is completely normal, then by definition, T is normal and all its invariant subspaces are reducing. Therefore from Proposition 3.7(ii), T admits spectral synthesis.

(iv) Immediate from (iii), since every polynomially compact normal operator is completely normal. □

An immediate consequence of Proposition 3.8(iii) is that if T is selfadjoint operator with closed range and non empty point spectrum, then $T^+ \in \mathcal{A}(T, I)$.

The next two results are well known and will be used to prove that for every normal operator its generalized inverse belongs to the weakly closed star algebra generated by the operator and the identity.

THEOREM 3.9. *If T is a normal operator on a Hilbert space $\mathcal{H}$ and if the operator A commutes with every projection that commutes with T then A belongs to the weakly closed star-algebra generated by T and the identity.*

THEOREM 3.10. *If A, B are operators with closed ranges, then $(AB)^+ = B^+A^+$ if and only if the following three conditions hold*

(i) *The range of AB is closed*

(ii) *$\mathcal{R}(A^\star)$ is invariant under $BB^\star$*

(iii) *$\mathcal{R}(A^\star) \cap \mathcal{N}(B^\star)$ is invariant under $A^\star A$*

PROPOSITION 3.11. *If T is a linear bounded normal operator on a Hilbert space $\mathcal{H}$ with closed range, then T^+ belongs to the weakly closed star algebra $\mathcal{A}(T, T^\star)$, generated by T and the identity.*

Proof. According to Theorem 3.9, it is sufficient to show that T^+ commutes with every projection that commutes with T. Let P be an orthogonal projection such that

$PT = TP$. We want to show $PT^+ = T^+P$. Since $P^+ = P$ (for P is a projection) it will be enough to show (i) $(PT)^+ = T^+P$ and (ii) $(TP)^+ = PT^+$. To do this, it is enough to confirm, each one of the three conditions of Theorem 3.10.

$\mathcal{R}(PT)$ is a closed subspace, since $\mathcal{R}(PT) = P(T(\mathcal{H})) = \mathcal{R}(PT) \cap \mathcal{R}(T)$. Indeed, obviously $\mathcal{R}(PT) \subseteq \mathcal{R}(PT) \cap \mathcal{R}(T)$ and if $y \in \mathcal{R}(PT) \cap \mathcal{R}(T)$ then $y = Tx$ for some $x \in \mathcal{H}$ and $y = Py = PTx \in \mathcal{R}(PT)$.

$\mathcal{R}(P)$ is invariant reducing subspace of T and hence invariant under $TT^\star$.

Obviously $\mathcal{R}(P) \cap \mathcal{N}(T^\star)$ is invariant under $P^\star P = P$.

Therefore $(PT)^+ = T^+P$. Similarly $(TP)^+ = PT^+$. Hence $T^+ \in \mathcal{A}(T, T^\star)$ and the proof is completed. □

PROPOSITION 3.12. *For any operator $T \in \mathcal{B}(\mathcal{H})$ with closed range, its generalized inverse T^+ belongs to Von-Neumann algebra generated by T.*

Proof. From Theorem 4.4 [1], we have $T^+ = (T^\star T)^+ T^\star$. But $T^\star T$ is selfadjoint and hence, from Proposition 3.11, $(T^\star T)^+$ belongs to the von-Neumann algebra $\{T^\star T\}''$, which is a subalgebra of the von-Neumann algebra $\{T, T^\star\}''$. Hence $T^+ \in \{T, T^\star\}''$. □

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