PARTIAL ISOMETRIES AND EXTREME POINTS OF CERTAIN OPERATOR ALGEBRAS

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Abstract

In this paper we try to find classes of operators in Alg$\mathcal{L}$, where $\mathcal{L}$ is a completely distributive subspace lattice, which are extreme points. To this direction we analyze a special example and give sufficient conditions, concerning the initial and final spaces of a partial isometry, so that to be an extreme point of $(\mathrm{Alg}\mathcal{L})_1$. We prove that a partial isometry is extreme if and only if it is maximal within Alg$\mathcal{L}$.

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1 Introduction

In what follows $\mathcal{H}$ will be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ will denote the algebra of bounded linear operators on $\mathcal{H}$. Let $\mathcal{L}$ be a completely distributive lattice of subspaces in $\mathcal{H}$ and consider the algebra $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ of operators in $\mathcal{B}(\mathcal{H})$ leaving invariant each subspace in $\mathcal{L}$. Moore and Trent [7] gave necessary and sufficient conditions for an operator in $\mathcal{A}$ to be an extreme point of the unit ball $\mathcal{A}_1 = \{A \in \mathcal{A} \mid \|A\| \leq 1\}$. Since these conditions refer to all projections in $\mathcal{L}$ it is of interest to know classes of operators in $\mathcal{A}$ which are extreme points in $\mathcal{A}_1$. For example isometries or co-isometries in $\mathcal{A}_1$ are extreme points. In fact we see that these operators have a stronger property. They are strong extreme points of $\mathcal{A}_1$ (and any norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ to which they belong). To this end, we discuss a simple example and give sufficient conditions for a partial isometry to be an extreme point in the unit ball of $\mathrm{Alg}\mathcal{L}$, in terms of its initial and final spaces. We prove that a partial isometry is an extreme point of $\mathcal{A}_1$ if and only if it is maximal within $\mathcal{A}$. Finally we give sufficient conditions for a partial isometry to belong to a reflexive operator algebra.

By a *subspace* of $\mathcal{H}$ we mean a closed subspace and all the *projections* are assumed self adjoint. For a projection E let $E^{\perp} = I - E$. For two vectors $e, f \in \mathcal{H}$ the operator $e \otimes f^*$ is defined by $(e \otimes f^*)x = (x, f)e$ for $x \in \mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ and if $\|A \pm B\| = \|A\|$ for an operator $B \in \mathcal{B}(\mathcal{H})$ we will say that B is a *perturbation* of A. Note that if $\|A \pm B\| \leq \|A\|$ then $\|A \pm B\| = \|A\|$. The set of *extreme points* in $\mathcal{A}_1$ is denoted by $\mathrm{ext}\mathcal{A}_1$. If $A \in \mathcal{A}_1$ then $A \in \mathrm{ext}\mathcal{A}_1$ if and only if there exists a non zero perturbation of A. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called *reflexive* if $\mathcal{A} = \mathrm{AlgLat}\mathcal{A}$, where $\mathrm{Lat}\mathcal{A}$ denotes the set of all projections P for which $AP = PAP$ for every $A \in \mathcal{A}$. If S is a subset of $\mathcal{H}$, $[S]$ denotes the closed linear span of S. Throughout this paper when it is convenient we identify a subspace with the projection on it.

2 Example

Let $\{e_1, e_2\}$ be an orthonormal basis of a two dimensional Hilbert space $\mathcal{H}$ and consider the nest $\mathcal{N} = \{0,\ P = [e_1],\ I\}$. The corresponding nest algebra $\mathcal{A} = \mathrm{Alg}\mathcal{N}$ consists of the 2×2 upper triangular matrices,

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ a, b, c \in C \right\}.$$

For $A \in \mathcal{A}$, given by a matrix as above, it is easy to check by standard calculations that

$$\|A\| = 1 \Longleftrightarrow |b|^2 = (1 - |a|^2)(1 - |c|^2)$$

Moreover, it is not difficult to show that $A \in \mathcal{A}_1$ if and only if $\|A\| = 1$ and either $|a| = |c| = 1$ or $\max\{|a|, |c|\} < 1$. Finally, A is a partial isometry if and only if $abc = 0$. Thus by taking $a = c = 0,\ b = 1$ we have a partial isometry (actually a projection) which is not an extreme point, and by taking for example $a = c = 1/2,\ b = 3/4$ we get an extreme point which is not a partial isometry.

The following proposition characterizes when a rank one operator is an extreme point.

Proposition 1 *A rank one operator $e \otimes f^*$ in a nest algebra $\mathcal{A} = \mathrm{Alg}\mathcal{N}$ is an extreme point of $\mathcal{A}_1$ if and only if the nest $\mathcal{N}$ is of the form $\{0,\ P,\ I\}$, where $\dim P = 1$ and $\dim I = 2$.*

Proof. The proof, aside from the Theorem 7,[7], is easy and is omitted. (See also remark 2).

In the sequel we find sufficient conditions for a partial isometry in Alg$\mathcal{A}$ to be extreme. For this we need the following result which is a restatement of Lemma 2 in [3] and the definition of a strong extreme point in [2].

Lemma 2 *Let $\mathcal{A}$ be an algebra of operators in a Hilbert space and $S \in \mathcal{A}$. If either S or S^* is an isometry then for each $W \in \mathcal{A}$,*

$$\max\{\|S + W\|,\ \|S - W\|\} \geq \sqrt{1 + \|W\|^2}.$$

Definition 3 *A unit vector x in a Banach space X is said to be a* strong extreme point *of the unit ball X_1 if and only if, for every $\varepsilon > 0$ there exist $\delta > 0$ such that $\|y\| \leq \varepsilon$ whenever* $\max\{\|x+y\|, \|x-y\|\} \leq 1+\delta$.

Note that every strong extreme point is an extreme point. We show that in a norm-closed algebra of operators, that contains the compact operators, the converse is also true.

Proposition 4 *Every extreme point in a norm-closed algebra $\mathcal{A}$ of operators that contains the compact operators is a strong extreme point.*

Proof. From Theorem 5 in [7] we know that the extreme points of the unit ball of $\mathcal{A}$ are the isometries and co-isometries in $\mathcal{A}$. Let $U \in \text{ext}\mathcal{A}_1$ and for $\varepsilon > 0$ take $\delta = \sqrt{1+\varepsilon^2} - 1$. For an operator $A \in \mathcal{A}$ such that $\max\{\|U+A\|, \|U-A\|\} \leq 1+\delta$, using Lemma 2, we have

$$1 + \|A\|^2 \leq \max\{\|U+A\|^2, \|U-A\|^2\} \leq (1+\delta)^2 = 1+\varepsilon^2.$$

Therefore $\|A\| \leq \varepsilon$ and hence U is a strong extreme point of $\mathcal{A}_1$.

Corollary 5 *Let $\mathcal{A}$ be as in Proposition 4 and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Then*

1. *Every isometry or co-isometry in $\mathcal{B}$ is a strong extreme point of $\mathcal{B}_1$.*
2. *Every partial isometry $V \in \mathcal{B}$ with $P = V^*V$ and $Q = VV^*$ is a strong extreme point of the unit ball of the space $Q\mathcal{B}P$.*

Proof. The operator $V = QVP$ is an isometry on the range of P.

Proposition 6 *Let $\mathcal{A} = \text{Alg}\mathcal{L}$, where $\mathcal{L}$ is a completely distributive lattice of projections. If V is a partial isometry in $\mathcal{A}$ with initial space $P = V^*V$ and final space $Q = VV^*$ such that either*

1. *$Q \in \mathcal{L}$ and $[range(I - Q^\perp P^\perp)] = \mathcal{H}$ or*

2. *$P^\perp \in \mathcal{L}$ and $I - Q^\perp P^\perp$ is one to one*

then $V \in \text{ext}\mathcal{A}_1$.

Proof. Let V be a partial isometry in $\mathcal{A}$ such that $V = \frac{1}{2}(X+Y)$ where X, $Y \in \mathcal{A}_1$. Then $V = VP = \frac{1}{2}(XP+YP)$. Since V is an isometry on the range of P, from Corollary 5, V is an extreme point of $(\mathcal{A}P)_1$. Therefore $V = XP = YP$ and hence

$$(X - Y)P = 0. \tag{1}$$

Similarly since V^* is an isometry in the range of Q it is an extreme point of $(\mathcal{A}^*Q)_1$ and therefore from $V^* = \frac{1}{2}(X^*Q + Y^*Q)$ we have $V^* = X^*Q = Y^*Q$ and hence

$$Q(X - Y) = 0. \tag{2}$$

Combining (1) and (2) we get

$$X - Y = Q^\perp(X - Y)P^\perp. \tag{3}$$

Now if $Q \in \mathcal{L}$ and the range of $(I - Q^\perp P^\perp)$ is dense, then from (3) we have $X - Y = Q^\perp(X - Y)Q^\perp P^\perp$, therefore $(X - Y)(I - Q^\perp P^\perp) = 0$ and hence $X = Y$.

Analogously, if $P^\perp \in \mathcal{L}$ and $I - Q^\perp P^\perp$ is one to one then, from (3), $X - Y = Q^\perp P^\perp(X - Y)$, therefore $(I - Q^\perp P^\perp)(X - Y) = 0$ and hence $X = Y$.

Therefore $V \in \text{ext}\mathcal{A}_1$.

Remark 1 Every partial isometry $V \in \text{Alg}\mathcal{L}$ such that $V^*V = E^\perp$ and $VV^* = E$ for some $E \in \mathcal{L}$ is an extreme point of $(\text{Alg}\mathcal{L})_1$.

Remark 2 Conditions 1 and 2 of Proposition 6 are not necessary for a partial isometry to be in $\text{ext}(\text{Alg}\mathcal{L})_1$. To see this we give the following example.

Example. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces and $\mathcal{N}_1$, $\mathcal{N}_2$ be two nests of subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ correspondingly. If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{N} = \{N \oplus 0,\, N \in \mathcal{N}_2\} \cup \{\mathcal{H}_2 \oplus M,\, M \in \mathcal{N}_1\}$ then the nest algebra corresponding to $\mathcal{N}$ is

$$\mathrm{Alg}\mathcal{N} = \left\{ \begin{pmatrix} A & X \\ 0 & B \end{pmatrix},\, A \in \mathrm{Alg}\mathcal{N}_2,\, B \in \mathrm{Alg}\mathcal{N}_1,\, X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \right\}.$$

Let also $N \in \mathcal{N}_2$, $M \in \mathcal{N}_1$ and consider the partial isometries V_1 with initial space $P_N^\perp$ and final space P_N and V_2 with initial space $P_M^\perp$ and final space P_M. Then $V_1 \in \mathrm{Alg}\mathcal{N}_2$, $V_2 \in \mathrm{Alg}\mathcal{N}_1$ and the operator

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \in \mathrm{Alg}\mathcal{N}$$

is a partial isometry with initial space $N^\perp \oplus M^\perp$ and final space $M \oplus N$. It is ease to prove that $V \in \mathrm{ext}(\mathrm{Alg}\mathcal{N})_1$. Obviously V satisfies none of the conditions 1 and 2 of Proposition 6.

3 Maximal partial isometries within $\mathrm{Alg}\mathcal{L}$

Halmos in [4] defines a partial order for partial isometries as follows: If U and V are partial isometries write $V < U$ in case U agrees with V on the initial space of V. This implies that $(\ker V)^\perp \subseteq (\ker U)^\perp$.

Definition 7 *Let $\mathcal{A}$ be an algebra of operators. We say that a partial isometry $U \in \mathcal{A}$ is* maximal within $\mathcal{A}$ *if for any partial isometry $U \in \mathcal{A}$ such that $V < U$ implies $V = U$.*

Theorem 8 *Let $\mathcal{L}$ be a completely distributive lattice and $\mathcal{A} =$ $\mathrm{Alg}\mathcal{L}$. A partial isometry $V \in \mathcal{A}_1$ is an extreme point of $\mathcal{A}_1$ if and only if V is maximal within $\mathcal{A}$.*

Proof. Let V be a partial isometry, maximal within $\mathcal{A}$ and $V \notin \text{ext}\mathcal{A}_1$. Then there exists a projection $E \in \mathcal{L}$ such that $P^\perp \cap E_-^\perp \neq 0$ and $Q^\perp \cap E \neq 0$ where $P = V^*V$ and $Q = VV^*$. Consider unit vectors $f \in P^\perp \cap E_-^\perp$ and $e \in Q^\perp \cap E$. Then the rank one operator $e \otimes f^*$ belongs to $\mathcal{A}_1$, the operator $V + e \otimes f^*$ is also in $\mathcal{A}_1$, it is a partial isometry and $V < V + e \otimes f^*$ which contradicts the maximality of V within $\mathcal{A}$. Therefore $V \in \text{ext}\mathcal{A}_1$.

Conversely. Let V, U be partial isometries in $\mathcal{A}$ such that $V \in \text{ext}\mathcal{A}_1$ and $V \leq U$. We show that $V = U$. Put $A = U - V$. Then $\|A\| = 1$, $\|V + A\| = \|U\| = 1$ and $\|V - A\| = \|2V - U\| = 1$. Therefore A is a perturbation of V and since $V \in \text{ext}\mathcal{A}_1$ we have $A = 0$. Hence $U = V$ and so V is maximal within $\mathcal{A}$.

Remark 3 Partial isometries that satisfy the hypothesis of Proposition 6 are maximal within $\text{Alg}\mathcal{L}$.

Remark 4 If $\mathcal{N}$ is a nest then a rank one operator is maximal within $\text{Alg}\mathcal{N}$ if and only if $\mathcal{N} = \{0, P, I\}$, $\dim P = 1$ and $\dim I = 2$. This is an immediate consequence of Proposition 6 and Theorem 8 above. Note that this can also be proved directly without any difficulty.

Remark 5 In the finite dimensional case every extreme point is a strong extreme point. Indeed McGuigan in [6] has shown that if X is a separable conjugate space such that the norm and the weak star convergence of sequences agree on the surface of the unit ball, then every extreme point of the unit ball of X is strong extreme. Now if X is finite dimensional then its conjugate space X^* is also finite dimensional and the weak star topology coincides with the norm topology on X^* (and this happens if and only if X is finite dimensional). Since a finite dimensional space is reflexive the result follows from McGuigan's Theorem. A consequence of this is that all the extreme points in the example of section 2 are strong extreme.

Remark 6 There is a notion of an inner operator, parallel to that of an inner function, for operators in a special type of nest algebras. This notion is given by Areveson in [1]. Let $\mathcal{N}$ be a $\overline{\mathbb{Z}}$ ordered nest of projections where $\overline{\mathbb{Z}} = (-\infty) \cup \mathbb{Z} \cup (+\infty)$.This means that there exists an order lattice isomorphism between $\mathcal{N}$ and a subset of $\overline{\mathbb{Z}}$. Then an operator $U \in \mathrm{Alg}\mathcal{N}$ is called *inner* if U is a partial isometry whose the initial space U^*U commutes with every $P_n \in \mathcal{N}$. Theorem 1 in [2] says that if $f \in H^\infty(U)$, where U is the open unit disc in $\mathbb{C}$, then f is a strong extreme point of the unit ball of $H^\infty(U)$ if and only if f is an inner function. As a corollary from this Theorem we have the following Proposition:

Proposition 9 *Let $\mathcal{N}$ be an N-ordered nest of projections and $\mathcal{A} = \mathrm{Alg}\mathcal{N}$. Then a Toeplitz operator $T_f \in \mathcal{A}$ is a strong extreme point of $\mathcal{A}_1$ if and only if $\|f\|_\infty = 1$ and $\overline{f} \in H^\infty(U)$ is inner.*

Proof. For each $f \in L^\infty$ the corresponding Toeplitz operator T_f is defined by the equation $(T_f e_j, e_{j+n}) = \hat{f}(n)$ for all $j \geq 0$ and all integers n , where $\{e_j\}$ is an orthogonal basis in the Hilbert space $\mathcal{H}$ and $\hat{f}(n)$ is the nth Fourier coefficient of the function f. It is well known that $T_f \in \mathrm{Alg}\mathcal{N}$ if and only if $\overline{f} \in H^\infty$. Also for $f \in L^\infty, \|T_f\| = \|f\|_\infty$. Let $T_f \in \mathcal{A}_1$ be a strong extreme point of $\mathcal{A}_1$. If $\overline{f}$ is not inner then $\overline{f}$ is not a strong extreme point of the unit ball of H^∞. Therefore there exists $\varepsilon > 0$ and a function $g \in H^\infty(U)$ such that for any $\delta > 0$,

$\|f \pm g\| \leq 1+\delta$ and $\|g\|_\infty > \varepsilon$. Consider the Toeplitz operator $T_{\overline{g}}$. Then $T_{\overline{g}} \in \mathcal{A}$,

$$\|T_f \pm T_{\overline{g}}\| = \|f \pm \overline{g}\|_\infty = \|\overline{f} \pm g\|_\infty \leq 1 + \delta$$

and $\|T_{\overline{g}}\| = \|\overline{g}\|_\infty > \varepsilon$, which is a contradiction. Hence $\overline{f}$ is an inner function.

Conversely, if $\overline{f}$ is inner, $\|f\|_\infty = 1$ then T_f is an isometry and hence a strong extreme point of $(\mathrm{Alg}\mathcal{N})_1$.

The following result is of independent interest and gives sufficient conditions for a partial isometry to belong to a given reflexive algebra.

Proposition 10 *Let $\mathcal{A}$ be a reflexive algebra of operators and $\mathcal{L} = \mathrm{Lat}\mathcal{A}$. Suppose $V \in \mathcal{B}(\mathcal{H})$ is a partial isometry such that either $P = V^*V$ commutes with $\mathcal{L}$ and $\|EVx\| \geq \|Ex\|$ for every $E \in \mathcal{L}$ and for every $x \in P\mathcal{H}$ or $Q = VV^*$ commutes with $\mathcal{L}$ and $\|E^\perp V^* x\| \geq \|E^\perp x\|$ for every $E \in \mathcal{L}$ and for every $x \in Q\mathcal{H}$. Then $V \in \mathcal{A}$.*

Proof. Suppose that P commutes with $\mathcal{L}$ and $\|EVx\| \geq \|Ex\|$ for every $E \in \mathcal{L}$ and for every $x \in P\mathcal{H}$. (We work similarly when Q commutes with $\mathcal{L}$ and $\|E^\perp V^*\| \geq \|E^\perp x\|$ for every $x \in Q\mathcal{H}$ and $E \in \mathcal{L}$). We have

$$\|EVEx\| = \|EVPEx\| = \|EVEPx\| \geq \|EPx\| = \|Ex\|,\ x \in P\mathcal{H},\ E \in \mathcal{L}.$$

Hence

$$\begin{aligned} \|E^\perp VEx\|^2 &= \|VEx\|^2 - \|EVEx\|^2 \leq \|VEx\|^2 - \|Ex\|^2 \\ &\leq \|V\|^2\|Ex\|^2 - \|Ex\|^2 = 0,\ x \in P\mathcal{H},\ E \in \mathcal{L}. \end{aligned}$$

Therefore $\|E^\perp VEx\| = 0$ for every $x \in P\mathcal{H}$ and $E \in \mathcal{L}$. Since $EVEx = 0$ for every $x \in P^\perp\mathcal{H}$ and $E \in \mathcal{L}$ we have $\|E^\perp VE\| = 0$ for all $E \in \mathcal{L}$. Equivalently $E^\perp VE = 0$ for all $E \in \mathcal{L}$ and hence $V \in \mathcal{A}$.

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