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ON THE NEST ALGEBRA-MODULE FACTORIZATIONS

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Abstract

In this paper, we define special type nest algebra module factorizations of a bounded operator and prove a factorization result when the module is determined by a decreasing order homomorphism. This is a generalization of the corresponding nest case.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$, $\mathcal{E}$ be a complete nest of projections, $\mathcal{A} = \text{Alg}\,\mathcal{E}$ be the corresponding algebra and $\mathcal{U}$ be an $\mathcal{A}$-module determined by the order homomorphism $\phi : E \to \widetilde{E}$, $E \in \mathcal{E}$. Then

$$\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H}) : XE = \widetilde{E}XE, \text{ for all } E \in \mathcal{E}\}.$$

The module $\mathcal{U}$ is an algebra if and only if $\widetilde{\widetilde{E}} \leq \widetilde{E}$ for all $E \in \mathcal{E}$, (see [2, Lemma 1.8]).

We recall the following definition from [4].

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Definition 1. The set $\mathcal{R}(\mathcal{E}, \sim)$ consists of all the operators $X \in \mathcal{U}$ such that for every $\varepsilon > 0$ there exists a partition $\mathcal{P}$ of $\mathcal{E}$ such that $\| \Delta\widetilde{E}_i X \Delta E_i \| < \varepsilon$, $1 \leq i \leq n$.

The set $\mathcal{R}(\mathcal{E}, \sim)$ is norm closed and has the property that for every $X \in \mathcal{R}(\mathcal{E}, \sim)$ the module diagonal integral $\mathcal{D}(X)$ is zero. Hence

$$X = (m)\int_{(\mathcal{E}, \sim)} \widetilde{E} X dE.$$

If we restrict the order homomorphism ϕ suitably we can get a bijection map ϕ_1. Indeed for each element Q in $\phi(\mathcal{E})$ there is a largest element $E \in \mathcal{E}$ such that $\phi(E) = Q$. Let $\mathcal{E}_1$ be the nest consisting of these largest elements corresponding to the members of $\phi(\mathcal{E})$. The order homomorphism $\phi_1 = \phi|_{\mathcal{E}_1} : \mathcal{E}_1 \to \phi(\mathcal{E})$ is a bijection and $\mathcal{U} = \mathrm{Op}\,\phi = \mathrm{Op}\,\phi_1$. It is proved in [3] that $\mathrm{Op}\,\phi = \{X \in \mathcal{B}(\mathcal{H}) : XE \subseteq \phi(E) \text{ for every } E \in \mathcal{E}_1\}$. Let ψ be the inverse of ϕ_1. Define

$$\mathcal{V} = \mathrm{Op}\,\psi = \{X \in \mathcal{B}(\mathcal{H}) : (I - E)X\widetilde{E} = 0 \text{ for every } E \in \mathcal{E}_1\}.$$

Then $\mathcal{V}$ contains $(\mathcal{U}^{\perp})^*$. When the order homomorphism is decreasing, $\mathcal{U}$ is an algebra and $(\mathcal{U}^{\perp})^*$ contains the identity. The following set $\mathcal{V}_{nil}(\mathcal{U})$ is defined in [6]:

$$\mathcal{V}_{nil}(\mathcal{U}) = \{X \in \mathcal{U} : YX \text{ is quasinilpotent for each } Y \in \mathcal{V}\}.$$

We also recall a part of Theorem 3.10, [6] and restate it for our use.

Theorem 1. *For an* $X \in \mathcal{U}$ *the following are equivalent:*

(i) $X \in \mathcal{V}_{nil}(\mathcal{U})$.

(ii) $X \in \mathcal{R}(\mathcal{E}, \sim)$.

A consequence of the above is that when the order homomorphism, which describes the module $\mathcal{U}$, is decreasing, then the set $\mathcal{R}(\mathcal{E}, \sim)$ consists of quasinilpotent operators.

2. The Main Result

In the sequel, we define special type factorizations concerning an operator $A \in \mathcal{B}(\mathcal{H})$, according to the set $\mathcal{R}(\mathcal{E}, \sim)$.

Definition 2. Let A be an invertible operator. A representation

$$A = (I + R)S$$

is called a *left-*$\mathcal{R}(\mathcal{E}, \sim)$ *factorization* of A with respect to the module $\mathcal{U}$ if $R \in \mathcal{R}(\mathcal{E}, \sim)$ and $S \in \mathcal{U}^{\perp}$. A left-$\mathcal{R}(\mathcal{E}, \sim)$ factorization of A is called *left regular* if $I + R$ is invertible and $(I + R)^{-1} \in (\mathcal{U}^{\perp})^{*}$, *right regular* if S is invertible and $S^{-1} \in \mathcal{U}^{*}$ and *regular* if it is both left and right regular. A *right-*$\mathcal{R}(\mathcal{E}, \sim)$ *factorization* is defined similarly. A left-$\mathcal{R}(\mathcal{E}, \sim)$ factorization of A is called *left compact* if R is a compact operator.

For other types of module factorizations, see [5]. Note that the notions left regular and regular are different from those defined in Definition 2, [5], since $I + R \notin \mathcal{U}$ because $\mathcal{U}$ does not contain the identity. When the order homomorphism is the identity map, the module $\mathcal{U}$ is a nest algebra and the $\mathcal{R}(\mathcal{E}, \sim)$-factorizations defined above coincide with the radical factorizations defined in [1].

Lemma 1. *If $\mathcal{U}$ is determined by a decreasing order homomorphism, then a left-$\mathcal{R}(\mathcal{E}, \sim)$ right regular factorization of A with respect to $\mathcal{U}$ is regular.*

Proof. Since $\tilde{E} \leq E$, every $E \in \mathcal{E}$ implies $\tilde{\tilde{E}} \leq \tilde{E}$. For every $E \in \mathcal{E}$, the module $\mathcal{U}$ is an algebra. Also $(I + R)E = E + \tilde{E}RE = E(I + R)E$ for every $E \in \mathcal{E}$ and hence $I + R \in \operatorname{Alg} \mathcal{E}$. So $R \in \operatorname{Alg} \mathcal{E}$. Since R is quasinilpotent $(I + R)^{-1}$ exists and belongs to $\operatorname{Alg} \mathcal{E}$. The module $\mathcal{U}^{\perp}$ contains the identity and, therefore, $\operatorname{Alg} \mathcal{E}^{\perp} = (\operatorname{Alg} \mathcal{E})^{*} \subseteq \mathcal{U}^{\perp}$ which implies $\operatorname{Alg} \mathcal{E} \subseteq (\mathcal{U}^{\perp})^{*}$. Hence $(I + R)^{-1} \in (\mathcal{U}^{\perp})^{*}$ and, therefore, the factorization is regular.

Theorem 2. *Let A be an invertible operator, $\mathcal{E}$ be a complete nest and $\mathcal{U}$ be an $\operatorname{Alg}\mathcal{E}$ module which is determined by a decreasing order homomorphism $\phi : \mathcal{E} \to \mathcal{E}$, $\phi(E) = \tilde{E}$. Suppose that A has a left-$\mathcal{R}(\mathcal{E}, \sim)$ right regular factorization and the operators $EA^{-1}\tilde{\tilde{E}}$ and $\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]^{-1}$ are invertible for every $E \in \mathcal{E}$, $\tilde{\tilde{E}} \neq 0$. Then the integral*

$$(m)\int_{(\mathcal{E},\, \sim)}\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]^{-1} EA^{-1}dE$$

exists and the factorization is given by $A = (I + R)S$, where

$$R = -(m)\int_{(\mathcal{E},\, \sim)}\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]^{-1} EA^{-1}dE$$

and $S \in \mathcal{U}^{\perp}$.

If, in addition, the factorization is left compact, then the operator $\tilde{E}AE^{\perp}$ is compact for every $E \in \mathcal{E}$.

Proof. Suppose that $A = (I + R)S$ is the left-$\mathcal{R}(\mathcal{E}, \sim)$ right regular factorization of A. By Lemma 1, the factorization is regular and hence $R \in \mathcal{R}(\mathcal{E}, \sim)$, $(I + R)^{-1} \in (\mathcal{U}^{\perp})^{*}$, $S \in \mathcal{U}^{\perp}$ and $S^{-1} \in \mathcal{U}^{*}$. From Lemma 4, [5] the operator $\tilde{E}SE$ is invertible with inverse the operator $ES^{-1}\tilde{E}$. Now from $EA^{-1}\tilde{\tilde{E}} = ES^{-1}(I + R)^{-1}\tilde{\tilde{E}} = (ES^{-1}\tilde{E})\left[\tilde{E}(I + R)^{-1}\tilde{\tilde{E}}\right]$, we have that the operator $\tilde{E}(I + R)^{-1}\tilde{\tilde{E}}$ is invertible for $E \in \mathcal{E}$, $\tilde{\tilde{E}} \neq 0$ and since $\left[\tilde{\tilde{E}}(I + R)\tilde{E}\right]\left[\tilde{E}(I + R)^{-1}\tilde{\tilde{E}}\right] = \tilde{\tilde{E}}$, its inverse is the operator $\tilde{\tilde{E}}(I + R)\tilde{E}$. Hence $\tilde{E}(I + R)^{-1}\tilde{\tilde{E}}(I + R)\tilde{E} = \tilde{E}$ for every $E \in \mathcal{E}$, $\tilde{\tilde{E}} \neq 0$. Therefore, since

$$\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]\left[I - \tilde{E} + (I + R)\tilde{E}S\right] = I - E$$

$$= ES^{-1}(I + R)^{-1}\tilde{\tilde{E}}(I + R)\tilde{E}S$$

$$= I - E + ES^{-1}\tilde{E}S$$

$$= I - E + ES^{-1}\tilde{E}S = I,$$

we have that

$$\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]^{-1} = I - \tilde{E} + (I + R)\tilde{E}S.$$

Now if we put $X = (I + R)^{-1} - I$, then $X \in \mathcal{R}(\mathcal{E}, \sim)$ and

$$\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]^{-1}\tilde{\tilde{E}}A^{-1} = \left[I - \tilde{E} + (I + R)\tilde{E}S\right]\tilde{\tilde{E}}S^{-1}(I + R)^{-1}$$

$$= (I + R)\tilde{E}(X + I).$$

Hence

$$(m)\int_{(\mathcal{E}, \sim)}\left[I - E + EA^{-1}\tilde{\tilde{E}}\right]^{-1}\tilde{\tilde{E}}A^{-1}dE = (m)\int_{(\mathcal{E}, \sim)}(I + R)\tilde{E}(X + I)\,dE$$

$$= (I + R)(m)\int_{(\mathcal{E}, \sim)}\tilde{E}X\,dE$$

$$= (I + R)X = -R.$$

Suppose now that the factorization is left compact. Then the operator R is compact. But $\tilde{E}AE^{\perp} = \tilde{E}(I + R)SE^{\perp} = \tilde{E}(I + R)\tilde{E}^{\perp}SE^{\perp} = \tilde{E}R\tilde{E}^{\perp}SE^{\perp}$. Hence the operator $\tilde{E}AE^{\perp}$ is compact.

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