FULL OPERATORS AND APPROXIMATION OF INVERSES

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1. *Introduction*

Let X be a Banach space. If X is finite-dimensional and T is an invertible linear operator on X then there is a polynomial p such that $T^{-1} = p(T)$. Since the analogue in the infinite-dimensional case is generally false even in the weak operator topology (see [**5**, §1]) it is of interest to look for conditions sufficient to ensure that T^{-1} is a limit of polynomials in T in the weak (equivalently, strong) operator topology. Such conditions were given in [**4**, **6**] when X is a Hilbert space. Also, necessary and sufficient conditions for an invertible operator T on a Hilbert space such that its inverse T^{-1} is a weak limit of polynomials of T were given in [**3**, **7**].

Here we give sufficient, and necessary and sufficient conditions for such a T when X is a uniformly convex Banach space.

In Section 2 we prove a lemma which gives a sufficient condition for a bounded operator on a uniformly convex Banach space to be full. Then we use this to get analogous results, as in [**3**], on uniformly convex Banach spaces. We also prove stronger and more general results than the results in [**7**] and generalize Lemma 3 of Section 3 in [**5**].

A result due to Erdos in [**3**] states that if T is an injective quasinilpotent dissipative operator on a Hilbert space then every maximal nest of subspaces invariant under T is continuous. It is shown below that this result is also true in a uniformly convex Banach space when we replace the hypothesis 'T is an injective dissipative operator' by '$0 \notin V(T)$ where $V(T)$ is the spatial numerical range of T'.

Finally we apply these results to find necessary and sufficient conditions for the operator $T^{(n)}$, the direct sum of n copies of T, to be full for all positive integers n.

In this paper, the term *Banach space* will mean complex Banach space, *subspace* will mean closed subspace and *operator* will mean bounded linear operator. Let X be a Banach space. We denote by $\mathscr{B}(X)$ the algebra of all bounded operators on X. We use the symbol X^* for the (continuous) dual space of X and $\setminus$ means 'set-theoretic difference'. If $S(X)$ is the unit sphere of X (that is, $S(X) = \{x \in X : \|x\| = 1\}$) then the *spatial numerical range* of an operator $T \in \mathscr{B}(X)$ denoted by $V(T)$ is defined by $V(T) = \{f(Tx) : f \in S(X^*), x \in S(X), f(x) = 1\}$. The *numerical range* of $T \in \mathscr{B}(X)$ is defined by $V(\mathscr{B}(X), T) = \{f(TA) : f \in S(\mathscr{B}(X)^*), A \in S(\mathscr{B}(X)), f(A) = 1\}$. By $\sigma(T)$ we denote the *spectrum* of T. For a subset M of X the *closure* of M is denoted by $\bar{M}$ or $\operatorname{cl} M$, the *convex hull* of M is denoted by $\operatorname{co} M$ and the *closed convex hull* of M is denoted by $\overline{\operatorname{co}}\, M$. It is shown in [**1**] that $\overline{\operatorname{co}}\, V(T) = V(\mathscr{B}(X), T)$. The smallest subspace containing each member of a subset M of X (that is, the closed linear span of M) will be denoted by $\operatorname{cls} M$.

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The operator $T \in \mathscr{B}(X)$ is called *full* if $\overline{T(M)} = M$ for every invariant subspace M of T. A Banach space X is called *strictly convex* if and only if x and y are linearly dependent whenever $\|x+y\| = \|x\|+\|y\|$; and X is said to be *uniformly convex* if for each ε, $0 < \varepsilon \leqslant 2$, there corresponds $\delta(\varepsilon) > 0$ such that the conditions $\|x\| = \|y\| = 1$, $\|x-y\| \geqslant \varepsilon$ imply that $\|\frac{1}{2}(x+y)\| \leqslant 1-\delta(\varepsilon)$. For any subset $\mathscr{A}$ of $\mathscr{B}(X)$ we denote by Lat $\mathscr{A}$ the set of all subspaces of X that are invariant under each member of $\mathscr{A}$. A set $\mathscr{N}$ of subspaces of X is said to be a *nest* if it is totally ordered by inclusion. A nest $\mathscr{N}$ is called *complete* if it contains (0) and X and is complete as a lattice. If $N \neq (0)$, $N \in \mathscr{N}$, the subspace N_- is defined by

$$N_- = \mathrm{cl}\left\{\bigcup\{M \in \mathscr{N} : M \subseteq N,\ M \neq N\}\right\}.$$

A complete nest is said to be *continuous* if $N_- = N$ for every non-zero member N. A nest is said to be *maximal* if it is not a proper subnest of any nest of subspaces of X.

2. *Approximation of inverses*

The following lemma is a restatement of Lemma 1 in [**10**].

LEMMA 1. *Let X be a Banach space and let $\mathscr{N}$ be a nest of subspaces of X. Then $\mathscr{N}$ is maximal if and only if $\mathscr{N}$ is complete and $N \in \mathscr{N}$, $N \neq (0)$ implies that the quotient space N/N_- is at most one-dimensional.*

LEMMA 2. *Let X be a uniformly convex Banach space. If T is a bounded linear operator on X with $0 \notin V(T)$, where $V(T)$ denotes the spatial numerical range of T, then T is full.*

Proof. Suppose that T is not full. Then there exists an invariant subspace N of T such that $\overline{T(N)} = M$ is a proper subspace of N. Let $x \in N \backslash M$ be a unit vector. Then $Tx \in M$ and by the Hahn–Banach theorem, there exists a continuous linear functional $f \in X^*$ such that

$$f(M) = 0 \qquad \text{and} \qquad f(x) = 1\,.$$

If $\|f\| \leqslant 1$ then, since $f(x) = 1 = \|x\|$, $\|f\| = 1$ and hence

$$0 = f(Tx) \in V(T),$$

which is a contradiction. Thus $\|f\| > 1$. Put $g = f|_N$, the restriction of f to N, and let $h = g\|g\|^{-1}$. Then $\|h\| = 1$ and $h \in N^*$.

Since N is a uniformly convex Banach space, there exists a (unique) unit vector $y \in N$ such that $h(y) = 1 = \|h\|$. But $h(M) = 0$. Hence $y \notin M$ and so $y \in N\backslash M$. Using the Hahn–Banach theorem again, there exists an extension ϕ of h to X such that $\|\phi\| = \|h\| = 1$ and so $\phi(y) = 1$ and $\phi(M) = 0$. Since $Ty \in M$ we have $\phi(Ty) = 0$ and hence

$$0 = \phi(Ty) \in V(T),$$

which is a contradiction. Therefore $\overline{T(N)} = N$ and so T is full.

Corollary 3. *If T is a bounded invertible operator on a uniformly convex Banach space then $0 \notin V(T)$ implies that* $\operatorname{Lat} T \subseteq \operatorname{Lat} T^{-1}$.

Proof. Let $M \in \operatorname{Lat} T$. Since T is an invertible operator, $T(M)$ is a closed subspace of M. Using Lemma 2, $0 \notin V(T)$ implies that T is full. Hence $T(M) = M$ and therefore $M \in \operatorname{Lat} T^{-1}$.

Theorem 4. *Let X be a uniformly convex Banach space and T a bounded linear quasinilpotent operator on X with $0 \notin V(T)$. If M and N are invariant subspaces of T such that $M \subset N$, then* $\dim(N/M) \neq 1$.

Proof. From Lemma 2, T is full and so $\overline{T(N)} = N$. Also T is injective, for if x is a unit vector such that $Tx = 0$ then, since by the Hahn–Banach theorem there exists a linear functional $f \in X^*$ such that $\|f\| = 1$ with $f(x) = 1$, we have $0 = f(Tx) \in V(T)$, contradicting our hypothesis.

Suppose that $\dim(N/M) = 1$. Then there exists a unit vector $x \in N \backslash M$ such that

$$\operatorname{cls}\{x+M\} = N/M .$$

Since N is invariant under T we have $Tx \in N$ and hence Tx can be expressed (uniquely) in the form

$$Tx = \alpha x + y, \tag{1}$$

where α is a scalar and $y \in M$.

From (1) we have $(T-\alpha I)x \in M$ and so $(T-\alpha I)N \subseteq M$. If $\alpha = 0$ we get a contradiction. Suppose that $\alpha \neq 0$. Then, since T is quasinilpotent, $(T-\alpha I)^{-1}$ exists and is a norm limit of polynomials in T. Hence

$$N = (T-\alpha I)^{-1}(T-\alpha I)(N) \subseteq (T-\alpha I)^{-1} M \subseteq M ,$$

which is also a contradiction. Therefore $\dim(N/M) \neq 1$.

Corollary 5. *If T is a bounded quasinilpotent operator on a uniformly convex Banach space with $0 \notin V(T)$ then any maximal nest of subspaces invariant under T is continuous.*

Proof. This is immediate from Lemma 1 and Theorem 4.

Remark. It is well known that if T is a bounded operator on a strictly convex (and hence on a uniformly convex) Banach space and if $\lambda \in V(T)$ such that $|\lambda| = \|T\|$ then λ is an eigenvalue of T (see [**1**, Theorem 8, p. 93]). It follows from this that if λ is a number in $\overline{V(T)}$ such that $|\lambda| = \|T\|$ and λ is not an eigenvalue of T (and, in particular if T has no eigenvalues) then λ does not belong to $V(T)$. It is easy to see that any eigenvalue of T is in the $V(T)$. We know also that $\sigma(T) \subseteq \overline{V(T)}$ [**11**]. Therefore any quasinilpotent operator such that $0 \notin V(T)$ is also an example of an operator with numerical range not closed. The following example from [**8**, problem 170], shows that the numerical range is not closed even for compact operators. Furthermore the example shows that Corollary 5 above describes a non-empty class of quasinilpotent operators for which any maximal nest of invariant subspaces (if it exists) is continuous.

EXAMPLE. Let $X = L^2[0,1]$ and let B be the Volterra integration operator on $L^2[0,1]$,

$$(Bf)(x) = \int_0^x f(t)dt\,.$$

Put $A = I-(I+B)^{-1} = B(I+B)^{-1}$. Then A is a compact quasinilpotent operator and $0 \notin V(A)$ (see [**8**, problem 170]). Therefore $V(A) \neq \overline{V(A)}$ and by Corollary 5 every maximal nest of subspaces invariant under A is continuous.

It follows easily from [**9**, Theorem 2.14, p. 33] that $\operatorname{Lat} A = \operatorname{Lat} B$. Hence we recapture the well-known result, concerning the Volterra integration operator, that every maximal nest of subspaces invariant under the Volterra operator is continuous.

Let $X^{(n)}$ be the direct sum of n copies of a uniformly convex Banach space. Then $X^{(n)}$ with norm

$$\|x\| = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}, \qquad 1 < p < \infty\,,$$

$$x = \sum_{i=1}^{n} \oplus x_i, \qquad x_i \in X, \;\; i = 1, 2, \ldots, n$$

is also a uniformly convex Banach space (see [**2**, Theorem 1]).

Let $(X^{(n)})^*$ be the dual space of $X^{(n)}$ and let $(X^*)^{(n)}$ have norm

$$\|\phi\| = \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1\,,$$

$$\phi = \sum_{i=1}^{n} \oplus f_i \in (X^*)^{(n)}\,.$$

PROPOSITION 6. *The Banach spaces $(X^{(n)})^*$ and $(X^*)^{(n)}$ are isometrically isomorphic under the correspondence $\mathscr{T} : (X^*)^{(n)} \to (X^{(n)})^*$ defined by*

$$\mathscr{T}\left(\sum_{i=1}^{n} \oplus f_i\right) = \Lambda\,,$$

$$\Lambda\left(\sum_{i=1}^{n} \oplus x_i\right) = \sum_{i=1}^{n} f_i(x_i), \qquad x_i \in X,\; f_i \in X^*,\; i = 1, 2, \ldots, n\,.$$

Proof. It is easy to see that $\mathscr{T}$ is one to one. Also, if $\psi \in (X^{(n)})^*$, then, defining $g_i(x) = \psi\left(\sum_{j=1}^{n} \oplus \delta_{ij} x\right)$ for every $x \in X$ and $i = 1, 2, \ldots, n$, we have $g_i \in X^*$ and $\psi = \sum_{i=1}^{n} \oplus g_i$. This implies that $\mathscr{T}$ is surjective. Now let $\phi = \sum_{i=1}^{n} \oplus f_i \in (X^*)^{(n)}$, $\mathscr{T}(\phi) = \Lambda$ and $x = \sum_{i=1}^{n} \oplus x_i$, $x_i \in X$, $i = 1, 2, \ldots, n$. Then

$$|\Lambda(x)| = \left|\sum_{i=1}^{n} f_i(x_i)\right| \leqslant \sum_{i=1}^{n} \|f_i\|\|x_i\| \leqslant \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q} \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}.$$

Hence $\|\Lambda\| \leqslant \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q}$. Since X is a uniformly convex Banach space for each f_i there exists a vector x_i, $\|x_i\| = 1$, such that $f_i(x_i) = \|f_i\|$ (x_i is unique if $f_i \neq 0$). Put

$$y = \sum_{i=1}^{n} \oplus(\|f_i\|^{q/p} x_i).$$

Then

$$\|y\| = \left(\sum_{i=1}^{n} (\|f_i\|^{q/p}\|x_i\|)^p\right)^{1/p} = \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/p}$$

and

$$\Lambda(y) = \sum_{i=1}^{n} f_i(\|f_i\|^{q/p} x_i) = \sum_{i=1}^{n} \|f_i\|^{(q/p+1)} = \sum_{i=1}^{n} \|f_i\|^q$$

$$= \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q} \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/p} = \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q} \|y\|.$$

Hence $\|\Lambda\| = \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q}$ and so the map $\mathscr{T} : (X^*)^{(n)} \to (X^{(n)})^*$ is an isometric isomorphism.

LEMMA 7. *$V(T^{(n)}) \subseteq$ co $V(T)$ for all positive integers n, where $T \in \mathscr{B}(X)$ and $T^{(n)}$ denotes the direct sum of n copies of T regarded as an operator on $X^{(n)}$.*

Proof. Let $\lambda \in V(T^{(n)})$. Then there exists a linear functional $\phi \in (X^{(n)})^*$ and $x \in X^{(n)}$ such that

$$\phi(x) = \|\phi\| = \|x\| = 1 \quad \text{and} \quad \phi(T^{(n)}x) = \lambda.$$

Equivalently there exists $f_i \in X^*$, $x_i \in X$, $i = 1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} f_i(x_i) = \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q} = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} = 1$$

and $\sum_{i=1}^{n} f_i(Tx_i) = \lambda$. But

$$1 = \sum_{i=1}^{n} f_i(x_i) \leqslant \sum_{i=1}^{n} |f_i(x_i)| \leqslant \sum_{i=1}^{n} \|f_i\|\|x_i\|$$

$$\leqslant \left(\sum_{i=1}^{n} \|f_i\|^q\right)^{1/q} \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} = 1.$$

Therefore $\sum_{i=1}^{n} f_i(x_i) = \sum_{i=1}^{n} |f_i(x_i)| = \sum_{i=1}^{n} \|f_i\|\|x_i\| = 1$. Let

$$\mathscr{I} = \{i : \|f_i\|\|x_i\| \neq 0\},$$

Then $\mathscr{I} \neq \varnothing$ (since $\sum_{i=1}^{n} \|f_i\|\|x_i\| = 1$), $\frac{f_i}{\|f_i\|}\left(T\left(\frac{x_i}{\|x_i\|}\right)\right) \in V(T)$ for every $i \in \mathscr{I}$ and

$$\sum_{i \in \mathscr{I}} \|f_i\|\|x_i\| \frac{f_i(Tx_i)}{\|f_i\|\|x_i\|} = \sum_{i \in \mathscr{I}} f_i(Tx_i) = \sum_{i=1}^{n} f_i(Tx_i) = \lambda \in \operatorname{co} V(T).$$

Hence $V(T^{(n)}) \subseteq \operatorname{co} V(T)$ for every integer n.

REMARK. Let T be a bounded linear operator on a uniformly convex Banach space X such that $0 \notin \operatorname{co} V(T)$. Then by Lemma 7, $0 \notin V(T^{(n)})$ for all n and so Lemma 2 implies that $T(n)$ is a full operator for all integers n.

In the sequel we use this fact to find conditions under which the inverse of an invertible operator is a weak limit of polynomials of the operator.

First, we give a result of independent interest in itself, and which is a generalization of Feintuch's Lemma 2 in [5].

COROLLARY 8. *Let X be a uniformly convex Banach space and let $X^{(n)}$ be the direct sum of n copies of X, under the norm*

$$\|x\| = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p},$$

where $1 < p < \infty$. Then $\operatorname{co} V(T) = \operatorname{co} V(T^{(n)})$ *for any bounded operator $T \in \mathscr{B}(X)$ and hence $V(\mathscr{B}(X), T) = V(\mathscr{B}(X^{(n)}), T^{(n)})$.*

Proof. Let $\lambda \in V(T)$. Then there exist a unit vector $x \in X$ and a continuous linear functional $f \in X^*$ with norm one such that $f(x) = 1$ and $\lambda = f(Tx)$. Put $\phi = f \oplus 0 \oplus \ldots \oplus 0$ and $y = x \oplus 0 \oplus \ldots \oplus 0$. Then, using Proposition 6, $\|\phi\| = 1$, $\|y\| = 1$, $\phi(y) = f(x) = 1$, $\phi \in (X^{(n)})^*$ and

$$\phi(T^{(n)}y) \in V(T^{(n)}).$$

But $\phi(T^{(n)}y) = f(Tx) = \lambda$. Hence $\lambda \in V(T^{(n)})$ and so $V(T) \subseteq V(T^{(n)})$. Therefore

$$\operatorname{co} V(T) \subseteq \operatorname{co} V(T^{(n)}).$$

The reverse inclusion is an immediate consequence of Lemma 7. Hence

$$\operatorname{co} V(T) = \operatorname{co} V(T^{(n)}).$$

Now since $\overline{\operatorname{co}}\, V(T) = V(\mathscr{B}(X), T)$ it is clear that

$$V(\mathscr{B}(X), T) = V(\mathscr{B}(X^{(n)}), T^{(n)}).$$

THEOREM 9. *Let $\mathscr{A}(A, I)$ be the weakly closed algebra generated by the identity I and an invertible operator A acting on a uniformly convex Banach space. Then $A^{-1} \in \mathscr{A}(A, I)$ if and only if there exists an operator T in $\mathscr{A}(A, I)$ such that $0 \notin \operatorname{co} V(TA)$.*

Proof. If $A^{-1} \in \mathscr{A}(A, I)$ then the required operator T is the A^{-1}.

Conversely, suppose that there exists $T \in \mathscr{A}(A, I)$ such that $0 \notin \operatorname{co} V(TA)$. First we show that

$$\operatorname{Lat} \mathscr{A}(A, I) \subseteq \operatorname{Lat} A^{-1}.$$

Let $N \in \operatorname{Lat} \mathscr{A}(A, I)$. Then $N \in \operatorname{Lat} T \cap \operatorname{Lat}(AT)$. Since A is invertible, the subspace $A(N)$ is closed. From Lemma 2 we have that TA is full. Hence

$$N = \overline{TA(N)} = \overline{AT(N)} \subseteq A(N).$$

Therefore $A^{-1}(N) \subseteq N$ and so $N \in \operatorname{Lat} A^{-1}$. Thus $\operatorname{Lat} \mathscr{A}(A, I) \subseteq \operatorname{Lat} A^{-1}$.

For any integer n the weakly closed algebra generated by $A^{(n)}$ and $I^{(n)}$ is $[\mathscr{A}(A, I)]^{(n)}$. Also, using Lemma 7, $0 \notin \operatorname{co} V(TA)$ implies that $0 \notin V(T^{(n)}A^{(n)})$ and thus $T^{(n)}A^{(n)}$ is full for all n. Therefore

$$\operatorname{Lat} [\mathscr{A}(A, I)]^{(n)} \subseteq \operatorname{Lat} (A^{-1})^{(n)}$$

for all $n \in \mathbb{Z}^+$. Hence by [9, Theorem 7.1], this implies that $A^{-1} \in \mathscr{A}(A, I)$ and the proof is complete.

Note that the proof of Theorem 7.1 of [9] is given by the authors for Hilbert spaces but it is not difficult to see that it is also valid for Banach spaces.

REMARK. An operator T on a Hilbert space H is strictly positive if there exists a real number $\delta > 0$ such that for all $x \in H$, $\operatorname{Re}\langle Tx, x\rangle \geqslant \delta\|x\|^2$. A. Feintuch [7] proved that if A is an invertible operator on a Hilbert space then $A^{-1} \in \mathscr{A}(A, I)$ if and only if there exists an operator $T \in \mathscr{A}(A, I)$ such that $T^{-1} \in \mathscr{A}(A, I)$ and TA is strictly positive. He also mentioned as a corollary the following: $A^{-1} \in \mathscr{A}(A, I)$ if and only if there exists an invertible operator T such that T and T^{-1} are in $\mathscr{A}(A, I)$ and $0 \notin V(TA)$. But this in fact is obvious since $0 \notin V(TA)$ implies that $0 \notin V(T^{-1}A^{-1})$, and hence $\mathscr{A}(TA, I) = \mathscr{A}(T^{-1}A^{-1}, I) \subseteq \mathscr{A}(A, I)$ (see [5, Theorem 1]). Therefore $A^{-1} = T(T^{-1}A^{-1}) \in \mathscr{A}(A, I)$.

Now since the numerical range of an operator on a Hilbert space is always convex and never contains zero when the operator is strictly positive, it follows that Theorem 9 is stronger as well as being more general than Feintuch's results.

THEOREM 10. *Let A be an invertible operator on a uniformly convex Banach space and let $\mathscr{A}(A, I)$ be the weak closure of the algebra of polynomials in A. If $\mathscr{A}(A, I)$ contains a quasinilpotent operator T such that $0 \notin \operatorname{co} V(T)$ then $A^{-1} \in \mathscr{A}(A, I)$.*

Proof. Let $N \in \operatorname{Lat} \mathscr{A}(A, I)$. Since A is invertible, $A(N)$ is closed. Suppose that $A(N) \neq N$. We shall show that this leads to a contradiction. Let $x \in N \backslash A(N)$. Define

$$L = \operatorname{cls}\{A^n x : n \geqslant 0\}, \qquad M = \operatorname{cls}\{A^{n+1} x : n \geqslant 0\}.$$

Then L, M are invariant under A and so they belong to $\operatorname{Lat} \mathscr{A}(A, I)$. Consequently, $L, M \in \operatorname{Lat} T$. But $M \subset L$ and $\dim(L/M) = 1$. Since T is a quasinilpotent operator with $0 \notin \operatorname{co} V(T)$, Theorem 4 implies that this is a contradiction. Hence $A(N) = N$ and so $\operatorname{Lat} \mathscr{A}(A, I) \subseteq \operatorname{Lat} A^{-1}$.

We complete the proof by noting that the same argument applies to the algebra $[\mathcal{A}(A, I)]^{(n)}$. Indeed, since by Lemma 7, $0 \notin \operatorname{co} V(T)$ implies that $0 \notin V(T^{(n)})$ for all n we have

$$\operatorname{Lat}[\mathcal{A}(A, I)]^{(n)} \subseteq \operatorname{Lat}(A^{-1})^{(n)}$$

and [9, Theorem 7.1] completes the proof.

3. *Full operators*

It is clear that a necessary condition for the inverse of an invertible operator A on a Banach space to be in the weakly closed algebra $\mathcal{A}(A, I)$ generated by A and the identity I, is that $\operatorname{Lat} A \subseteq \operatorname{Lat} A^{-1}$. This is equivalent to A being a full operator. It is an open question whether this condition is also sufficient. From [9, Theorem 7.1] $A^{-1} \in \mathcal{A}(A, I)$ if and only if $\operatorname{Lat} A^{(n)} \subseteq \operatorname{Lat}(A^{-1})^{(n)}$ for all n. Equivalently $A^{-1} \in \mathcal{A}(A, I)$ if and only if $A^{(n)}$ is full for all n.

Let $\mathcal{A}(A)$ be the weakly closed algebra generated by A (and not by A and the identity I), in other words $\mathcal{A}(A)$ is the weak closure of the algebra of polynomials in A without constant terms. In the sequel we give conditions on $\mathcal{A}(A)$ each of which is necessary and sufficient for the operator $A^{(n)}$ to be full for all n.

PROPOSITION 11. *Let A be a bounded operator on a Banach space X and let $\mathcal{A}(A)$ be the weakly closed algebra generated by A. Then $A^{(n)}$ is full for every positive integer n if and only if $I \in \mathcal{A}(A)$.*

Proof. If $I \in \mathcal{A}(A)$ then obviously $A^{(n)}$ is full for all $n \in \mathbb{Z}^+$. For the converse, suppose that $A^{(n)}$ is full for every n. Let $\{x_1, x_2, \ldots, x_k\}$ be any vectors in X and

$$U_I = \{B \in \mathcal{B}(X) : \|Bx_i - x_i\| < \varepsilon \text{ for } i = 1, 2, \ldots, k\}$$

be any basic strong neighbourhood of the identity I. We have to show that $\mathcal{A}(A)$ contains an operator in U_I.

Let M be the smallest invariant subspace of $\mathcal{A}(A^{(k)})$ containing the vector $x_1 \oplus x_2 \oplus \ldots \oplus x_k$. Consider the set

$$N = \{Tx_1 \oplus Tx_2 \oplus \ldots \oplus Tx_k : T \in \mathcal{A}(A)\}$$

and let $L = \operatorname{cls}\{x_1 \oplus x_2 \oplus \ldots \oplus x_k, N\}$. Then since $M \in \operatorname{Lat} \mathcal{A}(A^{(k)})$ and $x_1 \oplus x_2 \oplus \ldots \oplus x_k \in M$ we have $N \subseteq M$. Also $L \in \operatorname{Lat} \mathcal{A}(A^{(k)})$ and L contains the vector $x_1 \oplus x_2 \oplus \ldots \oplus x_k$. Hence $L = M$. Moreover it is easy to see that $A^{(k)}(L) \subseteq \bar{N}$. But $A^{(k)}$ is full, hence

$$M = \overline{A^{(k)}(M)} = \overline{A^{(k)}(L)} \subseteq \bar{N},$$

which implies that $M = \bar{N}$, and so $x_1 \oplus x_2 \oplus \ldots \oplus x_k \in \bar{N}$. Hence there exists $T \in \mathcal{A}(A)$ such that

$$\|(Tx_1 \oplus Tx_2 \oplus \ldots \oplus Tx_k) - (x_1 \oplus x_2 \oplus \ldots \oplus x_k)\| < \varepsilon.$$

This implies that $T \in U_I$. Therefore $I \in \mathcal{A}(A)$.

Let X be a uniformly convex Banach space, let $X^{(n)}$ be the direct sum of n copies of X with norm $\|x\| = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}$, $1 < p < \infty$, $x_i \in X$, $i = 1, 2, \ldots, n$, and let A be a bounded operator on X. The subsequent results include necessary and sufficient conditions on $\mathscr{A}(A)$ such that $A^{(n)}$ be full for all n.

Proposition 12. *Let A be a bounded operator on a uniformly convex Banach space and let $\mathscr{A}(A)$ be the weakly closed algebra generated by A. Then $A^{(n)}$ is full for every n if and only if there exists an operator $T \in \mathscr{A}(A)$ such that $0 \notin \operatorname{co} V(T)$.*

Proof. If $A^{(n)}$ is full for all n then, by Proposition 11, this is equivalent to $I \in \mathscr{A}(A)$. Hence in this case take $T = I$.

Conversely, suppose that there exists $T \in \mathscr{A}(A)$ such that $0 \notin \operatorname{co} V(T)$. Then Lemma 7 implies that $0 \notin V(T^{(n)})$ for all n and thus by Lemma 2, $T^{(n)}$ is full for all n. Hence $I \in \mathscr{A}(T) \subseteq \mathscr{A}(A)$ and therefore $A^{(n)}$ is full for every $n \in \mathbb{Z}^+$.

Corollary 13. *Let A be an invertible operator on a uniformly convex Banach space. Then $A^{(n)}$ is full for all n if and only if there exists $T \in \mathscr{A}(A)$ such that $0 \notin \operatorname{co} V(TA)$.*

Proof. If $A^{(n)}$ is full for all n then $I \in \mathscr{A}(A)$ and

$$\operatorname{Lat}[\mathscr{A}(A)]^{(n)} = \operatorname{Lat}[\mathscr{A}(A, I)]^{(n)} \subseteq \operatorname{Lat}(A^{-1})^{(n)} \qquad \text{for all } n.$$

Hence by [**9**, Theorem 7.1], $A^{-1} \in \mathscr{A}(A)$ and so $T = A^{-1}$ satisfies the conditions. Conversely, if there exists $T \in \mathscr{A}(A)$ such that $0 \notin \operatorname{co} V(TA)$ then $T^{(n)}A^{(n)}$ is full for all n and hence $I \in \mathscr{A}(TA) \subseteq \mathscr{A}(A)$. Therefore $A^{(n)}$ is full for every $n \in \mathbb{Z}^+$.

Corollary 14. *Let A be an invertible operator on a uniformly convex Banach space. The following are equivalent.*

(i) $A^{-1} \in \mathscr{A}(A)$.

(ii) *There exists $T \in \mathscr{A}(A)$ such that $0 \notin \operatorname{co} V(T)$.*

(iii) *There exists $T \in \mathscr{A}(A)$ such that $0 \notin \operatorname{co} V(TA)$.*

Proof. This uses the equivalence of the statements $A^{-1} \in \mathscr{A}(A)$ and $A^{(n)}$ is full for all $n \in \mathbb{Z}^+$.

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